ENUMERATIVE PROOFS OF CERTAIN q–IDENTITIES

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1. Introduction. Many q-identities have been proved combinatorially. For example,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)^2 \dots (1-q^n)^2} = \prod_{n=1}^{\infty} (1-q^n)^{-1}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} z^n}{(1-q) \dots (1-q^n)} = \prod_{n=0}^{\infty} (1+zq^n), \tag{1.2}$$

$$\sum_{n=0}^{\infty} \frac{z^n}{(1-q) \dots (1-q^n)} = \prod_{n=0}^{\infty} (1-zq^n)^{-1}, \tag{1.3}$$

$$\prod_{n=0}^{\infty} \{(1-q^{2n+2})(1+q^{2n+1}z)(1+q^{2n+1}z^{-1})\} = \sum_{n=-\infty}^{\infty} q^{n^2} z^n. \tag{1.4}$$

Combinatorial proofs of (1.1), (1.2), and (1.3) are either given or indicated in Hardy and Wright [**4**; Ch. XIX]. (1.4) has been proved combinatorially by Sylvester [**8**; pp. 34–36], Cheema [**2**; p. 415], and Wright [**10**]; Professor Wright also informs me that C. Sudler has a combinatorial proof of (1.4).

The main object of this paper is to give partition-theoretic proofs of other famous q-identities. In particular, in § 2 we shall prove that

$$\sum_{n=0}^{\infty} \frac{(1+a) \dots (1+aq^{n-1}) z^n q^n}{(1-q) \dots (1-q^n)} = \prod_{j=1}^{\infty} \frac{(1+azq^j)}{(1-zq^j)}, \tag{1.5}$$

and in § 3 we shall prove that

$$\sum_{n=0}^{\infty} \prod_{j=0}^{n-1} \left\{ \frac{(1-\alpha q^j)(1-\beta q^j)}{(1-q^{j+1})(1-\gamma q^j)} \right\} \tau^n = \prod_{j=0}^{\infty} \frac{(1-\beta q^j)(1-\alpha\tau q^j)}{(1-\gamma q^j)(1-\tau q^j)} \cdot \sum_{n=0}^{\infty} \prod_{j=0}^{n-1} \left\{ \frac{(1-\gamma\beta^{-1} q^j)(1-\tau q^j)}{(1-q^{j+1})(1-\alpha\tau q^j)} \right\} \beta^n. \tag{1.6}$$

(1.5) dates back to Euler [**3**; p. 223], and in fact (1.2) and (1.3) are special cases of (1.5). (1.6) is the fundamental transformation of basic hypergeometric series given by Heine [**5**; p. 106].

In § 4, we briefly indicate enumerative proofs of several other lesser known identities.

2. Proof of (1.5). In this section we shall be concerned with the following type of partitions, namely,

$$N = \sum_{j=1}^{s} a_j + \sum_{k=1}^{t} b_k \quad (a_1 \leqq \dots \leqq a_s,\ b_1 > \dots > b_t). \tag{2.1}$$

In the remainder of this section, we shall abbreviate our notation for such partitions to $a_1 \dots a_s \mid b_1 \dots b_t$.

Let $\pi_1(n,m;N)$ denote the number of partitions of N given in (2.1) subject to the further restrictions that $a_s = n$, $a_s > b_1$, and t is either m or $m-1$.

Let $\pi_2(n,m;N)$ denote the number of partitions of N given in (2.1) subject only to the further restrictions that $t = m$, $s+t = n$.

Now

$$\sum_{n=0}^{\infty} \frac{(1+a)\dots(1+aq^{n-1})z^n q^n}{(1-q)\dots(1-q^n)} = \sum_{N=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \pi_1(n,m;N)\, a^m z^n q^N,$$

and

$$\prod_{j=1}^{\infty} \frac{(1+azq^j)}{(1-zq^j)} = \sum_{N=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \pi_2(n,m;N)\, a^m z^n q^N.$$

Thus, defining $\pi_1(0,0;0) = \pi_2(0,0;0) = 1$, we must show that

$$\pi_1(n,m;N) = \pi_2(n,m;N)$$

in order to establish (1.5).

