THE LEGENDRE-STIRLING NUMBERS

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ABSTRACT. The Legendre-Stirling numbers are the coefficients in the integral Lagrangian symmetric powers of the classical Legendre second-order differential expression. In many ways, these numbers mimic the classical Stirling numbers of the second kind which play a similar role in the integral powers of the classical second-order Laguerre differential expression. In a recent paper, Andrews and Littlejohn gave a combinatorial interpretation of the Legendre-Stirling numbers. In this paper, we establish several properties of the Legendre-Stirling numbers; as with the Stirling numbers of the second kind, they have interesting generating functions and recurrence relations. Moreover, there are some surprising and intriguing results relating these numbers to some classical results in algebraic number theory.

1. INTRODUCTION

The Legendre-Stirling numbers $\{PS_n^{(j)}\}$ were introduced into the literature in [7] as the coefficients of the integral powers of the second-order Legendre differential expression $\ell[\cdot]$ defined by

$$\ell[y](x) := -\left((1-x^2)y'(x)\right)' = -(1-x^2)y''(x) + 2xy'(x). \tag{1.1}$$

These numbers are specifically defined by

$$PS_n^{(0)} = 0 \text{ and } PS_0^{(j)} = 0 \text{ except } PS_0^{(0)} = 1 \quad (n, j \in \mathbb{N}), \tag{1.2}$$

and

$$PS_n^{(j)} = \sum_{r=0}^{j} (-1)^{r+j} \frac{(2r+1)(r^2+r)^n}{(r+j+1)!(j-r)!} \quad (n \in \mathbb{N}; j = 1, 2, \ldots, n). \tag{1.3}$$

These powers of the Legendre expression (1.1) are used to develop the left-definite spectral theory (see [12]) of the classical Legendre differential operator A in $L^2(-1,1)$, generated by $\ell[\cdot]$, that has the Legendre polynomials $\{P_m\}_{m=0}^{\infty}$ as eigenfunctions; we briefly discuss these results in Section 2.

We note that the classical Stirling numbers of the second kind $\{S_n^{(j)}\}$ play a similar role as coefficients in the integral powers of the second-order Laguerre differential expression; it is partly for this reason that we call the numbers $\{PS_n^{(j)}\}$ the Legendre-Stirling numbers. In fact, as we will see in this paper, the Legendre-Stirling numbers have several properties that are strikingly similar to the Stirling numbers of the second kind. We note that Comtet [4] further generalized the

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classical Stirling numbers of the second kind in [4]. For an excellent account of Stirling numbers of the first and second kind, see Comtet's text [5, Chapter V].

This paper is a continuation of the recent work of Andrews and Littlejohn in [3] where the authors obtained a combinatorial interpretation of the Legendre-Stirling numbers. The contents of this paper are as follows. In Section 2, we briefly describe the connection between Legendre-Stirling numbers, the integral powers of the Legendre expression (1.1), and the left-definite operator theory associated with the Legendre polynomials. In this section, we will also briefly show how the classical Stirling numbers of the second kind arise in the powers of the classical Laguerre and Hermite differential equations, a fact that was only recently noticed by the authors in [8] and [12]. Section 3 deals with an alternative formula for the Legendre-Stirling numbers; as we will see, this formula has its advantages over the one given in (1.3). In Section 4, we give the combinatorial interpretation of the Legendre-Stirling numbers that is developed in [3]. Section 5 deals with specific combinatorial properties of the Legendre-Stirling numbers and how these properties compare with the Stirling numbers of the second kind. Among several combinatorial properties developed in this section, we define Legendre-Stirling numbers of the first kind through matrix inversion. In this respect, we refer to the two important contributions by Milne [13] and by Milne and Bhatnagar [14]. In Section 6, we establish an interesting divisibility property of the Legendre-Stirling numbers $PS_p^{(j)}$, when p is a prime number; see Theorem 6.1 below. Remarkably, we show in Section 7, that this theorem is equivalent to the well known Euler Criterion in algebraic number theory.

