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A POLYNOMIAL IDENTITY WHICH IMPLIES THE ROGERS–RAMANUJAN IDENTITIES

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1. Introduction and history

In [37; pp. 90–99] Hardy gives a thorough account of the history of the Rogers–Ramanujan identities up to about 1930. In particular he cites the proofs of these identities appearing in [48], [49], [47], [50], and [60]. Since no recent paper known to me attempts any extensive account of work since then, I should like to give a brief resume of recent work.

F. H. Jackson did much pioneering work in q-series especially in [39]; a list of his publications is given in [30]. In the late 1920's, Schur [51] and Gleissberg [33] proved a result similar to the Rogers–Ramanujan identities for the modulus 6. Schur's theorem has also been proved in [10] and [14]. H. Gollnitz [34] proved a related result for the modulus 12; a second proof of Gollnitz's theorem appears in [18], and general theorems of this nature are proved in [9], [16], and [17]. Starcher [59] developed an extensive technique for obtaining q-identities and proved the Rogers–Ramanujan identities. Selberg [52] gave q-series theorems for the modulus 7, and Dyson [32] simplified the proofs of Selberg's identities and remarked that the results were originally due to Rogers [49]. In the papers [63] and [64], Watson rediscovered several formulae of Rogers [48; p. 330] very closely related to the Rogers–Ramanujan identities. Watson's results have been extended in [40], [4], [5], [6], and [1]. In the early 1940's, Lehner [42] obtained asymptotic formulae for the partition functions involved in the Rogers–Ramanujan identities; Niven [44] obtained similar formulae for the partition functions related to the modulus 6. Next Alder [2] and Lehmer [41] proved certain non-existence theorems related to possible generalizations of the Rogers–Ramanujan identities. Also in this period, Bailey and his student, Slater, obtained large numbers of q-series identities related to the Rogers–Ramanujan identities [22], [23], [24], [25], [56], [57]. Alder gave generalizations of the Rogers–Ramanujan identities involving certain polynomials [3]; his generalizations are also studied in [28], [53], [54], [55]. In 1961, Gordon gave a partition-theoretic generalization of the Rogers–Ramanujan identities [35]; the proof of this theorem is simplified in [8]. Partition theorems of the type studied by Gordon in [35] have been proved in [11], [12], and [13]; in [20] the main result includes Gordon's theorems in [35]

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and [36; p. 741], Schur's theorem [51], and the result in [13] as special cases. Gordon also gave some new continued fraction theorems related to the Rogers–Ramanujan identities in [36]; these results were extended by Carlitz [29], and a general theorem on such continued fractions was proved in [19]. A partition theorem of Sylvester's related to the series considered by Carlitz in [29] was treated in [7]. Dobbie [31] and Carlitz [26], [27] gave new proofs for the Rogers–Ramanujan identities. Finally the following comprehensive works study the Rogers–Ramanujan identities in detail: [43; pp. 33–48], [38; pp. 290–296], [21; pp. 70–72), [46; pp. 68–85], [45; pp. 46–49], [58; pp. 103–105, 199–203].

It should be remarked that the above references do not take into account the studies made of modular equations related to the Rogers-Ramanujan identities; for work in this area see [61], and [62].

The proof of the Rogers–Ramanujan identities to be presented in this paper is related to Schur's second proof in [50], in that one of the polynomials in our theorem was studied by Schur. However, the details here are somewhat simpler. We shall prove the following result.

THEOREM. *If* $\alpha = 0$, *or* -1, *then*

$$\sum_{j=0}^{\infty} q^{j^2-\alpha j} \begin{bmatrix} n+1+\alpha-j \\ j \end{bmatrix} = \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} q^{\frac{1}{2}\lambda(5\lambda+1)+2\alpha\lambda} \begin{bmatrix} n+1 \\ [\frac{1}{2}(n+1-5\lambda)]-\alpha \end{bmatrix},$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} = \prod_{j=1}^{m} \frac{(1-q^{n-j+1})}{(1-q^j)} \quad \text{if} \quad n \geqslant m \geqslant 0$$

$$= 0 \quad \textit{otherwise},$$

and $[x]$ *means the largest integer* $\leqslant x$.

In §2, we shall prove the above theorem. In §3, some corollaries of this theorem will be discussed.