Suppose $a_1 \dots a_s \mid b_1 \dots b_t$ is a partition of N enumerated by $\pi_1(n,m;N)$. Then b rearranging terms we may form an ordinary partition of N of the form $f_1c_1 + \dots + f_rc_r$, wher $c_1 < \dots < c_r = n$ (f_j denotes the number of times c_j occurs in the partition). We now not that there may be several partitions enumerated by $\pi_1(n,m;N)$ that yield upon rearrange ment the same ordinary partition $f_1c_1 + \dots + f_rc_r$. In fact all we need do is pick either m o $m-1$ distinct parts from among the c's (excluding c_r) to form the b's with the remainder form ing the a's. Thus there are

$$\binom{r-1}{m} + \binom{r-1}{m-1} = \binom{r}{m}$$

partitions enumerated by $\pi_1(n,m;N)$ that correspond to the ordinary partition $f_1c_1 + \dots + f_rc_r$ $(c_1 < \dots < c_r = n)$.

Now, by considering conjugate partitions, we see that there is a one-to-one correspondence between ordinary partitions of the form $f_1c_1 + \dots + f_rc_r$ $(c_1 < \dots < c_r = n)$ and ordinary partitions of the form $f_1'c_1' + \dots + f_r'c_r'$ $(f_1' + \dots + f_r' = n)$.

Suppose that $a_1' \dots a_s' \mid b_1' \dots b_t'$ is a partition of N enumerated by $\pi_2(n,m;N)$. Then by rearranging terms we may form an ordinary partition of N of the form $f_1'c_1' + \dots + f_r'c_r'$ $(f_1' + \dots + f_r' = n)$. As above, several partitions enumerated by $\pi_2(n,m;N)$ may yield the same ordinary partition. Now to form a partition enumerated by $\pi_2(n,m;N)$ from the given ordinary partition, we need only choose m distinct parts from among the c's to form the b's; the remaining summands make up the a's. Thus, in this case as well, there are

$$\binom{r}{m}$$

partitions enumerated by $\pi_2(n,m;N)$ that correspond to the ordinary partition $f_1'c_1' + \dots + f_r'c_r'$ (with $c_1' < \dots < c_r'$, $f_1' + \dots + f_r' = n$).

Consequently we have $\pi_1(n,m;N) = \pi_2(n,m;N)$.

To illustrate, we enumerate all cases for $n = 4$, $m = 2$, $N = 9$. Column I gives the partitions enumerated by $\pi_1(4,2;9)$. Column II gives the related ordinary partitions. Column III gives the ordinary partitions conjugate to those of Column II. Column IV gives the corresponding partitions enumerated by $\pi_2(4,2;9)$.

I	II	III	IV
44 | 1	441	3222	22 | 32
134 | 1 114 | 3 14 | 31	4311	4221	22 | 41 24 | 21 12 | 42
1124 | 1 1114 | 2 114 | 21	42111	5211	11 | 52 12 | 51 15 | 21
11114 | 1	411111	6111	11 | 61
224 | 1 124 | 2 24 | 21	4221	4311	14 | 31 13 | 41 11 | 43
34 | 2 4 | 32 24 | 3	432	3321	33 | 21 23 | 31 13 | 32

Thus $\pi_1(4,2;9) = \pi_2(4,2;9) = 14$.

