THE FINITE HEINE TRANSFORMATION

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Abstract. We shall present finite summations that converge to the Heine ${}_2\phi_1$ transformations in the limit as $n \to \infty$. We shall investigate their partition-theoretic implications.

1. Introduction

In an expository article describing Euler's pioneering work on partitions, I was particularly drawn to Euler's assertion [6; p. 566, eq. (5.2) corrected]

$$\prod_{n=0}^{\infty}\left(q^{-3^n}+1+q^{3^n}\right)=\sum_{n=-\infty}^{\infty} q^n, \tag{1.1}$$

an identity valid only in a formal sense in that neither the series nor the product converges for any value of q.

This led to my comparisons of the two infinite series identities ([6; p. 567, eq. (5.5)] and [6; p. 567, eq. (5.6)] respectively):

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, \tag{1.2}$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^2}\sum_{j=0}^{\infty}(-1)^j q^{j(j+1)/2}. \tag{1.3}$$

Each of the left-hand series is analytic inside $|q|<1$ with $|q|=1$ as a natural boundary, and the second series is formally transformable into the first by the mapping $q \to 1/q$. The fact that $|q|=1$ is a natural boundary means we should not be surprised when the same transformation applied to the right-hand side produces only nonsense.

However, it was observed in [4] that it is sometimes possible to find polynomial or rational function identities that converge to infinite q-series in the limit. This observation in [7] was the secret to dealing with Regime II of Baxter's generalized hard-hexagon model (cf. [5; Ch. 8]).

So this led to the question: Are there finite identities that would both (A) simplify (1.2) and (1.3) in the limit, and (B) allow the mapping $q \to 1/q$ prior to taking limits?

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The answer to this question is yes. In Section 2 we provide q-analogs of the Heine transformations of the ${}_2\phi_1$. In Section 3, we shall derive generalizations of the following corollaries.

$$\sum_{n=0}^{N} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2} = \prod_{n=1}^{N} \frac{1}{(1-q^n)} \sum_{j=0}^{N} \frac{q^{(N+1)j}}{(1-q)(1-q^2)\cdots(1-q^j)}, \tag{1.4}$$

and

$$\sum_{n=0}^{N} \frac{q^{n}}{(1-q)^2(1-q^2)^2\dots(1-q^n)^2} = \prod_{n=1}^{N} \frac{1}{(1-q^n)} \sum_{j=0}^{N} \frac{(-1)^j q^{j(j+1)/2}}{(1-q)(1-q^2)\cdots(1-q^{N-j})}. \tag{1.5}$$

Clearly (1.4) and (1.5) converge to (1.2) and (1.3) as $N \to \infty$, and by reversing the sum on the right-hand side it is a simple matter to see that (1.4) becomes (1.5) under the now legitimate mapping $q \to 1/q$.

In Section 4, we shall note quite transparent combinatorial proofs of (1.4) and (1.5).

2. Finite Heine Transformations

We shall employ the following standard notation

$$(a)_n = (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \tag{2.1}$$

$$(a_1,\dots,a_r;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n, \tag{2.2}$$

and

$${}_{r+1}\phi_r = \left(\begin{matrix} a_0,a_1,\dots,a_r;q,t \\ b_1,\dots,b_r \end{matrix}\right) = \sum_{j=0}^{\infty} \frac{(a_0,a_1,\dots,a_r;q)_n t^n}{(q,b_1,\dots,b_r;q)_n}. \tag{2.3}$$

Lemma 1. *For non-negative integers n,*

$${}_3\phi_2\left(\begin{matrix} q^{-n},\alpha,\beta;q,q \\ \gamma,q^{1-n}/\tau \end{matrix}\right) = \frac{(\alpha\tau;q)_n}{(\tau;q)_n}{}_3\phi_2\left(\begin{matrix} q^{-n},\gamma/\beta,\alpha;q,\beta\tau q^n \\ \gamma,\alpha\tau \end{matrix}\right). \tag{2.4}$$

Proof. In (III.13) of [8; p. 242] $b=\gamma/\beta$, $c=\alpha$, $d=\gamma$, $e=\alpha\tau$. The result after simplification is (2.4). □

We note in passing that Lemma 1 is, in fact, a finite version of Jackson's summation [9] (cf. [8; p. 11, eq. (1.54)], [2; p. 527, Lemma]).

Theorem 2.

$${}_3\phi_2\left(\begin{matrix} q^{-n},\alpha,\beta;q,q \\ \gamma,q^{1-n}/\tau \end{matrix}\right) = \frac{(\beta,\alpha\tau;q)_n}{(\gamma,\tau;q)_n}{}_3\phi_2\left(\begin{matrix} q^{-n},\gamma/\beta,\tau;q,q \\ \alpha\tau,q^{1-n}/\beta \end{matrix}\right). \tag{2.5}$$

Remark. When $n \to \infty$, this is Heine's classic ${}_2\phi_1$ transformation [8; p. 9, eq. (1.4.1)], [3; p. 28, Cor. 2.3].

Proof. If in Lemma 1, we replace α, β, γ, and τ by γ/β, τ, $\alpha\tau$ and β respectively, we find that

$${}_3\phi_2\left(\begin{matrix} q^{-n},\gamma/\beta,\alpha;q,\beta\tau q^n \\ \gamma,\alpha\tau \end{matrix}\right) = \frac{(\beta;q)_n}{(\gamma,q)_n}{}_3\phi_2\left(\begin{matrix} q^{-n},\gamma/\beta,\tau;q,q \\ \alpha\tau,q^{1-n}/\beta \end{matrix}\right). \tag{2.6}$$

Now substituting the left-hand side of (2.6) into the right-hand side of (2.4) we deduce (2.5). □

Corollary 3.

