PARTITIONS WITH DISTINCT EVENS

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In honor of the 70th birthday of Georgy Egorychev

ABSTRACT. Partitions with no repeated even parts (DE-partitions) are considered. A DE-rank for DE-partitions is defined to be the integer part of half the largest part minus the number of even parts. $\Delta(n)$ denotes the excess of the number of DE-partitions with even DE-rank over those with odd DE-rank. Surprisingly $\Delta(n)$ is (1) always non-negative, (2) almost always zero, and (3) assumes every positive integer value infinitely often. The main results follow from the work of Corson, Favero, Liesinger and Zubairy. Companion theorems for DE-partitions counted by exceptional parts conclude the paper.

1. INTRODUCTION

In [4] Ramanujan's series [16; p. 14]

$$R(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n} = \sum_{n=0}^{\infty} S(n) q^n \tag{1.1}$$

was examined. Here

$$(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}). \tag{1.2}$$

It was shown [4] that $S(n)$ is almost always equal to zero and also assumes every integral value infinitely often. Combinatorially $S(n)$ is the excess of the number of partitions of n into distinct parts with even rank over those with odd rank. The *rank* of a partition is the largest part minus the number of parts [7], [5]. A similar theorem was proven [4; Sec. 5] for partitions into odd parts without gaps.

The results for $S(n)$ rely crucially on the identity [4; p. 392]

$$R(q) = \sum_{\substack{n \geqq 0 \\ |j|<n}} (-1)^{n+j} q^{n(3n+1)/2-j^2}(1-q^{2n+1}). \tag{1.3}$$

It was noted at the end of [4] that there are numerous series similar in form to the right-hand expression in (1.3).

Indeed, results of this nature were given for Ramanujan's fifth order mock theta functions [2] (c.f. [17]), and such identities formed the basis for pathbreaking work by Zwegers [18] and Bringmann, Ono and Rhoades [6].

The object of this paper is to reveal a similar phenomenon connected to DE-partitions, i.e. partitions with no repeated even parts. Now DE-partitions have been examined previously. R. Honsberger [13] proved the following Euler-type theorem.

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Theorem 1. *Let $P_{DE}(n)$ denote the number of partitions of n with no repeated even parts. Let $P_{<4}(n)$ denote the number of partitions of n in which no part appears more than thrice. Let $P_{\nmid 4}(n)$ denote the number of partitions of n into parts not divisible by* 4. *Then*

$$P_{DE}(n) = P_{<4}(n) = P_{\nmid 4}(n)$$

for each $n \geqq 0$.

Honsberger's proof is immediate from the following identification of the related generating functions

$$\begin{aligned}\sum_{n\geqq 0} P_{DE}(n)q^n = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} &= \frac{(q^4;q^4)_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty} \\ &= \frac{(q^4;q^4)_\infty}{(q;q)_\infty} = \sum_{n\geqq 0} P_{\nmid 4}(n)q^n \\ &= \prod_{n=1}^{\infty}(1+q^n+q^{2n}+q^{3n}) = \sum_{n\geqq 0} P_{<4}(n)q^n.\end{aligned}$$

The fact that $P_{<4}(n) = P_{\nmid 4}(n)$ is due to J. W. L. Glaisher [10], and the asymptotics of these partition functions has been completely examined by P. Hagis [11].

From here on, our focus will be on the DE-rank of DE-partitions which is defined to be the integer part of half the largest part minus the number of even parts.

We let $\delta(m,n)$ denote the number of DE-partitions of n with DE-rank m.

Theorem 2.

$$\sum_{m,n\geqq 0} \delta(m,n) z^m q^n = 1 + \sum_{j\geqq 0} \frac{(-z^{-1}q^2;q^2)_j z^j q^{2j+1}(1+q)}{(q;q^2)_{j+1}}. \tag{1.4}$$

Next we write

$$\Delta(n) = \sum_{m\geqq 0} (-1)^m \delta(m,n). \tag{1.5}$$

Theorem 3.

$$\sum_{n\geqq 0} \Delta(n)(-q)^n = \sum_{n\geqq 0} \frac{(-1)^n q^{n(n+1)/2}(q;q)_n}{(-q)_n}. \tag{1.6}$$

Fortunately, the expression on the right-hand side of (1.6) is, in fact, $W_1(-q)$, a function studied extensively by Corson et al. in [7]. In particular, their Theorem 2.3 combined with our Theorem 3 yields

Theorem 4.

$$\sum_{n\geqq 0} \Delta(n) q^n = \sum_{n=0}^{\infty} \left(q^{\binom{2n+1}{2}} + q^{\binom{2n+2}{2}}\right) \sum_{j=-n}^{n} q^{-j^2}. \tag{1.7}$$

Theorem 3.2 of Corson et al. [7], may be restated here as:

Theorem 5. $\Delta(n)$ *is the number of inequivalent elements of the ring of integers of $Q(\sqrt{2})$ with norm $8n+1$.*

