A GENERAL THEOREM ON PARTITIONS WITH DIFFERENCE CONDITIONS.

By George E. Andrews.[1]

1. Introduction. The Rogers-Ramanujan identities may be stated partition-theoretically as follows [5; Ch. 3].

If $c = 1$ or 2, then the number of partitions of n into parts $\equiv \pm c \pmod 5$ equals the number of partitions of n into parts $\geqq c$ with minimal difference 2 between summands.

The discovery of the partition-theoretic statement of the Rogers- Ramanujan identities by MacMahon and Schur [6] prompted a search for similar partition theorems with a difference other than 2 required of the summands in one class of the partitions considered. In 1926, Schur [7] proved the following theorem.

Theorem 2. *Let* $C_1(n)$ *denote the number of partitions of* n *into parts* $\equiv \pm 1 \pmod 6$. *Let* $D_1(n)$ *denote the number of partitions of* n *into distinct parts* $\equiv \pm 1 \pmod 3$. *Let* $E_1(n)$ *denote the number of partitions of* n *of the form* $n = b_1 + \cdots + b_s$, *where* $b_i - b_{i+1} \geqq 3$ *with strict inequality if* $3 \mid b_{i+1}$. Then $C_1(n) = D_1(n) = E_1(n)$.

The difficult part of this theorem lies in proving $D_1(n) = E_1(n)$. The fact that $C_1(n) = D_1(n)$ follows directly from

$$\prod_{j=0}^{\infty} (1 + q^{3j+1})(1 + q^{3j+2}) = \prod_{j=0}^{\infty} (1 - q^{6j+1})^{-1}(1 - q^{6j+5})^{-1}.$$

In 1948, Alder [1] proved that if further identities exist with difference $d > 3$, then more complex conditions than those enunciated in Theorem 2 must hold. In this paper we shall prove a general partition theorem (Theorem 1) which contains Theorem 2 as a special case. Other special cases of Theorem 1 are the following new results.

Theorem 3. *Let* $C_2(n)$ *denote the number of partitions of* n *into parts*

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$\equiv 1, 9, 11 \pmod{14}$. *Let* $D_2(n)$ *denote the number of partitions of* n *into distinct parts* $\equiv 1, 2, 4 \pmod 7$. *Let* $E_2(n)$ *denote the number of partitions of* n *of the form* $n = b_1 + \cdots + b_s$, *where* $b_i - b_{i+1} \geqq 7$ *if* $b_{i+1} \equiv 1, 2, 4 \pmod 7$, $b_i - b_{i+1} \geqq 12$ *if* $b_{i+1} \equiv 3 \pmod 7$, $b_i - b_{i+1} \geqq 10$ *if* $b_{i+1} \equiv 5, 6 \pmod 7$ *and* $b_i - b_{i+1} \geqq 15$ *if* $b_{i+1} \equiv 0 \pmod 7$. Then $C_2(n) = D_2(n) = E_2(n)$.

Again here the fact that $C_2(n) = D_2(n)$ follows directly from

$$\prod_{j=0}^{\infty} (1 + q^{7j+1})(1 + q^{7j+2})(1 + q^{7j+4}) = \prod_{j=0}^{\infty} (1 - q^{14j+1})^{-1}(1 - q^{14j+9})^{-1}(1 - q^{14j+11})^{-1}.$$

THEOREM 4. *Let* $C_3(n)$ *denote the number of partitions of* n *into parts* $\equiv 1, 17, 19, 23 \pmod{30}$. *Let* $D_3(n)$ *denote the number of partitions of* n *into distinct parts* $\equiv 1, 2, 4, 8 \pmod{15}$. *Let* $E_3(n)$ *denote the number of partitions of* n *of the form* $n = b_1 + \cdots + b_s$, *where* $b_i - b_{i+1} \geqq 15$ *if* $b_{i+1} \equiv 1, 2, 4, 8 \pmod{15}$, $b_i - b_{i+1} \geqq 28$ *if* $b_{i+1} \equiv 3 \pmod{15}$, $b_i - b_{i+1} \geqq 26$ *if* $b_{i+1} \equiv 5, 6 \pmod{15}$, $b_i - b_{i+1} \geqq 22$ *if* $b_{i+1} \equiv 9, 10 \pmod{15}$, $b_i - b_{i+1} \geqq 39$ *if* $b_{i+1} \equiv 7 \pmod{15}$, $b_i - b_{i+1} \geqq 35$ *if* $b_{i+1} \equiv 11 \pmod{15}$, $b_i - b_{i+1} \geqq 33$ *if* $b_{i+1} \equiv 13, 14 \pmod{15}$, $b_i - b_{i+1} \geqq 46$ *if* $b_{i+1} \equiv 0 \pmod{15}$. *Then* $C_3(n) = D_3(n) = E_3(n)$.