We thank the two anonymous referees for bringing to our attention the important recent contribution by E. S. Egge [6] on new properties of Legendre-Stirling numbers and Legendre-Stirling numbers of the first kind. We comment further on this remarkable paper in Section 5.6. Based on notation promoted by Knuth in [9] for the classical Stirling numbers of the second kind, Egge adopts the notation

$$\left\{\left\{ \begin{matrix} n \\ j \end{matrix} \right\}\right\},$$

for the Legendre-Stirling number $PS_n^{(j)}$. This is sensible notation but we resist the temptation to change; Legendre polynomials are typically written $\{P_n\}$ and, in many texts, the notation $S_n^{(j)}$ is used to denote Stirling numbers of the second kind so it seems natural, and appropriate, to use our notation. Furthermore, for real numbers $\alpha, \beta > -1$, the authors in [10] define the Jacobi-Stirling numbers $\{P^{(\alpha,\beta)}S_n^{(j)}\}$ which further generalize Legendre-Stirling numbers and are motivated by integral powers of the classical second-order Jacobi differential expression. Since the Jacobi polynomials are usually written as $\{P_n^{(\alpha,\beta)}\}$, with the special case $P_n = P_n^{(0,0)}$, our notation is convenient.

2. Background

In [7], the authors prove the following result which is the key prerequisite in establishing the left-definite theory of the Legendre differential expression; it is in this result that the Legendre-Stirling numbers are first introduced.

Theorem 2.1. *Let $\ell[\cdot]$ be the classical Legendre differential expression defined in (1.1). Then, for each $n \in \mathbb{N}$, the n^{th} composite power of the Legendre differential expression (1.1), in Lagrangian* *symmetric form, is given by*

$$\ell^n[y](x) = \sum_{j=1}^{n} (-1)^j \left(PS_n^{(j)} (1-x^2)^j y^{(j)}(x) \right)^{(j)} \quad (x \in (-1,1)), \tag{2.1}$$

where the coefficients $PS_n^{(j)}$ are the Legendre-Stirling numbers defined in (1.3).

We list a few Legendre-Stirling numbers:

j/n	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$j=1$	1	2	4	8	16	32	64	128	256	512
$j=2$	–	1	8	52	320	1936	11648	69952	419840	2519296
$j=3$	–	–	1	20	292	3824	47824	585536	7096384	85576448
$j=4$	–	–	–	1	40	1092	25664	561104	11807616	243248704
$j=5$	–	–	–	–	1	70	3192	121424	4203824	137922336
$j=6$	–	–	–	–	–	1	112	7896	453056	23232176
$j=7$	–	–	–	–	–	–	1	168	17304	1422080
$j=8$	–	–	–	–	–	–	–	1	240	34584
$j=9$	–	–	–	–	–	–	–	–	1	330
$j=10$	–	–	–	–	–	–	–	–	–	1

Table 1: Legendre-Stirling numbers of the second kind ($PS_4^{(2)} = 52$, $PS_6^{(4)} = 1092$)

As we will see in this paper, the Legendre-Stirling numbers have several properties that are similar to the classical Stirling numbers of the second kind $\{S_n^{(j)}\}$ (see [1, pp. 824-825] and [5, Chapter V] for a comprehensive list of properties of $\{S_n^{(j)}\}$).

From (2.1), it is shown in [7] that, for each positive integer n, the classical Legendre polynomials $\{P_m\}_{m=0}^{\infty}$, where

$$P_m(x) := \sqrt{\frac{2m+1}{2}} \sum_{j=0}^{[m/2]} \frac{(-1)^j (2m-2j)!}{2^m j!(m-j)!(m-2j)!} x^{m-2j} \quad (m \in \mathbb{N}_0),$$

are orthogonal with respect to the Sobolev inner product

$$(f,g)_n := \sum_{j=0}^{n} \int_{-1}^{1} PS_n^{(j)} (1-x^2)^j f^{(j)}(x) \overline{g}^{(j)}(x) dx;$$

in fact,

$$(P_m, P_r)_n = (m(m+1))^n \delta_{m,r} \quad (m, r \in \mathbb{N}_0).$$

Moreover, these Legendre polynomials form a complete orthogonal set in the Hilbert-Sobolev space H_n, endowed with this inner product $(\cdot,\cdot)_n$, where

$$H_n = \{ f : (-1,1) \to \mathbb{C} \mid f, f', \ldots, f^{(n-1)} \in AC_{\text{loc}}(-1,1); \\ (1-x^2)^{j/2} f^{(j)} \in L^2(-1,1) \ (j = 0, 1, \ldots n) \}.$$