2. Proof of theorem. We first remark that both expressions in the theorem are polynomials; this is because all but a finite number of terms in each sum are zero and each term is a polynomial. Let $E_n(\alpha; q)$ denote the left hand side of our identity, and $D_n(\alpha; q)$ the right. We shall prove that

$$E_0(0; q) = D_0(0; q) = 1 \tag{2.1}$$

$$E_1(0; q) = D_1(0; q) = 1 + q \tag{2.2}$$

$$E_0(-1;q) = D_0(-1;q) = 1 \tag{2.3}$$

$$E_1(-1;q) = D_1(-1;q) = 1, \tag{2.4}$$

and that

$$E_n(\alpha;q) = E_{n-1}(\alpha;q) + q^n E_{n-2}(\alpha;q) \tag{2.5}$$

$$D_n(\alpha;q) = D_{n-1}(\alpha;q) + q^n D_{n-2}(\alpha;q), \tag{2.6}$$

for $n \geqslant 2$. This will establish the theorem.

Now (2.1)-(2.4) are easily verified. (2.5) is proved as follows. We shall utilize the following two identities.

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix} \tag{2.7}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}. \tag{2.8}$$

By (2.8),

$$\begin{aligned} E_n(\alpha;q) &= \sum_{j=0}^{\infty} q^{j^2-\alpha j} \left(\begin{bmatrix} (n-1)+1+\alpha-j \\ j \end{bmatrix} + q^{n+1+\alpha-2j} \begin{bmatrix} n+\alpha-j \\ j-1 \end{bmatrix} \right) \\ &= E_{n-1}(\alpha;q) + q^n \sum_{j=0}^{\infty} q^{(j-1)^2-\alpha(j-1)} \begin{bmatrix} n+\alpha-j \\ j-1 \end{bmatrix} \\ &= E_{n-1}(\alpha;q) + q^n \sum_{j=0}^{\infty} q^{j^2-\alpha j} \begin{bmatrix} (n-2)+1+\alpha-j \\ j \end{bmatrix} \\ &= E_{n-1}(\alpha;q) + q^n E_{n-2}(\alpha;q). \end{aligned}$$

Thus (2.5) is established.

The proof of (2.6) is somewhat harder. First we note that

$$\begin{aligned} D_{2n}(\alpha;q) = &\sum_{\lambda=-\infty}^{\infty} q^{\lambda(10\lambda+1)+4\alpha\lambda} \begin{bmatrix} 2n+1 \\ n-5\lambda-\alpha \end{bmatrix} \\ &- \sum_{\lambda=-\infty}^{\infty} q^{(2\lambda+1)(5\lambda+3)+4\alpha\lambda+2\alpha} \begin{bmatrix} 2n+1 \\ n-2-5\lambda-\alpha \end{bmatrix} \end{aligned}$$

$$\begin{aligned} D_{2n+1}(\alpha;q) = &\sum_{\lambda=-\infty}^{\infty} q^{\lambda(10\lambda+1)+4\alpha\lambda} \begin{bmatrix} 2n+2 \\ n+1-5\lambda-\alpha \end{bmatrix} \\ &- \sum_{\gamma=-\infty}^{\infty} q^{(2\lambda+1)(5\lambda+3)+4\alpha\lambda+2\alpha} \begin{bmatrix} 2n+2 \\ n-2-5\lambda-\alpha \end{bmatrix}. \end{aligned}$$

Now

$$D_{2n}(\alpha; q) = \sum_{\lambda=-\infty}^{\infty} q^{\lambda(10\lambda+1)+4\alpha\lambda}\left(\begin{bmatrix} 2n \\ n-5\lambda-\alpha \end{bmatrix} + q^{n+1+5\lambda+\alpha}\begin{bmatrix} 2n \\ n-1-5\lambda-\alpha \end{bmatrix}\right)$$

$$- \sum_{\lambda=-\infty}^{\infty} q^{(2\lambda+1)(5\lambda+3)+4\alpha\lambda+2\alpha}\left(\begin{bmatrix} 2n \\ n-3-5\lambda-\alpha \end{bmatrix} + q^{n-2-5\lambda-\alpha}\begin{bmatrix} 2n \\ n-2-5\lambda-\alpha \end{bmatrix}\right)$$

$$= D_{2n-1}(\alpha; q) + q^{n+1+\alpha}\left(\sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2+6\lambda+4\alpha\lambda}\begin{bmatrix} 2n \\ n-1-5\lambda-\alpha \end{bmatrix} - \sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2+6\lambda+4\alpha\lambda}\begin{bmatrix} 2n \\ n-2-5\lambda-\alpha \end{bmatrix}\right)$$