As before we deduce that $C_3(n) = D_3(n)$ from

$$\prod_{j=0}^{\infty} (1 + q^{15j+1})(1 + q^{15j+2})(1 + q^{15j+4})(1 + q^{15j+8}) = \prod_{j=0}^{\infty} (1 - q^{30j+1})^{-1}(1 - q^{30j+17})^{-1}(1 - q^{30j+19})^{-1}(1 - q^{30j+23})^{-1}.$$

In Section 2, we shall make certain definitions and state Theorem 1. In Section 3, we shall prove Theorem 1 combining techniques developed in [2] and [3].

2. Preliminaries. Throughout this paper we shall write $2(n)$ for 2^n. We consider a set $A = \{a(1), \cdots, a(r)\}$ of r distinct positive integers which will be fixed throughout our discussion and which satisfy $\sum_{i=1}^{k-1} a(i) < a(k)$, $1 \leqq k \leqq r$. We require that A be such that the $2(r) - 1$ possible sums of distinct elements of A are also distinct; we denote this set of sums by A' and its elements by $\alpha(1) < \alpha(2) < \cdots < \alpha(2(r) - 1)$. From the previously stated inequalities for the a's, it is clear that $\alpha(2(i)) = a(i + 1)$ and that all α's with $a(k-1) \leqq \alpha < a(k)$ have $a(k-1)$ as the largest summand in their defining sum. We let N be a positive integer with

$$N \geqq \alpha(2(r)-1) = a(1) + a(2) + \cdots + a(r).$$

Let A_N be the set of all positive integers which are congruent to some $a(i)$ $(\text{mod}\, N)$. Let A'_N be the set of all positive integers which are congruent to some $\alpha(i)$ $(\text{mod}\, N)$. Let $\beta_N(m)$ denote the least positive residue of m $(\text{mod}\, N)$. If $m \in A'$, let $w(m)$ be the number of terms appearing in the defining sum of m, and let $v(m)$ denote the smallest $a(i)$ appearing in this sum. With these definitions, we are now prepared to state Theorem 1.

THEOREM 1. *Let $D(A_N;n)$ denote the number of partitions of n into distinct parts taken from A_N. Let $E(A'_N;n)$ denote the number of partitions of n into parts taken from A'_N of the form $n = b_1 + \cdots + b_s$, $b_i \geqq b_{i+1}$,*

$$b_i - b_{i+1} \geqq Nw(\beta_N(b_{i+1})) + v(\beta_N(b_{i+1})) - \beta_N(b_{i+1}).$$

Then $D(A_N;n) = E(A'_N;n)$.

Let us now note how Theorems 2-4 are derived from Theorem 1.

To prove Theorem 2, take $N=3$, $a(1)=1$, $a(2)=2$. Then immediately $D(A_N;n)$ becomes $D_1(n)$. Also we note that A'_N is the set of all positive integers. Finally if $b_{i+1} \equiv 1 \pmod 3$, then $b_i - b_{i+1} \geqq 3\cdot 1 + 1 - 1 = 3$; if $b_{i+1} \equiv 2 \pmod 3$, then $b_i - b_{i+1} \geqq 3\cdot 1 + 2 - 2 = 3$; if $b_{i+1} \equiv 3 \pmod 3$, then $b_i - b_{i+1} \geqq 3 \cdot 2 + 1 - 3 = 4$. Thus $E(A'_N;n) = E_1(n)$.