This space H_n is called the n^{th} left-definite space associated with the self-adjoint operator A in $L^2(-1,1)$, generated by the Legendre expression (1.1), that has the Legendre polynomials $\{P_m\}_{m=0}^{\infty}$

as eigenfunctions. In [7], the authors construct a self-adjoint operator A_n in H_n, for each $n \in \mathbb{N}$, that has the Legendre polynomials as eigenfunctions.

It is well known (see [5, Chapter V, p. 220]) that the Stirling numbers of the second kind are the coefficients of integral powers of the simple first-order differential expression

$$r[y](x) := xy'(x) \quad (x \in (0,\infty));$$

in fact, with $r^2[y](x) = r[r[y]](x)$, $r^3[y](x) = r^2[r[y]](x)$, etc., it is the case that

$$r^n[y](x) = \sum_{j=1}^{n} PS_n^{(j)} x^j y^{(j)}(x).$$

As indicated earlier, it was discovered by Everitt, Littlejohn and Wellman (see [8] and [12]) that the Stirling numbers of the second kind arise quite naturally in the study of the classical Laguerre and Hermite differential equations. Indeed, the second-order Laguerre and Hermite differential expressions are given, respectively, by

$$m[y](x) := \frac{-1}{x^\alpha e^{-x}} \left(x^{\alpha+1} e^{-x} y'(x)\right)' \quad (x \in (0,\infty)),$$

and

$$h[y](x) := \frac{-1}{\exp(-x^2)} (\exp(-x^2) y'(x))' \quad (x \in \mathbb{R});$$

the authors in [8] and [12] show that, for each $n \in \mathbb{N}$,

$$m^n[y](x) = \frac{1}{x^\alpha e^{-x}} \sum_{j=1}^{n} (-1)^j S_n^{(j)} \left(x^{\alpha+j} e^{-x} y^{(j)}(x)\right)^{(j)},$$

and

$$h^n[y](x) = \frac{1}{\exp(-x^2)} \sum_{j=1}^{n} (-1)^j 2^{n-j} S_n^{(j)} (\exp(-x^2) y^{(j)}(x))^{(j)}.$$

3. An Alternative Formula for the Legendre-Stirling Numbers

In reference to the Legendre-Stirling numbers defined in (1.3), we note that since

$$\frac{2r+1}{(r+j+1)!(j-r)!} = \frac{1}{(2j)!}\left(\binom{2j}{j-r} - \binom{2j}{j-r-1}\right) \quad (r = 0, 1\ldots, j;\ \binom{2j}{-1} = 0),$$

we find that

$$\begin{aligned}
PS_n^{(j)} &= \frac{1}{(2j)!}\left\{\sum_{r=0}^{j-1}(-1)^{j+r}\binom{2j}{j-r} - \binom{2j}{j-r-1}(r^2+r)^n\right\} + \frac{2j+1}{(2j+1)!}(j^2+j)^n \\
&= \frac{1}{(2j)!}\left\{\sum_{r=0}^{j}(-1)^{j+r}\binom{2j}{j-r}(r^2+r)^n - \sum_{r=0}^{j-1}(-1)^{j+r}\binom{2j}{j-r-1}(r^2+r)^n\right\} \\
&= \frac{1}{(2j)!}\left\{\sum_{m=0}^{j}(-1)^{m}\binom{2j}{m}\left((j-m)^2+j-m\right)^n - \sum_{r=0}^{j-1}(-1)^{j+r}\binom{2j}{j+r+1}(r^2+r)^n\right\} \\
&= \frac{1}{(2j)!}\left\{\sum_{m=0}^{j}(-1)^{m}\binom{2j}{m}\left((j-m)(j+1-m)\right)^n + \sum_{m=j+1}^{2j}(-1)^{m}\binom{2j}{m}\left((j-m)(j+1-m)\right)^n\right\}
\end{aligned}$$

and so

$$PS_n^{(j)} = \frac{1}{(2j)!}\sum_{m=0}^{2j}(-1)^m\binom{2j}{m}\left((j-m)(j+1-m)\right)^n. \tag{3.1}$$

