q-CATALAN IDENTITIES

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dedicated to the memory of Professor Alladi Ramakrishnan

Abstract. q-Analogs of the Catalan number identities of Touchard, Jonah and Koshy are derived.

1. Introduction

Alladi Ramakrishnan was a grand man. I mostly knew him in the last two decades of his life. One always took away from conversation with him a sense of his joy in living and his excitement over mathematics and physics.

In my visits with him in the winter of 2008 shortly before his death, he was enthralled with the implications of and extensions of Pascal's triangle. He had prepared an expository article titled: Magic Lattice Imbedding Pascal Triangles. I was a very receptive audience. Now that Professor Alladi Ramakrishnan is gone, I propose to remember him with some observations about the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{1.1}$$

a topic closely related to Pascal's triangle. These famous integers are, by their very definition, slight variations on the central binomial coefficients. In addition, J. Koshy has just published a 422 page book [10] titled Catalan Numbers. R. Stanley [11] and [12] has devoted extensive attention to Catalan numbers, and H. W. Gould [8] has provided an extensive bibliography. These are just a few of the many works on the Catalan numbers.

My interest in the Catalan numbers has arisen from looking at various q-analogs (cf. [6]), i.e. polynomials or rational funtions in a variable q that reduce naturally to the Catalan numbers when $q = 1$.

To provide the flavor of q-analogs we recall Lagrange's identity for the sum of the squares of the binomial coefficients [10; p. 89]

$$\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}. \tag{1.2}$$

Let us recall the Gaussian polynomials (a.k.a. q-binomial coefficients):

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \begin{cases} 0 & \text{if } j < 0 \text{ or } j > n \\ \frac{(q;q)_n}{(q;q)_j (q;q)_{n-j}} & 0 \leqq j \leqq n \end{cases}, \tag{1.3}$$

where

$$(a;q)_N = (1-a)(1-aq)\cdots(1-aq^{N-1}). \tag{1.4}$$

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The q-analog of (1.2) is well-known [2; p. 37, (33.10), $m = n = h$, $k \to n - k$]

$$\sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}_q^2 = \begin{bmatrix} 2n \\ n \end{bmatrix}_q . \tag{1.5}$$

While there are a number of q-analogs of the Catalan numbers (cf. [5]), we shall be primarily interested in the following two.

First

$$C_n(q) = \frac{(1-q)}{(1-q^{n+1})} \begin{bmatrix} 2n \\ n \end{bmatrix}_q . \tag{1.6}$$

Clearly by l'Hôpital's rule,

$$C_n(1) = C_n.$$

(Actually $C_n(q)$ is a polynomial in q so that we may take $q = 1$ directly without invoking l'Hôpital.) $C_n(q)$ was shown [4] to be related to partitions as follows:

The partition $5+5+4+2+1+1$ has Ferrers graph

$$\begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & & & \\ \bullet & & & & \\ \bullet & & & & \end{array}$$

and conjugate $6+4+3+3+2$ (read columns instead of rows). The largest square of nodes in a partition (in this case, a 3×3 square) is called the Durfee square.

We say that a partition

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r \quad (\lambda_i \geqq \lambda_{i+1})$$

with conjugate

$$\lambda'_1 + \lambda'_2 + \cdots + \lambda'_t \quad (\lambda'_i \geqq \lambda'_{i+1})$$

is *Catalan* provided $\lambda_i < \lambda'_i$ for $1 \leq i \leq s$ where s is the side of the Durfee square.

It was proved in [4; Corollary 1] that $C_N(q)$ is the generating function for Catalan partitions with largest part $< N$ and number of parts $\leqq N$.

For example, $C_3(q) = 1 + q^2 + q^3 + q^4 + q^6$, and the partitions being generated are $1+1$, $1+1+1$, $2+1+1$, $2+2+2$.

In another paper [3], we considered

$$\mathcal{C}_n(\lambda, q) = \frac{q^{2n}(-\lambda/q; q^2)_n}{(q^2;q^2)_n}. \tag{1.7}$$

There it was shown that

$$\lim_{q \to 1} \mathcal{C}_n(-1, q) = \lim_{q \to 1} \mathcal{C}_n(1, -q) = -2^{1-2n} C_{n-1}.$$

In this case [3; (3.2)], $\mathcal{C}_n(\lambda, q)$ is the two variable generating function for partitions without repeated odd parts whose total number of parts is n with the exponent on λ counting the number of odd parts and the exponent on q exhibiting the number being partitioned.