To prove Theorem 3, take $N=7$, $a(1)=1$, $a(2)=2$, $a(3)=4$. Then immediately $D(A_N;n) = D_2(n)$. Also we note again that A'_N is the set of all positive integers. Finally if $b_{i+1} \equiv 1 \pmod 7$, then $b_i - b_{i+1} \geqq 7\cdot 1 + 1 - 1 = 7$; if $b_{i+1} \equiv 2 \pmod 7$, then $b_i - b_{i+1} \geqq 7 \cdot 1 + 2 - 2 = 7$; if $b_{i+1} \equiv 3 \pmod 7$, then $b_i - b_{i+1} \geqq 7\cdot 2 + 1 - 3 = 12$; if $b_{i+1} \equiv 4 \pmod 7$, then $b_i - b_{i+1} \geqq 7 \cdot 1 + 4 - 4 = 7$; if $b_{i+1} \equiv 5 \pmod 7$, then $b_i - b_{i+1} \geqq 7 \cdot 2 + 1 - 5 = 10$; if $b_{i+1} \equiv 6 \pmod 7$, then $b_i - b_{i+1} \geqq 7\cdot 2 + 2 - 6 = 10$; if $b_{i+1} \equiv 7 \pmod 7$, then $b_i - b_{i+1} \geqq 7 \cdot 3 + 1 - 7 = 15$. Thus $E(A'_N;n) = E_2(n)$.

Theorem 4 is proved similarly.

3. Proof of Theorem 1. Let $P_{\alpha(i)}(m,n)$ denote the number of partitions of n into m parts of the type enumerated by $E(A'_N;n)$ with the added restriction that the smallest part appearing in any partition is $\geqq \alpha(i)$, $1 \leqq i \leqq 2(r)$ (defining $\alpha(2(r)) = a(r+1) = N + a(1)$).

Define

$$f_{\alpha(i)}(x) \equiv f_{\alpha(i)}(x;q) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} p_{\alpha(i)}(m,n) x^m q^n, \qquad q < 1.$$

LEMMA 1. *If* $1 \leqq i \leqq 2(r) - 1$, *then*

$$(3.1) \quad p_{\alpha(i)}(m,n) - p_{\alpha(i+1)}(m,n) = p_{v(\alpha(i))}(m-1, n-(m-1)Nw(\alpha(i)) - \alpha(i)),$$

$$(3.2) \quad p_{\alpha(2(r))}(m,n) = p_{\alpha(1)}(m, n - Nm).$$

Proof. We note that $p_{\alpha(i)}(m,n) - p_{\alpha(i+1)}(m,n)$ enumerates all those partitions enumerated by $p_{\alpha(i)}(m,n)$ in which $\alpha(i)$ actually appears. Hence by the conditions defining these partitions, we know that if $\alpha(i) = b_s$ is the least part appearing then the next largest part

$$b_{s-1} \geqq \alpha(i) + Nw(\alpha(i)) + v(\alpha(i)) - \alpha(i) = Nw(\alpha(i)) + v(\alpha(i)).$$

We now consider a general partition of the type enumerated by $p_{\alpha(i)}(m,n) - p_{\alpha(i+1)}(m,n)$; we delete $\alpha(i)$ and subtract $Nw(\alpha(i))$ from all other parts. We are now partitioning $n-(m-1)Nw(\alpha(i)) - \alpha(i)$ into $m-1$ parts and the smallest part is $\geqq v(\alpha(i))$. Thus we have a partition of the type enumerated by $p_{v(\alpha(i))}(m-1, n-(m-1)Nw(\alpha(i)) - \alpha(i))$. The above procedure establishes a one-to-one correspondence between the partitions of the type enumerated by $p_{\alpha(i)}(m,n) - p_{\alpha(i+1)}(m,n)$ and those enumerated by $p_{v(\alpha(i))}(m-1, n-(m-1)Nw(\alpha(i)) - \alpha(i))$. Hence (3.1) is established.

For (3.2) we consider any partition of the type enumerated by $p_{\alpha(2(r))}(m,n)$, and we subtract N from every summand. We now have a partition of the type enumerated by $p_{\alpha(1)}(m, n-Nm)$. As above, this is sufficient to establish (3.2).