For a further discussion of this alternative formulation in (3.1), see Remark 5.2 below. We note that the formula given in (3.1) mimics the definition of the classical Stirling numbers of the second kind:

$$S_n^{(j)} := \frac{1}{j!}\sum_{m=0}^{j}(-1)^m\binom{j}{m}(j-m)^n. \tag{3.2}$$

4. A Combinatorial Interpretation of the Legendre-Stirling Numbers

It is well known that the Stirling number of the second kind $S_n^{(j)}$ counts the number of ways of putting n objects into j non-empty, indistinguishable sets. It is natural to ask: what do the Legendre-Stirling numbers count? Andrews and Littlejohn gave a combinatorial interpretation of these numbers in [3].

To describe this interpretation of the Legendre-Stirling number $PS_n^{(j)}$, for each $n \in \mathbb{N}$, we consider two copies (colors) of each positive integer between 1 and n:

$$1_1, 1_2, 2_1, 2_2, \ldots, n_1, n_2.$$

For positive integers $p, q \leq n$ and $i, j \in \{1, 2\}$, we say that $p_i > q_j$ if $p > q$. We now describe two rules on how to fill $j+1$ 'boxes' with the numbers $\{1_1, 1_2, 2_1, 2_2, \ldots, n_1, n_2\}$:

(1) the 'zero box' is the only box that may be empty and it may not contain both copies of any number.
(2) the other j boxes are indistinguishable and each is non-empty; for each such box, the smallest element in that box must contain both copies (colors) of this smallest number but no other elements have both copies in that box.

In [3], the authors show:

Theorem 4.1. *For $n, j \in \mathbb{N}_0$ and $j \leq n$, the Legendre-Stirling number $PS_n^{(j)}$ is the number of different distributions according to the above two rules.*

5. Combinatorial Properties of the Legendre Stirling Numbers

In this section, which consists of multiple subsections, we establish several combinatorial properties of the Legendre-Stirling numbers. At the end of this section, we summarize these properties and compare them to their 'first cousins', the classical Stirling numbers of the second kind.

As discussed in Sections 1 and 2, the Legendre-Stirling numbers (see Table 1 in Section 2) are the coefficients in the Lagrangian symmetric form of the n^{th} composite power of the Legendre differential expression $\ell[\cdot]$ given in (1.1). For example,

$$\ell^1[y](x) = \ell[y](x) = -(\mathbf{1}(1-x^2)y')',$$

$$\ell^4[y](x) = -\left(\mathbf{8}(1-x^2)y'\right)' + \left(\mathbf{52}(1-x^2)^2y''\right)'' - \left(\mathbf{20}(1-x^2)^3y'''\right)''' + \left(\mathbf{1}(1-x^2)^4y^{(4)}\right)^{(4)},$$

and

$$\begin{aligned}\ell^8[y](x) = &-\left(\mathbf{128}(1-x^2)y'(x)\right)' + \left(\mathbf{69952}(1-x^2)^2y''(x)\right)'' - \left(\mathbf{585536}(1-x^2)^3y'''(x)\right)''' \\ &+ \left(\mathbf{561104}(1-x^2)^4y^{(4)}(x)\right)^{(4)} - \left(\mathbf{121424}(1-x^2)^5y^{(5)}(x)\right)^{(5)} + \left(\mathbf{7896}(1-x^2)^6y^{(6)}(x)\right)^{(6)} \\ &- \left(\mathbf{168}(1-x^2)^7y^{(7)}(x)\right)^{(7)} + \left(\mathbf{1}(1-x^2)^8y^{(8)}(x)\right)^{(8)}.\end{aligned}$$

Later, in this section (see Subsection 5.7), we discuss another interpretation of the non-zero entries in each column of Table 1.