3. Proof of (1.6). We shall now consider partitions of N of the form

$$N = \sum_{i=1}^{p} a_i + \sum_{h=1}^{r} t_h + \sum_{j=1}^{s} b_j + \sum_{k=1}^{w} c_k, \tag{3.1}$$

where $a_1 < \ldots < a_p$, $t_1 \leqq \ldots \leqq t_r$, $b_1 \leqq \ldots \leqq b_s$, $c_1 > \ldots > c_w$. In the remainder of this section, we shall abbreviate our notation for such partitions to

$$a_1 \ldots a_p \mid t_1 \ldots t_r \mid b_1 \ldots b_s \mid c_1 \ldots c_w.$$

Denote by $\pi(M_1, M_2, M_3, M_4; N)$ the number of partitions given by (3.1) subject to the further restrictions that $a_p \leqq M_2 - 1$, p is either M_1 or $M_1 - 1$, $t_r = M_2$, $s = M_3 - M_4$, $b_1 \geqq M_2 + 1$, $w = M_4$, $c_w \geqq M_2 + 1$.

Now, if

$$\begin{aligned} F(\alpha, \tau, \beta, \gamma) &= \sum_{n=0}^{\infty} \prod_{j=0}^{n-1} \left\{ \frac{(1+\alpha q^j)}{(1-q^{j+1})} \right\} \prod_{k=1}^{\infty} \left\{ \frac{(1+\gamma\beta q^{n+k})}{(1-\beta q^{n+k})} \right\} \tau^n q^n \\ &= \sum_{N=0}^{\infty} \sum_{M_1=0}^{\infty} \sum_{M_2=0}^{\infty} \sum_{M_3=0}^{\infty} \sum_{M_4=0}^{\infty} \pi(M_1, M_2, M_3, M_4; N) \alpha^{M_1} \tau^{M_2} \beta^{M_3} \gamma^{M_4} q^N, \end{aligned}$$

we see that (1.6) may be rewritten as

$$F(\alpha,\tau,\beta,\gamma) = F(\gamma,\beta,\tau,\alpha).$$

Thus, defining $\pi(0,0,0,0;0) = 1$, we must show that

$$\pi(M_1,M_2,M_3,M_4;N) = \pi(M_4,M_3,M_2,M_1;N).$$

Suppose that we are given a partition of N enumerated by $\pi(M_1, M_2, M_3, M_4;N)$; as in §2, we may by rearrangement of terms form an ordinary partition of N of the form

$$f_1e_1+\ldots+f_de_d+f_{d+1}g_1+\ldots+f_{d+u}g_u$$

($e_1 < \ldots < e_d = M_2$, $e_d < g_1 < \ldots < g_u$, $f_{d+1}+\ldots+f_{d+u} = M_3$). We now search for the number of ways that our ordinary partition may be rearranged into a partition enumerated by $\pi(M_1, M_2, M_3, M_4; N)$. We see that to get the a's we must choose either M_1 or M_1-1 distinct terms from among the e's (excluding e_d); the remaining summands among the e's form the t's. There are thus

$$\binom{d-1}{M_1}+\binom{d-1}{M_1-1}=\binom{d}{M_1}$$

ways of getting the a's and t's. Now we get the c's by choosing M_4 distinct parts from among the g's; the remaining terms from among the g's form the b's. There are thus

$$\binom{u}{M_4}$$

ways of getting the b's and c's. Hence there are

$$\binom{d}{M_1}\binom{u}{M_4}$$

ways of getting a partition enumerated by $\pi(M_1,M_2,M_3,M_4;N)$ from our given ordinary partition.

By considering conjugate partitions, we see that there is a one-to-one correspondence between ordinary partitions of N of the form

$$f_1e_1+\ldots+f_de_d+f_{d+1}g_1+\ldots+f_{d+u}g_u$$

($e_1 < \ldots < e_d = M_2$, $e_d < g_1 < \ldots < g_u$, $f_{d+1}+\ldots+f_{d+u} = M_3$) and those of the form

$$f'_1e'_1+\ldots+f'_ue'_u+f'_{u+1}g'_1+\ldots f'_{u+d}g'_d$$

($e'_1 < \ldots < e'_u = M_3$, $e'_u < g'_1 < \ldots < g'_d$, $f'_{u+1}+\ldots+f'_{u+d} = M_2$).