Lemma 1 directly implies

$$(3.3) \quad f_{\alpha(i)}(x) - f_{\alpha(i+1)}(x) = xq^{\alpha(i)} f_{v(\alpha(i))}(xq^{Nw(\alpha(i))}), \quad 1 \leqq i \leqq 2(r) - 1,$$

$$(3.4) \quad f_{\alpha(2(r))}(x) = f_{a(1)}(xq^N).$$

Since $\alpha(2(i)) = a(i+1)$, we may add the equations (3.3) together for $1 \leqq i \leqq 2(k-1) - 1$, and we obtain

$$(3.5) \quad f_{a(1)}(x) - f_{a(k)}(x) = \sum_{\alpha < a(k)} xq^{\alpha} f_{v(\alpha)}(xq^{Nw(\alpha)}).$$

If now we add the equations (3.3) together for $2(k-2) \leqq i < 2(k-1)$, we obtain

$$(3.6) \quad f_{a(k-1)}(x) - f_{a(k)}(x) = \sum_{a(k-1) \leqq \alpha < a(k)} xq^{\alpha} f_{v(\alpha)}(xq^{Nw(\alpha)}).$$

Now every α in the interval $(a(k-1), a(k))$ is of the form $a(k-1) + \alpha'$ where $\alpha' < a(k-1)$. Hence

$$(3.7)\quad f_{a(k-1)}(x)-f_{a(k)}(x)=xq^{a(k-1)}f_{a(k-1)}(xq^N)+q^{a(k-1)-N}\sum_{\alpha'<a(k-1)}xq^{\alpha'+N}f_{v(\alpha')}(xq^{N(w(\alpha')+1)})$$
$$=xq^{a(k-1)}f_{a(k-1)}(xq^N)+q^{a(k-1)-N}(f_{a(1)}(xq^N)-f_{a(k-1)}(xq^N))$$
$$=q^{a(k-1)-N}f_{a(1)}(xq^N)-q^{a(k-1)-N}(1-xq^N)f_{a(k-1)}(xq^N).$$

LEMMA 2. *If* $1\leqq k\leqq r+1$,

$$(3.8)\quad f_{a(1)}(x)=f_{a(k)}(x)+\sum_{j=1}^{k-1}\Big(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=j}}xq^{\alpha}\Big)\prod_{h=1}^{j-1}(1-xq^{hN})f_{\alpha(1)}(xq^{jN}).$$

Proof. For $k=1$, (3.8) reduces to $f_{\alpha(1)}(x)=f_{\alpha(1)}(x)$. Assume (3.8) true for a particular $k<r+1$. Then

$$f_{a(1)}(x)-f_{a(k+1)}(x)=(f_{a(1)}(x)-f_{a(k)}(x))+(f_{a(k)}(x)-f_{a(k+1)}(x))$$
$$=\sum_{j=1}^{k-1}\Big(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=j}}xq^{\alpha}\Big)\prod_{h=1}^{j-1}(1-xq^{hN})f_{a(1)}(xq^{jN})+q^{a(k)-N}f_{a(1)}(xq^N)-q^{a(k)-N}(1-xq^N)f_{a(k)}(xq^N)$$
$$=\sum_{j=1}^{k-1}\Big(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=j}}xq^{\alpha}\Big)\prod_{h=1}^{j-1}(1-xq^{hN})f_{a(1)}(xq^{jN})+q^{a(k)-N}f_{a(1)}(xq^N)$$
$$-q^{a(k)-N}(1-xq^N)\Big(f_{a(1)}(xq^N)-\sum_{j=1}^{k-1}\Big(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=j}}xq^{N+\alpha}\Big)\prod_{h=1}^{j-1}(1-xq^{hN+N})f_{a(1)}(xq^{jN+N})\Big)$$
$$=\sum_{j=1}^{k-1}\Big(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=j}}xq^{\alpha}\Big)\prod_{h=1}^{j-1}(1-xq^{jN})f_{a(1)}(xq^{jN})+xq^{a(k)}f_{a(1)}(xq^N)$$
$$+\sum_{j=1}^{k-1}\Big(\sum_{\substack{a(k)<\alpha'<a(k+1)\\ w(\alpha)=j+1}}xq^{\alpha'}\Big)\prod_{h=1}^{j}(1-xq^{N})f_{a(1)}(xq^{jN+N})$$
$$=\sum_{j=1}^{k-1}\Big(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=j}}xq^{\alpha}\Big)\prod_{h=1}^{j-1}(1-xq^{hN})f_{a(1)}(xq^{jN})$$
$$+\sum_{j=1}^{k}\Big(\sum_{\substack{a(k)\leqq\alpha'<a(k+1)\\ w(\alpha')=j}}xq^{\alpha'}\Big)\prod_{h=1}^{j-1}(1-xq^{hN})f_{a(1)}(xq^{jN})$$
$$=\sum_{j=1}^{k}\Big(\sum_{\substack{\alpha<a(k+1)\\ w(\alpha)=j}}xq^{\alpha}\Big)\prod_{h=1}^{j-1}(1-xq^{hN})f_{a(1)}(xq^{jN}).$$