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, \alpha, \beta; q, q \\ \gamma, q^{1-n}/\tau \end{matrix}\right) = \frac{(\gamma/\beta, \beta\tau; q)_n}{(\gamma, \tau; q)_n} {}_3\phi_2\left(\begin{matrix} q^{-n}, \alpha\beta\tau/\gamma, \beta; q, q \\ \beta\tau, \beta q^{1-n}/\gamma \end{matrix}\right). \tag{2.7} $$

Proof. Apply Theorem 2 (with α, β, γ and τ replaced by τ, γ/β, $\alpha\tau$ and β respectively) to transform the ${}_3\phi_2$ on the right -hand side of (2.5). □

Corollary 4.

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, \alpha, \beta; q, q \\ \gamma, \frac{q^{1-n}}{\tau} \end{matrix}\right) = \frac{(\frac{\alpha\beta\tau}{\gamma}; q)_n}{(\tau; q)_n} {}_3\phi_2\left(\begin{matrix} q^{-n}, \frac{\gamma}{\alpha}, \frac{\gamma}{\beta}; q, q \\ \gamma, \frac{\gamma q^{1-n}}{\alpha\beta\tau} \end{matrix}\right). $$

Proof. Apply Theorem 2 (with α, β, γ and τ replaced by β, $\alpha\beta\tau/\gamma$, $\beta\tau$, γ/β respectively) to transform the ${}_3\phi_2$ on the right-hand side of (2.7). □

Corollaries 3 and 4 reduce to the second and third Heine transformations [8; p. 10] when $n \to \infty$.

3. Identities (1.4) and (1.5)

Theorem 5.

$$ \sum_{j=0}^{n} \frac{q^j}{(q, \gamma; q)_j} = \frac{1}{(\gamma)_n} \sum_{j=0}^{n} \frac{(-1)^j \gamma^j q^{j(j-1)/2}}{(q)_{n-j}}. \tag{3.1} $$

Proof. Set $\alpha = 0$ and let $\beta \to 0$ in Theorem 2. The desired result follows after algebraic simplification. □

Theorem 6.

$$ \sum_{j=0}^{n} \frac{q^{j^2} \gamma^j}{(q, \gamma q; q)_j} = \frac{1}{(\gamma q)_n} \sum_{j=0}^{n} \frac{\gamma^j q^{j(n+1)}}{(q)_j}. \tag{3.2} $$

Proof. Replace q by $1/q$ and γ by $1/q\gamma$ in (3.1), then reverse the sum on the right-hand side and simplify. □

Identity (1.3) is Theorem 5 with $\gamma = q$, and (1.4) is Theorem 6 with $\gamma = 1$.

4. Combinatorial proofs

Replacing q by q^2 in Theorem 5 and then setting $\gamma = -zq$, we see that Theorem 5 is equivalent to the following assertion:

$$ \sum_{j=0}^{n} \frac{q^{2j} \left(-\gamma q^{2j+1}; q^2\right)_{n-j}}{(q^2; q^2)_j} = \sum_{j=0}^{n} \frac{\gamma^j q^{j^2}}{(q^2; q^2)_{n-j}}. \tag{4.1} $$

Proof of (4.1). The left-hand side of (4.1) is the generating function for partitions in which (1) all parts are $\leqq 2n$, (2) odd parts are distinct, and (3) each odd is $>$ each even. The general two-modular Ferrers graph [3; p. 13] for such partitions is thus

$$\begin{array}{ccccccccc}
2 & 2 & \cdots & 2 & \cdots & 2 & \cdots & 2 & 1 \\
2 & 2 & \cdots & 2 & \cdots & \cdots & 2 & 1 & \\
2 & 2 & \cdots & 2 & \cdots & 2 & 1 & & \\
\multicolumn{6}{l}{\dotfill} & & & \\
2 & 2 & \cdots & 2 & 2 & & & & \\
2 & 2 & \cdots & 2 & & & & & \\
\vdots & & & & & & & & \\
2 & & & & & & & &
\end{array}$$

Now remove the columns that have a 1 at the bottom. In light of the fact that the odds were distinct, we see that if there were originally j odd parts, then we have removed $1+3+5+\cdots+(2j-1)$ $(=j^2)$. The remaining parts are all even and the largest is at most $2n-2j$. Thus this transformation (which is clearly reversible) provides the partitions generated by the right-hand side of (4.1) and thus we have a bijective proof of Theorem 5. $\square$

Proof of (3.2). Classical arguments immediately reveal that the left-hand side of (3.2) is the generating function for partitions with Durfee square of side at most n. γ keeps track of the number of parts.

On the other hand, the side of the Durfee square is the largest j such that the j^{th} part is $\geqq j$. So we may replicate the partitions generated by the left-hand side of (3.2) by exhibiting the generating function for partitions in which the parts $> n$ are at most n in number. If there are j parts greater than n, the generating function is

$$\frac{\gamma^j q^{j(n+1)}}{(\gamma q)_n (q)_j}.$$

Hence summing on j from 0 to n we obtain a new expression for the generating function for partitions with Durfee square at most n, and this proves (3.2). $\square$

5. Conclusion

There are many other corollaries obtainable from the finite Heine transformations. The q-Pfaff-Saalschutz summation is merely [8; p. 13, eq. (1.7.2)] with $\tau = \gamma/\alpha\beta$. One can also obtain a finite version of the q-analog of Kummer's theorem [2], however, the result does not reduce to the hoped for "sum equals product" identity. Also it should be possible to provide a fully combinatorial proof of Theorem 2 along the lines given in [1] for the $n \to \infty$ case.

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