Thus, by the above reasoning, there are

$$\binom{u}{M_4}\binom{d}{M_1}$$

partitions enumerated by $\pi(M_4,M_3,M_2,M_1;N)$ that correspond to the conjugate of the ordinary partition considered earlier. Hence

$$\pi(M_1,M_2,M_3,M_4;N) = \pi(M_4,M_3,M_2,M_1;N).$$

To illustrate, we enumerate all cases for $M_1 = 3$, $M_2 = 4$, $M_3 = 3$, $M_4 = 2$, $N = 25$. Column I gives the partitions enumerated by $\pi(3,4,3,2;25)$. Column II gives the related ordinary partitions. Column III gives the ordinary partitions conjugate to those of Column II. Column IV gives the corresponding partitions enumerated by $\pi(2,3,4,3;25)$.

I	II	III	IV
23 | 4 | 5 | 65	655432	665431	1 | 3 | 6 | 654
12 | 24 | 5 | 65	6554221	764431	1 | 3 | 4 | 764
12 | 114 | 5 | 65	65542111	854431	1 | 3 | 4 | 854
12 | 4 | 7 | 65; 12 | 4 | 5 | 76; 12 | 4 | 6 | 75	765421	6544321	12 | 3 | 4 | 654; 1 | 23 | 4 | 654; 2 | 13 | 4 | 654
12 | 4 | 5 | 85	855421	65443111	1 | 113 | 4 | 654
13 | 14 | 5 | 65	6554311	755431	1 | 3 | 5 | 754
13 | 4 | 6 | 65	665431	655432	2 | 3 | 5 | 654
13 | 4 | 5 | 75	755431	6554311	1 | 13 | 5 | 654
12 | 14 | 6 | 65	6654211	754432	2 | 3 | 4 | 754
12 | 14 | 5 | 75	7554211	7544311	1 | 13 | 4 | 754

Thus $\pi(2,3,4,3;25) = \pi(3,4,3,2;25) = 12$.

4. Further identities. We shall deduce several identities from two combinatorial lemmas.

LEMMA 1. *Let $P_{a,b}(n)$ $(a = 0,1; b = 0,1)$ denote the number of partitions of n into distinct positive parts such that the number of parts is congruent to a* (mod 2) *and the largest part is congruent to b* (mod 2). *Let $Q_{a,b}(n)$ $(a = 0,1; b = 0,1)$ denote the number of partitions of n into distinct non-negative parts such that the number of parts is congruent to a* (mod 2) *and the largest part is congruent to b* (mod 2). *Then*

$$P_{0,b}(n)+P_{1,b}(n) = Q_{0,b}(n) = Q_{1,b}(n).$$

Proof. Since $P_{0,b}(n)+P_{1,b}(n)$ enumerates the number of partitions of n into distinct parts with largest part congruent to b (mod 2), add a zero to each partition enumerated by $P_{1,b}(n)$ and then the partitions enumerated are simply the partitions of n into an even number of non-negative parts with largest part congruent to b (mod 2); add a zero to each partition enumerated by $P_{0,b}(n)$ and then the partitions enumerated are simply the partitions of n into an odd number of non-negative parts with largest part congruent to b (mod 2).

Since

$$Q_{0,0}(n)+Q_{0,1}(n) = Q_{1,0}(n)+Q_{1,1}(n) = P_{0,0}(n)+P_{0,1}(n)+P_{1,0}(n)+P_{1,1}(n),$$

we deduce that

$$\sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(1-q)\dots(1-q^{2n})} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(1-q)\dots(1-q^{2n-1})} = \prod_{j=1}^{\infty} (1+q^j). \tag{4.1}$$

Since

$$Q_{0,0}(n)-Q_{0,1}(n) = Q_{1,0}(n)-Q_{1,1}(n) = P_{0,0}(n)+P_{1,0}(n)-P_{0,1}(n)-P_{1,1}(n),$$

we deduce that

$$2-\sum_{n=0}^{\infty}\frac{q^{n(2n-1)}}{(1+q)\dots(1+q^{2n})}=\sum_{n=0}^{\infty}\frac{q^{n(2n-1)}}{(1+q)\dots(1+q^{2n-1})}=\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(1+q)\dots(1+q^n)}. \tag{4.2}$$