$$= D_{2n-1}(\alpha; q)$$

$$+ q^{n+1+\alpha}\left(\sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2+6\lambda+4\alpha\lambda}\left(\begin{bmatrix} 2n-1 \\ n-2-5\lambda-\alpha \end{bmatrix} + q^{n-1-5\lambda-\alpha}\begin{bmatrix} 2n-1 \\ n-1-5\lambda-\alpha \end{bmatrix}\right)\right.$$

$$\left. - \sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2+6\lambda+4\alpha\lambda}\left(\begin{bmatrix} 2n-1 \\ n-2-5\lambda-\alpha \end{bmatrix} + q^{n+2+5\lambda+\alpha}\begin{bmatrix} 2n-1 \\ n-3-5\lambda-\alpha \end{bmatrix}\right)\right)$$

$= D_{2n-1}(\alpha; q) + q^{2n}D_{2n-2}(\alpha; q)$, since the first and third sums cancel each other.

On the other hand,

$$D_{2n+1}(\alpha; q) = \sum_{\lambda=-\infty}^{\infty} q^{\lambda(10\lambda+1)+4\alpha\lambda}\left(\begin{bmatrix} 2n+1 \\ n-5\lambda-\alpha \end{bmatrix} + q^{n+1-5\lambda-\alpha}\begin{bmatrix} 2n+1 \\ n+1-5\lambda-\alpha \end{bmatrix}\right)$$

$$- \sum_{\lambda=-\infty}^{\infty} q^{(2\lambda+1)(5\lambda+3)+4\alpha\lambda+2\alpha}\left(\begin{bmatrix} 2n+1 \\ n-2-5\lambda-\alpha \end{bmatrix} + q^{n+4+5\lambda+\alpha}\begin{bmatrix} 2n+1 \\ n-3-5\lambda-\alpha \end{bmatrix}\right)$$

$$= D_{2n}(\alpha; q) + q^{n+1-\alpha}\left(\sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2-4\lambda+4\alpha\lambda}\begin{bmatrix}2n+1\\ n+1-5\lambda-\alpha\end{bmatrix}\right.$$

$$\left. - \sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2+16\lambda+6+4\alpha\lambda+4\alpha}\begin{bmatrix}2n+1\\ n-3-5\lambda-\alpha\end{bmatrix}\right)$$

$$= D_{2n}(\alpha; q) +$$

$$q^{n+1-\alpha}\left(\sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2-4\lambda+4\alpha\lambda}\left(\begin{bmatrix}2n\\ n+1-5\lambda-\alpha\end{bmatrix} + q^{n+5\lambda+\alpha}\begin{bmatrix}2n\\ n-5\lambda-\alpha\end{bmatrix}\right)\right.$$

$$- \sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2+16\lambda+6+4\alpha\lambda+4\alpha}\left(\begin{bmatrix}2n\\ n-4-5\lambda-\alpha\end{bmatrix}\right.$$

$$\left.\left. + q^{n-3-5\lambda-\alpha}\begin{bmatrix}2n\\ n-3-5\lambda-\alpha\end{bmatrix}\right)\right)$$

$$= D_{2n}(\alpha; q) + q^{2n+1}D_{2n-1}(\alpha; q)$$

$$+ q^{n+1-\alpha}\left(\sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2-4\lambda+4\alpha\lambda}\begin{bmatrix}2n\\ n+1-5\lambda-\alpha\end{bmatrix}\right.$$

$$\left. - \sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2+16\lambda+6+4\alpha\lambda+4\alpha}\begin{bmatrix}2n\\ n-4-5\lambda-\alpha\end{bmatrix}\right)$$

(now replacing λ by $\lambda - 1$ in the second sum, we see that it is identical with the first)

$$= D_{2n}(\alpha; q) + q^{2n+1}D_{2n-1}(\alpha; q).$$

Hence in general

$$D_n(\alpha; q) = D_{n-1}(\alpha; q) + q^n D_{n-2}(\alpha; q).$$

Thus our theorem is established.

If we take $\alpha = 0$ and let $n \to \infty$, we obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)\ldots(1-q^n)} = \frac{\sum_{n=\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(5n+1)}}{\prod_{n=0}^{\infty}(1-q^{n+1})}$$

$$= \prod_{n=0}^{\infty}(1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1},$$

which is the first Rogers–Ramanujan identity [38; p. 290].