5.1. Rational generating function for the Legendre-Stirling numbers.

For each $j \in \mathbb{N}$, the authors prove in [7] that the Legendre-Stirling numbers have the following rational generating function:

$$\varphi_j(t) := \prod_{r=1}^{j} \frac{1}{1-r(r+1)t} = \sum_{n=0}^{\infty} PS_n^{(j)} t^{n-j} \quad \left(|t| < \frac{1}{j(j+1)}\right). \tag{5.1}$$

By comparison, the rational generating function for the classical Stirling numbers of the second kind $\{S_n^{(j)}\}$ is given by

$$\prod_{r=1}^{j} \frac{1}{1-rt} = \sum_{n=0}^{\infty} S_n^{(j)} t^{n-j} \quad \left(|x| < \frac{1}{j}\right); \tag{5.2}$$

see Comtet [5]. In regards to (5.1), since the coefficients of the Taylor (geometric) series of each function

$$\frac{1}{1-r(r+1)t} \quad (1 \le r \le j;\ \ |t| < \frac{1}{r(r+1)})$$

are positive and since each $PS_n^{(j)}$ is obtained from the Cauchy product of these j series, we see that $PS_n^{(j)} > 0$ when $1 \le j \le n$.

Remark 5.1. *We note that the coefficients r and $r(r+1)$ in the denominators of the rational generating functions given, respectively in (5.1) and (5.2) are, respectively, the eigenvalues that produce the Legendre and Laguerre polynomial solutions of degree r in, respectively, the Legendre and Laguerre differential equations, respectively. It is remarkable, and somewhat mysterious, that there* *is this intimate connection between spectral theory and combinatorics. For further information, see* [7] *and* [12].

5.2. **Forward differences of Legendre-Stirling numbers.** Recall that the forward difference of a sequence of numbers $\{x_n\}_{n=0}^{\infty}$ is the sequence $\{\Delta x_n\}_{n=0}^{\infty}$ given by

$$\Delta x_n := x_{n+1} - x_n \quad (n \in \mathbb{N}_0).$$

Higher order forward differences are defined recursively by

$$\Delta^k x_n := \Delta(\Delta^{k-1} x_n) = \sum_{m=0}^{k} \binom{k}{m} (-1)^m x_{n+k-m}.$$

In this section, we prove that

Theorem 5.1. *For $k \in \mathbb{N}_0$, we have*

$$\Delta^k \left(\frac{PS_n^{(j)}}{2^n} \right) \geq 0 \quad (n \geq j).$$

In particular, we see that for $n \geq j$,

$$\begin{aligned} PS_{n+1}^{(j)} - 2PS_n^{(j)} &\geq 0, \\ PS_{n+2}^{(j)} - 4PS_{n+1}^{(j)} + 4PS_n^{(j)} &\geq 0, \\ PS_{n+3}^{(j)} - 6PS_{n+2}^{(j)} + 12PS_{n+1}^{(j)} - 8PS_n^{(j)} &\geq 0, \ \textit{etc.} \end{aligned}$$

Proof. Fix $j \in \mathbb{N}$; from (5.1), replace t by $t/2$ to obtain

$$\sum_{n=j}^{\infty} \frac{PS_n^{(j)}}{2^n} t^n = \frac{t^j}{2^j} \psi_j(t) \quad \left(|t| < \frac{2}{j(j+1)} \right), \tag{5.3}$$

where

$$\psi_j(t) := \frac{1}{(1-t)(1-3t)\cdots(1-\frac{j(j+1)}{2}t)}.$$

Now

$$\sum_{n=j}^{\infty} \frac{PS_n^{(j)}}{2^n} t^n = \sum_{n=j-1}^{\infty} \frac{PS_{n+1}^{(j)}}{2^{n+1}} t^{n+1} = \frac{t^j}{2^j} + \sum_{n=j}^{\infty} \frac{PS_{n+1}^{(j)}}{2^{n+1}} t^{n+1}$$

since $PS_j^{(j)} = 1$. Hence, from (5.3), we see that

$$\sum_{n=j}^{\infty} \frac{PS_{n+1}^{(j)}}{2^{n+1}} t^n = \frac{t^{j-1}}{2^j} \psi_j(t) - \frac{t^{j-1}}{2^j}, \tag{5.4}$$

and consequently

$$\begin{aligned} \sum_{n=j}^{\infty} \Delta \left(\frac{PS_n^{(j)}}{2^n} \right) t^n &= \frac{t^{j-1}}{2^j} \psi_j(t) - \frac{t^{j-1}}{2^j} - \frac{t^j}{2^j} \psi_j(t) = \frac{t^{j-1}}{2^j} [\psi_j(t) - t\psi_j(t) - 1] \\ &= \frac{t^{j-1}}{2^j} [(1-t)\psi_j(t) - 1]. \end{aligned} \tag{5.5}$$