Thus we obtain (3.8) for $k+1$, and the lemma is proved.

We are now prepared to treat the main theorem.

Proof of Theorem 1. First we note that

$$(3.9)\qquad 1+\sum_{n=1}^{\infty} E(A'_N;n)q^n = f_{a(1)}(1).$$

Now by Lemma 2 and (3.4)

$$(3.10)\qquad f_{a(1)}(x) = f_{a(1)}(xq^N) + \sum_{j=1}^{r}\Big(\sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j}} xq^{\alpha}\Big)\prod_{h=1}^{j-1}(1-xq^{hN})f_{a(1)}(xq^{jN}).$$

Define

$$(3.11)\qquad G(x) = \prod_{t=0}^{\infty}(1-xq^{Nt})^{-1}f_{a(1)}(x).$$

If we divide equation (3.10) by $\prod_{t=1}^{\infty}(1-xq^{Nt})$, we find

$$(3.12)\qquad (1-x)G(x) = G(xq^N) + \sum_{j=1}^{r}\Big(\sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j}} xq^{\alpha}\Big)G(xq^{jN}).$$

We set $G(x) = \sum_{n=0}^{\infty} B_n x^n$; then $B_0 = 1$, and by comparing coefficients of x^n on both sides of (3.12), we obtain

$$(3.13)\qquad B_n - B_{n-1} = q^{Nn}B_n + \Big(\sum_{\alpha\in A'} q^{(n-1)w(\alpha)N+\alpha}\Big)B_{n-1}.$$

If we define $\alpha_0 = 0$, $w(\alpha_0) = 0$, $A^* = A'\cup\{\alpha_0\}$, then

$$\begin{aligned}(1-q^{Nn})B_n &= \Big(\sum_{\alpha\in A^*} q^{(n-1)w(\alpha)N+\alpha}\Big)B_{n-1}\\ (3.14)\qquad &= \Big(\sum q^{(n-1)jN+a(i_1)+\cdots+a(i_j)}\Big)B_{n-1}\\ &= (1+q^{(n-1)N+a(1)})(1+q^{(n-1)N+a(2)})\cdots(1+q^{(n-1)N+a(r)})B_{n-1},\end{aligned}$$

where the second sum is over all possible j-tuples of a's, $0\leqq j\leqq r$. Consequently

$$(3.15)\qquad B_n = \prod_{j=0}^{n=1}(1+q^{Nj+a(1)})(1+q^{Nj+a(2)})\cdots(1+q^{Nj+a(r)})(1-q^{Nj+N})^{-1}.$$

Finally by (3.9), (3.11), (3.15) and Appell's Comparison Theorem [4; p. 101], we obtain

$$\begin{aligned}1+\sum_{n-1}^{\infty}E(A'_N;n)q^n &= f_{a(1)}(1)\\ &= \lim_{x\to 1}\prod_{t=0}^{\infty}(1-xq^{Nt})G(x)\\ &= \prod_{t=1}^{\infty}(1-q^{Nt})\lim_{x\to 1}(1-x)G(x)\\ &= \prod_{t=1}^{\infty}(1-q^{Nt})\lim_{n\to\infty}B_n\end{aligned}$$

$$= \prod_{j=0}^{\infty} (1 + q^{Nj+a(1)}) (1 + q^{Nj+a(2)}) \cdots (1 + q^{Nj+a(r)})$$

$$= 1 + \sum_{n=1}^{\infty} D(A_N; n) q^n.$$

Thus $E(A'_N; n) = D(A_N; n)$.

THE PENNSYLVANIA STATE UNIVERSITY.

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