Since

$$Q_{0,1}(n)-Q_{0,0}(n)+2(P_{0,0}(n)+P_{1,0}(n)) = Q_{1,0}(n)-Q_{1,1}(n)+2(P_{0,1}(n)+P_{1,1}(n)) = P_{0,0}(n)+P_{0,1}(n)+P_{1,0}(n)+P_{1,1}(n),$$

we deduce that

$$\sum_{n=0}^{\infty}\frac{q^{n(2n-1)}}{(1+q)\dots(1+q^{2n})}+2\sum_{n=1}^{\infty}(1+q)\dots(1+q^{2n-1})q^{2n}$$
$$=\sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{(1+q)\dots(1+q^{2n-1})}+2\sum_{n=0}^{\infty}(1+q)\dots(1+q^{2n})q^{2n+1}=\prod_{j=1}^{\infty}(1+q^j). \tag{4.3}$$

Since

$$Q_{0,1}(n)+Q_{1,0}(n)-Q_{0,0}(n)-Q_{1,1}(n)=0,$$

we deduce that

$$\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}n(n-1)}}{(1+q)\dots(1+q^n)}=2. \tag{4.4}$$

Since

$$Q_{0,0}(n)+Q_{0,1}(n)-Q_{1,0}(n)-Q_{1,1}(n)=0,$$

we deduce that

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{(1-q)\dots(1-q^n)}=0. \tag{4.5}$$

We remark that (4.1) was originally proved by L. J. Slater [7; equations (84) and (85)]; (4.2) and (4.4) appear in [1], and (4.5) is a special case of (1.2).

LEMMA 2. *Let $a(n)$ denote the number of partitions of n with unique smallest part and largest part at most twice the smallest part. Let $b(n)$ denote the number of partitions of n in which the largest part is odd and the smallest part is larger than half the largest part. Then $a(n) = b(n)$.*

Proof. In Figure 1, we give a graphical representation of a typical partition of n enumerated by $b(n)$.

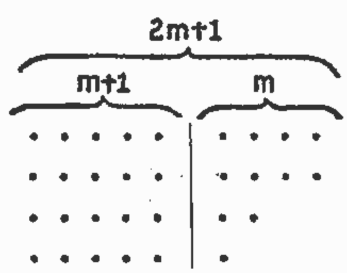

FIG. 1

We translate the set of nodes on the right of the vertical bar to a position directly below those nodes appearing on the left of the vertical bar. Our new graph is now pictured in Figure 2.

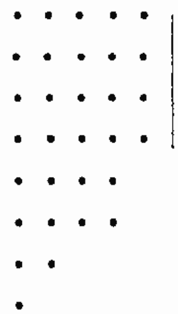

FIG. 2

Reading the graph in Figure 2 vertically, we see that now we have a partition of n which is of the type enumerated by $a(n)$. Clearly the process is reversible, and hence for every n, $a(n) = b(n)$.

Now

$$\sum_{n=0}^{\infty} a(n)q^n = \sum_{m=0}^{\infty} \frac{q^m}{(1-q^{m+1})\dots(1-q^{2m})},$$

and

$$\sum_{n=0}^{\infty} b(n)q^n = 1 + \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(1-q^{m+1})\dots(1-q^{2m+1})}.$$

Consequently,

$$\sum_{m=0}^{\infty} \frac{q^m}{(1-q^{m+1})\dots(1-q^{2m})} = 1 + \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(1-q^{m+1})\dots(1-q^{2m+1})}. \tag{4.6}$$

This identity was stated by Ramanujan in his last letter to Hardy [**6**; p. 354] and was later proved by Watson [**9**; p. 278].

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