By comparing powers of t on both sides of (5.5), we see that

$$(1-t)\psi_j(t) - 1 = \sum_{n=1}^{\infty} a_n(1)t^n,$$

where each of the coefficients $a_n(1)$ are non-negative as can easily be seen from the Taylor expansion of $(1-t)\psi_j(t) - 1$. Indeed, the coefficients in this Taylor series are the Cauchy product of the coefficients obtained from the products of the geometric series for

$$\frac{1}{1-t}, \frac{1}{1-3t}, \ldots, \frac{1}{1-\frac{j(j+1)}{2}t},$$

each of which have positive coefficients. It follows, then, from comparing coefficients on both sides of (5.5) that

$$\Delta\left(\frac{PS_n^{(j)}}{2^n}\right) \geq 0 \quad (n \geq j).$$

To see that

$$\Delta^2\left(\frac{PS_n^{(j)}}{2^n}\right) \geq 0 \quad (n \geq j), \tag{5.6}$$

we first notice that

$$\begin{aligned}\frac{t^j\psi_j(t)}{2^j} &= \sum_{n=j}^{\infty}\frac{PS_n^{(j)}}{2^n}t^n = \sum_{n=j-2}^{\infty}\frac{PS_{n+2}^{(j)}}{2^{n+2}}t^{n+2} \\ &= \frac{t^j}{2^j} + \frac{PS_{j+1}^{(j)}}{2^{j+1}}t^{j+1} + \sum_{n=j}^{\infty}\frac{PS_{n+2}^{(j)}}{2^{n+2}}t^{n+2}.\end{aligned}$$

Thus

$$\sum_{n=j}^{\infty}\frac{PS_{n+2}^{(j)}}{2^{n+2}}t^n = \frac{t^{j-2}}{2^j}\psi_j(t) - \frac{t^{j-2}}{2^j} - \frac{PS_{j+1}^{(j)}}{2^{j+1}}t^{j-1}. \tag{5.7}$$

Consequently, from (5.3), (5.4), and (5.7), we see that

$$\begin{aligned}\sum_{n=j}^{\infty}\Delta^2\left(\frac{PS_n^{(j)}}{2^n}\right)t^n &= \sum_{n=j}^{\infty}\left(\frac{PS_{n+2}^{(j)}}{2^{n+2}} - 2\frac{PS_{n+1}^{(j)}}{2^{n+1}} + \frac{PS_n^{(j)}}{2^n}\right)t^n \\ &= \frac{t^{j-2}}{2^j}\psi_j(t) - \frac{t^{j-2}}{2^j} - \frac{PS_{j+1}^{(j)}}{2^{j+1}}t^{j-1} - 2\frac{t^{j-1}}{2^j}\psi_j(t) + \frac{2t^{j-1}}{2^j} + \frac{t^j}{2^j}\psi_j(t) \\ &= \frac{t^{j-2}}{2^j}\left[(1-t)^2\psi_j(t) + \left(2 - \frac{PS_{j+1}^{(j)}}{2}\right)t - 1\right]. \qquad (5.8)\end{aligned}$$