It immediately follows that

Corollary 6. $\Delta(n)$ *is always non-negative.*

Finally, Corson et al. [7] in the Remark just before their Corollary 5.3 make an assertion equivalent to

Corollary 7. $\Delta(n)$ *is almost always equal to zero.*

The Corollary 5.3 of Corson et al. [7] is equivalent to

Corollary 8. $\Delta(n)$ *is equal to any given positive integer infinitely often.*

The above results in some sense relate only half of the Corson et al. [7] paper to DE-partitions. In order to consider their companion function $W_2(q)$, we need a new definition related to DE-partitions. We shall say that a part of a DE-partition is *exceptional* if it is either even or one of the smallest parts or both. For example, $5+4+2+1+1$ is a DE-partition with four exceptional parts.

We let $\epsilon(m,n)$ denote the number of DE-partitions of n with m exceptional parts, and we write

$$E(n) = \sum_{m \geqq 0} (-1)^{m-1} \epsilon(m,n). \tag{1.8}$$

Our main result for $E(n)$ requires the $W_2(q)$ of Corson et al. [7]:

$$W_2(q) = \sum_{n=1}^{\infty} \frac{(-1;q^2)_n (-q)^n}{(q;q^2)_n}. \tag{1.9}$$

Theorem 9.

$$\sum_{n=1}^{\infty} E(n) q^n = W_2(-q) - \sum_{n=1}^{\infty} q^{\binom{n+1}{2}}. \tag{1.10}$$

This assertion allows us to utilize Theorem 3.3 of Corson et al. [7] to establish immediately that

Theorem 10.

$$\sum_{n=1}^{\infty} E(n) q^n = \sum_{n \geqq 1} \left(q^{\binom{2n}{2}} + q^{\binom{2n+1}{2}} \right) \sum_{\substack{j=-n \\ j \neq 0}}^{n-1} q^{-j^2-j}.$$

The three results following Theorem 4 now have perfect analogs as consequences of Theorem 10. These follow for Theorem 3.3 of [7], the Remark preceeding Corollary 5.3 and the proof of Corollary 5.3.

Theorem 11. $E(n)$ *is the number of inequivalent elements of the ring of integers of* $Q(\sqrt{2})$ *with norm* $8n-1$ *or one less if* n *is a triangular number.*

Corollary 12. $E(n)$ *is always non-negative.*

Corollary 13. $E(n)$ *is almost always equal to* 0.

The analog of Corollary 8 is quite plausible but it does not follow directly because of the second term in (1.10).

The remainder of the paper will be devoted to proofs of theorems 3 and 9. All the other results are, as noted, direct consequences of these two results and results in Corson et al. [7].

I thank Dean Hickerson for an extensive set of comments on this paper. In particular he has noted that $\Delta(n)$ is also the number of divisors of $8n+1$ which are congruent to ± 1 modulo 8 minus the number which are congruent to 3 or 5 modulo 8. Consequently, $\Delta(n)$ is the coefficient of $8n+1$ in

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\left(\frac{2}{n}\right) q^n}{1-q^n},$$

where $\left(\frac{2}{n}\right)$ is the Legendre symbol.

Finally I note that A. Patkowski [15] has recently found two related theorems for DE-partitions. His theorems provide other lacunary series arising from DE-partition statistics other than the rank.

2. Proof Theorem 2

For those DE-partitions with largest part $2j+1$, the DE-rank generating function is

$$\frac{(1+z^{-1}q^2)(1+z^{-1}q^4)\cdots(1+z^{-1}q^{2j})z^j q^{2j+1}}{(1-q)(1-q^3)\cdots(1-q^{2j+1})}.$$

For those DE-partitions with largest part $2j+2$, the DE-rank generating function is

$$\frac{(1+z^{-1}q^2)(1+z^{-1}q^4)\cdots(1+z^{-1}q^{2j})z^j q^{2j+2}z^{-1}}{(1-q)(1-q^3)\cdots(1-q^{2j+1})}.$$

We take the empty partition to have DE-rank 0, and so adding together the empty case, the odd case and the even case we find

$$\sum_{m,n\geqq 0} \delta(m,n) z^m q^n = 1 + \sum_{j=0}^{\infty} \frac{(-z^{-1}q^2;q^2)_j z^j (q^{2j+1}+q^{2j+2})}{(q;q^2)_{j+1}},$$

which is equivalent to Theorem 2. □

3. Proof of Theorem 3

By Theorem 2 with z replaced by -1 and q replaced by $-q$, we see that

$$\begin{aligned}\sum_{n\geqq 0} \Delta(n)(-q)^n &= \sum_{m,n\geqq 0} \delta(m,n)(-1)^{m+n} q^n \\ &= 1 + \sum_{j\geqq 0} \frac{(q^2;q^2)_j (-1)^{j-1} q^{2j+1}(1-q)}{(-q;q^2)_{j+1}} \\ &= 1 - \frac{q(1-q)}{1+q} \sum_{j\geqq 0} \frac{(q^2;q^2)_j (q^2;q^2)_j (-q^2)^j}{(q^2;q^2)_j (-q^3;q^2)_j} \\ &= 1 + \sum_{j=0}^{\infty} \frac{(q;q^2)_{j+1}(-q)^{j+1}}{(-q^2;q^2)_{j+1}}\end{aligned}$$