If we take $\alpha = -1$ and let $n \to \infty$, we obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)\ldots(1-q^n)} = \frac{\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(5n+3)}}{\prod_{n=0}^{\infty} (1-q^{n+1})}$$

$$= \prod_{n=0}^{\infty} (1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1},$$

which is the second Rogers–Ramanujan identity [38; p. 290].

3. Further results. If one puts $q = 1$ in our main identity, one obtains

$$u_{n+2} = \sum_{j=0}^{\infty} \binom{n+1-j}{j} = \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} \binom{n+1}{[\frac{1}{2}(n+1-5\lambda)]} \quad (\alpha = 0),$$

where u_{n+2} is the $(n+2)$-nd Fibonacci number. The first part of this identity is well-known while the second appears to have first appeared in [15]. In [15], numbers of the form

$$F_{k,n} = \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} \binom{n}{[\frac{1}{2}(n-(2k+1)\lambda)]}$$

are studied in great detail, and tests for the primality of $(k^p - 1)/(k-1)$ are derived. Note that for any prime $p \not\equiv \pm 1 \pmod{(2k+1)}$, it is almost obvious that $p \mid F_{k,p}$.

If we define

$$\Delta_{k,n} = \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} q^{\frac{1}{2}\lambda((2k+1)\lambda+1)} \begin{bmatrix} n+1 \\ [\frac{1}{2}(n+1-(2k+1)\lambda)] \end{bmatrix},$$

it is possible to obtain recurrence formulae for these functions by generalizing the technique in Section 2. Thus for example, when $k = 3$,

$$\Delta_{3,n} = \Delta_{3,n-1} + (q^n + q^{n-1})\Delta_{3,n-2} - q^{n-1}\Delta_{3,n-3} + (q^{n-1} - q^{2n-3})\Delta_{3,n-4}.$$

It would be nice to deduce Gordon's generalization of the Rogers–Ramanujan identities [35] from a study of these polynomials. However the theorem in this paper is the only q-series identity I have been able to obtain.

The following result demonstrates how one may obtain results about the Rogers–Ramanujan identities by observing their analogy with the Fibonacci numbers.

THEOREM. *Let $\pi(s, n)$ denote the number of partitions of s with largest part n and minimal difference of at least 2 between summands. If $n + 1$ is a prime of the form $5m \pm 2$, then there exists a sequence $\{c_j^{(n)}\}_{j=0}^{\infty}$ such that for all $s \geqslant 0$*

$$\pi(s, n) = \sum_{j=0}^{\min(s,n)} c_{s-j}(n)$$

Proof: If p is a prime, then it is well-known that

$$1 + q + \dots + q^{p-1}$$

is irreducible over the rational numbers.

As in the case of ordinary binomial coefficients, it is easily verified that

$$1 + q + \dots + q^{p-1} \left| \begin{bmatrix} p \\ a \end{bmatrix} \right.$$

provided $a \neq 0, p$.

Consequently if $n + 1$ is a prime $\equiv \pm 2 \pmod 5$, then

$$[\tfrac{1}{2}(n + 1 - 5\lambda)] \neq 0, \mathrm{n} + 1$$

for any integral λ, and thus

$$1 + q + \dots + q^n \mid D_n(0; q)$$

since $1 + q + \dots + q^n$ divides each term. Thus

$$D_n(0; q) = 1 + \sum_{s=1}^{\infty} \pi(s, n)q^s = (1 + q + \dots + q^n)P_n(q) \tag{3.1}$$

where $P_n(q)$ is a polynomial in q. Letting

$$P_n(q) = \sum_{i=0}^{\infty} c_j^{(n)} q^j$$

and substituting into (3.1) we obtain the desired result.

As an example of this theorem, we find for $n = 6$ that $c_0^{(6)} = c_4^{(6)} = c_6^{(6)} = 1$, while all over c's are zero. Thus in particular

$$\begin{aligned}\pi(6, 6) &= c_6^{(6)} + c_5^{(6)} + c_4^{(6)} + c_3^{(6)} + c_2^{(6)} + c_1^{(6)} + c_0^{(6)} \\ &= 1 + 0 + 1 + 0 + 0 + 0 + 1 \\ &= 3,\end{aligned}$$

and indeed there are exactly three partitions of 6 of the desired type, namely 6, 5 + 1, 4 + 2.

It is possible to prove a less elegant theorem when $n + 1$ is a prime of the form $5m \pm 1$.

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