The overarching object of this paper is to emphasize the methods for finding q-analogs [1; Sec. 5]. Succinctly put, this method reduces binomial coefficient identities to identities for the generalized hypergeometric function [5; p. 8]

$$ {}_{n+1}F_n\left[\begin{matrix} a_0, a_1, \dots, a_n; t \\ b_1, \dots, b_n \end{matrix}\right] = \sum_{j=0}^{\infty} \frac{[a_0]_j [a_1]_j \cdots [a_n]_j t^j}{j! [b_1]_j \cdots [b_n]_j}, \tag{1.8} $$

where

$$ [A]_j = A(A+1)\cdots(A+j-1). $$

(We note that the symbol $[A]_j$ is unconventional but is necessary in a paper where the symbol $(A;q)_n$ also appears.)

Once this first step is complete there is generally a canonical q-analog from the world of generalized q-hypergeometric functions [7; p. 4]

$$ {}_{n+1}\phi_n\left(\begin{matrix} A_0, A_1, \dots, A_n; q, t \\ B_1, \dots, B_n \end{matrix}\right) = \sum_{j=0}^{\infty} \frac{(A_0;q)_j (A_1;q)_j \cdots (A_n;q)_j t^j}{(q;q)_j (B_1;q)_j \cdots (B_n;q)_j}. \tag{1.9} $$

The final step involves reversing the evaluation in step one to provide the perfect q-analog.

We have chosen three identities. First is Touchard's identity [10; p. 319]

$$ C_{n+1} = \sum_{r \geqq 0} \binom{n}{2r} 2^{n-2r} C_r. \tag{1.10} $$

We shall prove

Theorem 1.

$$ C_{n+1}(q) = \sum_{r \geqq 0} q^{2r^2+2r} \begin{bmatrix} n \\ 2r \end{bmatrix}_q C_r(q) \frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r}. \tag{1.11} $$

Note how, in this instance, the q-analog of 2^{n-2r} is $(-q^{r+2};q)_{n-r}/(-q;q)_r$. This is surely not something easily guessed.

Koshy provides another recursive formula for Catalan numbers [10; p. 322]

$$ C_n = \sum_{r=1}^{\infty} (-1)^{r-1} \binom{n-r+1}{r} C_{n-r}. \tag{1.12} $$

We shall prove

Theorem 2.

$$ C_n(q) = \sum_{r=1}^{n} (-1)^{r-1} q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix}_q C_{n-r}(q) \frac{(-q^{n-r+1};q)_r}{(-q;q)_r}. \tag{1.13} $$

Note that in this q-analog, the factor $(-q^{n-r+1};q)_r/(-q;q)_r$ is equal to 1 when we set $q = 1$.

Both Theorems 1 and 2 are deduced from the q-analog of the Chu-Vandermonde summation [7; p. 236, (II.6) and (II.7)].

Finally we consider Jonah's identity [10; p. 325]

$$ \binom{n+1}{r} = \sum_{j=0}^{r} \binom{n-2j}{r-j} C_j, \tag{1.14} $$

provide $2r \leqq n$.

We shall prove

Theorem 3.

$$\frac{(1+q^{n-r+1})}{(1+q^{r+1})}\begin{bmatrix} n+1 \\ r \end{bmatrix}_{q^2} = -(-q;q)_{n+1}\sum_{j=0}^{r}\begin{bmatrix} n-2j \\ r-j \end{bmatrix}_{q^2}\frac{\mathcal{C}_{j+1}(-1;q)}{(-q;q)_{n-2j}}q^{-j-1}. \tag{1.15}$$

Here we must rely on the q-analog of the Pfaff-Saalschütz summation [7; p. 237, (II.12)].

In our final section, we discuss some of the combinatorial questions that arise from these considerations.

2. q-Touchard Identity

Following the standard reduction rules given in [1; Sec. 5], we see that (1.10), Touchard's identity, may be reduced to the equivalent assertion

$$2^n{}_2F_1\left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}; 1 \\ 2 \end{matrix}\right] = \frac{2^{2n+2}\left[\frac{1}{2}\right]_{n+1}}{(n+2)!} = C_{n+1}. \tag{2.1}$$

Identity (2.1) is a specialization of the classic Chu-Vandermonde summation [5; p. 3]. Now we choose the natural q-analog of (2.1) [7; p. 236, eq. (II.7)]

$$(-q^2;q)_n{}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-n+1}; q^2, q^2 \\ q^4 \end{matrix}\right) = C_{n+1}(q), \tag{2.2}$$

and the standard reduction of the left-hand side of (2.2) following the rules given in [1; Sec. 5] yields (1.11) which is Theorem 1.