(by [9; eq. (III.2), p. 241 with $q \to q^2$, then $a = b = q^2$, $z = -q^2$, $c = -q^3$])

$$= \sum_{j=0}^{\infty} \frac{(q;q^2)_j(-q)^j}{(-q^2;q^2)_j}$$

$$= \sum_{j\geqq 0} \frac{(q;q^2)_j(q^2;q^2)_j}{(-q^2;q^2)_j(-q;q^2)_j}(q^3)^j q^{2j^2-2j}(1+q^{4j+2})$$

(by [3; eq. (9.1.1), p. 223, $q \to q^2$, then $\alpha = q$, $\beta = -q^2$, $\tau = -q$])

$$= \sum_{j\geqq 0} \frac{(q;q)_{2j}}{(-q;q)_{2j+1}} q^{2j^2+j}\big(1+q^{2j+1}-q^{2j+1}(1-q^{2j+1})\big)$$

$$= \sum_{j=0}^{\infty} \frac{(q;q)_{2j} q^{\binom{2j+1}{2}}}{(-q;q)_{2j}} - \sum_{j=0}^{\infty} \frac{(q;q)_{2j+1} q^{\binom{2j+2}{2}}}{(-q;q)_{2j+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(q;q)_n}{(-q;q)_n}. \qquad \square$$

4. Proof of Theorem 9

We require two results from the literature:

$$\sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \qquad [9;\ \text{p. 6, eq. (7.321)}] \tag{4.1}$$

and

$$ {}_2\phi_1\binom{a,b;q,t}{c} = \frac{\left(\frac{abt}{c};q\right)_\infty}{(t;q)_\infty} {}_2\phi_1\binom{\frac{c}{a},\frac{c}{b};q,t}{c} \qquad [10;\ \text{p. 10, weq. (1.4.6)}] \tag{4.2}$$

where

$$ {}_2\phi_1\binom{a,b;q,t}{c} = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n t^n}{(q;q)_n(c;q)_n}. \tag{4.3}$$

Thus starting from (1.9)

$$\begin{aligned} W_2(-q)+1 &= \sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^n}{(-q;q^2)_n} \\ &= {}_2\phi_1\binom{-1,q^2;q^2,q}{-q} \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n\geqq 0} \frac{(-q^{-1};q^2)_n(q;q^2)_n q^{2n}}{(q^2;q^2)_n(-q;q^2)_n} \qquad \text{by (4.2)} \end{aligned}$$

Consequently

$$
\begin{aligned}
(4.4)\quad W_2(-q) + 1 - \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} &= \sum_{n\geqq 1} \frac{(1+q^{-1})}{(1+q^{2n-1})} q^{2n} \frac{(q^{2n+2};q^2)_\infty}{(q^{2n+1};q^2)_\infty} \\
&= \sum_{n\geqq 1} \frac{\left(1+q^{2n-1}+q^{-1}(1-q^{2n})\right)}{(1+q^{2n-1})} q^{2n} \frac{(q^{2n+2};q^2)_\infty}{(q^{2n+1};q^2)_\infty} \\
&= \sum_{n\geqq 1} \frac{q^{2n}(q^{2n+2};q^2)_\infty}{(q^{2n+1};q^2)_\infty} + \sum_{n\geqq 1} \frac{q^{2n-1}(q^{2n};q^2)_\infty}{(1+q^{2n-1})(q^{2n+1};q^2)_\infty}.
\end{aligned}
$$

Now the first sum above counts DE-partitions with smallest part even and a weight of $+1$ if there are an odd number of exceptional parts and -1 if there are an even number. The second sum counts DE-partitions with smallest part odd and a weight of $+1$ if there are an odd number of exceptional parts and -1 if there are an even number. Thus the right-hand side of (4.4) is the generating function for $E(n)$. Invoking (4.1), we see that

$$\sum_{n=1}^{\infty} E(n)q^n = W_2(-q) - \sum_{n=1}^{\infty} q^{\binom{n+1}{2}}. \qquad \square$$

5. Conclusion

There are a number of natural questions that arise from this study. First, combinatorial proofs of Theorems 4 and 10 might be possible and are much to be desired.

In addition, the ordinary rank of Dyson has led both to explications of the Ramanujan congruence for $p(n)$ (cf. [5] and [8]) and to surprising and appealing combinatorial theorems (cf. [9; eqs. (2.3.91) and (2.4.6)]. These aspects of the DE-rank and of exceptional parts of DE-partitions are completely unexplored.

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