AN ANALYTIC PROOF OF THE ROGERS-RAMANUJAN-GORDON IDENTITIES.

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1. Introduction. In [2], Gordon gives a combinatorial proof of the following beautiful generalization of the Rogers-Ramanujan identities.

Theorem. *Let a and k be integers with $0 < a \leqq k$. Let $A_{k,a}(n)$ denote the number of partitions of n into parts not of the forms $(2k+1)m$, $(2k+1)m \pm a$; $A_{k,a}(0) = 1$. Let $B_{k,a}(n)$ denote the number of partitions of n of the form*

$$n = b_1 + \cdots + b_s$$

with $b_i \geqq b_{i+1}$, $b_i - b_{i+k-1} \geqq 2$, and with 1 appearing as a summand at most $a-1$ times; $B_{k,a}(0) = 1$. Then

$$A_{k,a}(n) = B_{k,a}(n).$$

Gordon states, however, [2; p. 394] that he has been unable to deduce this theorem from the corresponding identities of Alder [1] which generalize the Rogers-Ramanujan identities analytically.

The object of this paper is to give an analytic proof of Gordon's theorem along the lines of Ramanujan's proof of the original identities [4].

2. Proof of the theorem. We define

$$C_{k,i}(x) = 1 - x^i q^i + \sum_{\mu=1}^{\infty} (-1)^{\mu} x^{k\mu} q^{\frac{1}{2}(2k+1)\mu(\mu+1)-i\mu} (1 - x^i q^{(2\mu+1)i}) \frac{(1-xq)\cdots(1-xq^{\mu})}{(1-q)\cdots(1-q^{\mu})}.$$

Selberg [5; p. 4, equation 3] has proved

$$(2.1) \qquad C_{k,i}(x) = C_{k,i-1}(x) + x^{i-1}q^{i-1}(1-xq)C_{k,k-i+1}(xq).$$

If we define

$$Q_{k,i}(x) = C_{k,i}(x) \prod_{j=1}^{\infty} (1-xq^j)^{-1},$$

then

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(2.2) $$Q_{k,i}(x) = Q_{k,i-1}(x) + x^{i-1}q^{i-1}Q_{k,k-i+1}(xq).$$

We may expand $Q_{k,i}(x)$ as follows

(2.3) $$Q_{k,i}(x) = \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} c_{k,i}(M,N)x^M q^N, \ |x| \leqq 1, |q| < 1.$$

We then easily verify by means of the definition of $Q_{k,i}(x)$ and (2.2) that

(2.4) $$c_{k,0}(M,N) = 0 \text{ for all } k, M, N.$$

(2.5) $$c_{k,i}(M,N) = \begin{cases} 1 \text{ if } M = 0 \text{ and } N = 0, 1 \leqq i \leqq k \\ 0 \text{ if either } M \leqq 0 \text{ or } N \leqq 0 \text{ and not both are } 0 \end{cases}$$

(2.6) $$c_{k,i}(M,N) - c_{k,i-1}(M,N) = c_{k,k-i+1}(M-i+1, N-M), \ 1 \leqq i \leqq k.$$

One easily verifies by methematical induction that the $c_{k,i}(M,N)$ are uniquely determined by (2.4), (2.5), and (2.6).

Let $p_{k,i}(M,N)$ denote the number of partitions of N into M parts of the form $N = b_1 + \cdots + b_M$, with $b_i \geqq b_{i+1}$, $b_i - b_{i+k-1} \geqq 2$, and 1 appearing as a summand at most $i-1$ times. The $p_{k,i}(M,N)$ clearly satisfy (2.4) and (2.5). We now show that they satisfy (2.6).

$p_{k,i}(M,N) - p_{k,i-1}(M,N)$ enumerates the number of partitions of the form given in the previous paragraph with the added condition that 1 appears *exactly* $i-1$ times as a summand. We note that if 1 appears exactly $i-1$ times then 2 can appear at most $k-i$ times. Let us now subtract 1 from every summand of the partition under consideration. Since 1 appeared exactly $i-1$ times we have reduced the number of summands to $M-i+1$. Since we have subtracted M ones from our partition, we are now partitioning $N-M$. Since 2 could appear at most $k-i$ times formerly, now 1 appears at most $k-i$ times. Thus we now have a partition of the form enumerated by $p_{k,k-i+1}(M-i+1, N-M)$. The above procedure clearly establishes a one-to-one correspondence between the partitions enumerated by $p_{k,i}(M,N) - p_{k,i-1}(M,N)$ and the partitions enumerated by $p_{k,k-i+1}(M-i+1, N-M)$. Therefore

$$p_{k,i}(M,N) - p_{k,i-1}(M,N) = p_{k,k-i+1}(M-i+1, N-M).$$

Thus the $p_{k,i}(M,N)$ fulfill (2.4), (2.5), and (2.6). Therefore by the remark following (2.6) we have

(2.7) $$p_{k,i}(M,N) = c_{k,i}(M,N).$$

Thus

$$\sum_{N=0}^{\infty} A_{k,a}(N)\,q^N = \prod_{n \not\equiv 0, \pm a (\mathrm{mod}\ 2k+1)}^{\infty} (1-q^n)^{-1}$$
$$= Q_{k,a}(1) = \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} p_{k,a}(M,N)\,q^N = \sum_{N=0}^{\infty} B_{k,a}(N)\,q^N,$$

where the second equality follows from Jacobi's identity [3; p. 282]. Therefore $A_{k,a}(N) = B_{k,a}(N)$.

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REFERENCES.

[1] H. L. Alder, "Generalizations of the Rogers-Ramanujan identities," *Pacific Journal of Mathematics*, vol. 4 (1954), pp. 161-168.

[2] B. Gordon, "A combinatorial generalization of the Rogers-Ramanujan identities," *American Journal of Mathematics*, vol. 83 (1961), pp. 393-399.

[3] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford University Press, Oxford, 1960.

[4] S. Ramanujan, "Proof of certain identities in combinatory analysis," *Proceedings of the Cambridge Philosophical Society*, vol. 19 (1919), pp. 214-216.

[5] A. Selberg, "Uber einige arithmetische Identitäten," *Avhandlinger Norske Akademie*, 1936, no. 8.