3. Koshy's Identity

First we rewrite (1.2) as follows:

$$\sum_{r=0}^{n}(-1)^r\binom{n-r+1}{r}C_{n-r} = 0. \tag{3.1}$$

This identity is equivalent to the assertion that

$${}_2F_1\left(\begin{matrix} \frac{-n-1}{2}, \frac{-n}{2}; 1 \\ -n+\frac{1}{2} \end{matrix}\right) = 0, \tag{3.2}$$

and (3.2) is also a specialization of the Chu-Vandermonde summation [5; p. 3]. The corresponding q-Chu-Vandermonde summation [7; p. 236, eq. (II.7)] is

$${}_2\phi_1\left(\begin{matrix} q^{-n-1}, q^{-n}; q^2, q^2 \\ q^{1-2n} \end{matrix}\right) = 0. \tag{3.3}$$

After reversing the steps from (3.2) to (3.1) in the q-analogous procedure, we obtain

$$\sum_{r=0}^{n}(-1)^r q^{r^2-r}\begin{bmatrix} n-r+1 \\ r \end{bmatrix}_q C_{n-r}(q)\frac{(-q^{n-r+1};q)_r}{(-q;q)_r} = 0. \tag{3.4}$$

Equation (3.4) reduces to (1.13) once we move the $r=0$ term to the other side of the equation.

4. Jonah's Identity

Identity (1.15) is deeper than the previous results. In this case, assuming $0 \leqq 2r \leqq n$,

$$\sum_{j=0}^{r}\binom{n-2j}{r-j}C_j = -\frac{1}{2}\binom{n+2}{r+1}\left({}_3F_2\left(\begin{matrix}-r-1, -n+r-1, \frac{-1}{2}; 1\\ -\frac{n}{2}-1, \frac{n}{2}-\frac{1}{2}\end{matrix}\right)-1\right) \tag{4.1}$$

$$= -\frac{1}{2}\binom{n+2}{r+1}\left(\frac{n-2r}{n+2}-1\right) \qquad \text{(by [5; Sec. 2.2])}$$

$$= \binom{n+1}{r}.$$

The summation identity used was Pfaff-Saalschütz. The related q-analog [5; Sec. 8.4] is

$${}_3\phi_2\left(\begin{matrix}q^{-2r-2}, q^{-2n+2r-2}, q^{-1}; q^2, q^2\\ q^{-n-2}, q^{-n-1}\end{matrix}\right)-1 = -\frac{(1-q^{r+1})(1+q^{n-r+1})}{(1-q^{n+2})}. \tag{4.2}$$

Finally using (4.2) to produce the q-analog of (4.1), we obtain (1.15) which is Theorem 3.

5. Conclusion

First we note along with Koshy [10; p. 327] that Jonah's Theorem was generalized by Hilton and Pedersen [9] to remove the restrictions $2r \leqq n$. A q-analog of the Hilton-Pedersen extension can be obtained in exactly the way that the q-analog of Jonah's theorem was proved. Indeed the following identity is equivalent to a q-analog of the Hilton-Pedersen identity [10; p. 327]

$$\sum_{j\geqq 0}\frac{(a^2q^{2-2r-2j};q^2)_{r-j}(-aq^{1-2j};q)_{2j+1}\mathcal{C}_{j+1}(-1;q)q^{-j}}{(q^2;q^2)_{r-j}} \tag{5.1}$$

$$= \frac{q^{1+2r-r^2}a^{2r}(a^{-2}q^{-2};q^2)_r(aq+q^r)(-1)^{r-1}}{(1+q^{r+1})(q^2;q^2)_r}.$$

More intriguing are some obvious combinatorial questions that lie within some of our q-analogs. Suppose we rewrite (1.13) as

$$C_n(q) = \sum_{r=1}^{n}(-1)^{r-1}T_r(n,q), \tag{5.2}$$

where

$$T_r(n,q) = q^{r^2-r}\begin{bmatrix}n-r+1\\ r\end{bmatrix}_q C_{n-r}(q)\frac{(-q^{n-r+1};q)_r}{(-q;q)_r}. \tag{5.3}$$

Problem 1. Show that $T_r(n,q)$ is a polynomial.

Problem 2. If $2r \leqq n$, show that all the coefficients in $T_r(n,q)$ are non-negative.

Problem 3. Show that $T_{r+1}(2r+1,-q)$ has non-negative coefficients.

Problem 4. Provide a partition-theoretic interpretation of $T_r(n,q)$ for $2r \leqq n$ and for $T_{n+1}(2n+1,-q)$.

Problem 5. In light of the fact that $C_n(q)$ generates the Catalan partitions with largest part $< n$ and number of parts $\leqq n$, show by using Problem 4 to interpret the right-hand side of (5.2), that a sieve process eliminates all non-Catalan partitions.

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