Again, by comparing both sides of this identity, we see that

$$(1-t)^2\psi_j(t) + \left(2 - \frac{PS_{j+1}^{(j)}}{2}\right)t - 1 = \sum_{n=2}^{\infty} a_n(2)t^n,$$

where each $a_n(2)$ is non-negative, as can easily be seen from the Taylor series expansion of $(1-t)^2\psi_j(t)$. The inequality in (5.6) now follows. From (5.5) and (5.8), we can generalize to see that, for each $k \in \mathbb{N}$,

$$\sum_{n=j}^{\infty} \Delta^k \left(\frac{PS_n^{(j)}}{2^n} \right) t^n = \frac{t^{j-k}}{2^j} \left[(1-t)^k \psi_j(t) + p_{k-1}(t) \right], \tag{5.9}$$

where $p_{k-1}(t)$ is a polynomial of degree $\leq k-1$. Moreover, by comparing both sides of (5.9), we see that

$$(1-t)^k \psi_j(t) + p_{k-1}(t) = \sum_{n=k}^{\infty} a_n(k) t^n$$

where each $a_n(k) \geq 0$ since the coefficients in the Taylor expansion of $(1-t)^k\psi_j(t)$ are all non-negative. Consequently, it follows that

$$\Delta^k \left(\frac{PS_n^{(j)}}{2^n} \right) \geq 0 \quad (n \geq j).$$

This completes the proof of the theorem. □

We have an immediate application of Theorem 5.1. Indeed, the fourth-order Legendre type differential expression is defined by

$$\ell[y](x) = (1-x^2)^2 y^{(4)}(x) - 8x(1-x^2)y^{(3)}(x) + (4A+12)(x^2-1)y''(x) + 8Axy'(x);$$

here, A is a fixed, positive constant; this expression was discovered by H. L. Krall [11] in 1938. For each $n \in \mathbb{N}_0$, the equation

$$\ell[y] = n(n+1)(n^2+n+4A-2)y$$

has a unique monic polynomial solution $y = P_{n,A}(x)$ of degree n. The set $\{P_{n,A}\}_{n=0}^{\infty}$ is called the Legendre type polynomials and they form a complete orthogonal set of eigenfunctions in the Hilbert space $L^2_\mu[-1,1]$, where

$$d\mu = (\frac{1}{A}\delta(x+1) + \frac{1}{A}\delta(x-1) + 1)dx.$$

In order to complete a left-definite operator-theoretic analysis of this fourth-order expression $\ell[\cdot]$, it is important to know that the coefficient $a_{n,j}$ in the n^{th} composite power $\ell^n[y]$ is positive. This n^{th} power has the form

$$\ell^n[y](x) = \sum_{j=1}^{2n} (-1)^j \left(a_{n,j}(1-x^2)^j + b_{n,j}(1-x^2)^{j-1} \right) y^{(j)}(x);$$

in fact, it can be shown (see [16]) that

$$\begin{aligned}
a_{n,j} &:= \sum_{k=0}^{j} (-1)^{k+j} \frac{(2k+1)(k^2+k)^n(k^2+k-2+4A)^n}{(j-k)!(j+k+1)!} \\
&= \sum_{k=0}^{j} (-1)^{k+j} \frac{(2k+1)(k^2+k)^n}{(j-k)!(j+k+1)!} \sum_{r=0}^{n} \binom{n}{r} (4A)^{n-r}(k^2+k-2)^r \\
&= \sum_{r=0}^{n} \binom{n}{r} (4A)^{n-r} \sum_{k=0}^{j} (-1)^{k+j} \frac{(2k+1)(k^2+k)^n(k^2+k-2)^r}{(j-k)!(j+k+1)!}.
\end{aligned}$$

From this, it is easy to see that each $a_{n,j} > 0$ if

$$\widetilde{a}_{n,j,r} := \sum_{k=0}^{j} (-1)^{k+j} \frac{(2k+1)(k^2+k)^n(k^2+k-2)^r}{(j-k)!(j+k+1)!} > 0 \quad (r = 0, 1, \ldots, n).$$

By expanding $(k^2+k-2)^r$, we see that

$$\begin{aligned}
\widetilde{a}_{n,j,r} &= \sum_{m=0}^{r} \binom{r}{m} (-2)^m \left(\sum_{k=0}^{j} (-1)^{k+j} \frac{(2k+1)(k^2+k)^{n+r-m}}{(j-k)!(j+k+1)!} \right) \\
&= \sum_{m=0}^{r} \binom{r}{m} (-2)^m PS_{n+r-m}^{(j)} \\
&= \sum_{m=0}^{r} \binom{r}{m} (-1)^m 2^{n+r} \frac{PS_{n+r-m}^{(j)}}{2^{n+r-m}} = 2^{n+r} \Delta^r \left(\frac{PS_n^{(j)}}{2^n} \right).
\end{aligned}$$

consequently, from Theorem 5.1, we see that $a_{n,j} > 0$.

5.3. **A vertical generating function.** We recall that the vertical generating function for the Stirling numbers of the second kind $S_n^{(j)}$ is

$$\sum_{n=0}^{\infty} \frac{S_n^{(j)}}{n!} t^n = \frac{(\exp(t)-1)^j}{j!}. \tag{5.10}$$

From (1.2) and (1.3), we see that, for $j \in \mathbb{N}$,

$$\begin{aligned}
\phi_j(t) :&= \sum_{n=0}^{\infty} \frac{PS_n^{(j)}}{n!} t^n \\
&= \sum_{r=0}^{j} \left(\sum_{n=0}^{\infty} \frac{(r^2+r)^n}{n!} t^n \right) \frac{(-1)^{r+j}(2r+1)}{(r+j+1)!(j-r)!} \\
&= \sum_{r=0}^{j} \frac{(-1)^{r+j}(2r+1)\exp((r^2+r)t)}{(r+j+1)!(j-r)!}.
\end{aligned} \tag{5.11}$$

Alternatively, from (3.1), we see that

$$\phi_j(t) = \sum_{n=0}^{\infty} \frac{PS_n^{(j)}}{n!} t^n \quad (j = 1, 2, \ldots)$$

$$= \frac{1}{(2j)!} \sum_{m=0}^{2j} \left(\sum_{n=0}^{\infty} \frac{((j-m)(j+1-m))^n}{n!} t^n \right) (-1)^m \binom{2j}{m}$$

$$= \frac{1}{(2j)!} \sum_{m=0}^{2j} (-1)^m \binom{2j}{m} \exp((j-m)(j+1-m)t). \tag{5.12}$$

In particular,

$$\phi_j(0) = PS_0^{(j)} = \frac{1}{(2j)!} \sum_{m=0}^{2j} (-1)^m \binom{2j}{m} = 0.$$

Remark 5.2. *The formula for the Legendre-Stirling numbers given in (3.1), or equivalently, the vertical generating function given in (5.12) has its advantages over the original definition given in (1.2) and (1.3). More specifically, the formula given in (3.1) replaces <u>both</u> (1.2) and (1.3); indeed, it follows from (3.1) or (5.12) that*

$$PS_n^{(0)} = 0,\ PS_0^{(j)} = 0\ (j \in \mathbb{N}),\ PS_0^{(0)} = 1.$$

Moreover, it is not difficult to see that $PS_n^{(j)} = 0$ when $j > n$ (so each 'dash' in Table 1 can be replaced by 0).

Remark 5.3. *It can be shown, for $j \in \mathbb{N}$, that*

$$\phi_j(t) = \frac{1}{(2j)!} (e^{2t} - 1)^j p_j(e^{2t}),$$

where $p_j(x)$ is a polynomial of degree $(j^2 - j)/2$. At this point, however, we are unable to identify this polynomial p_j. By definition, $\phi_j(\cdot)$ is a real linear combination of $j+1$ exponentials. Such functions have at most j real zeros counting multiplicities (see [15]). Furthermore, we know that $t = 0$ is a j-fold zero of $\phi_j(\cdot)$. The very same property is shared by the classical Stirling vertical generating function given in (5.10). By elementary calculus, it follows that both generating functions have, essentially, the same graph subject to the considerations mentioned above.

5.4. **A vertical recurrence relation.** The Stirling numbers of the second kind $\{S_n^{(j)}\}$ satisfy the well-known vertical recurrence relation

$$S_n^{(j)} = \sum_{k=j}^{n} S_{k-1}^{(j-1)} j^{n-k}; \tag{5.13}$$

see [5, page 209]. In this section, we prove the following analogous result.

Theorem 5.2. *The Legendre-Stirling numbers of the second kind $\{PS_n^{(j)}\}$ satisfy the vertical recurrence relation*

$$PS_n^{(j)} = \sum_{k=j}^{n} PS_{k-1}^{(j-1)} (j(j+1))^{n-k} \quad (n, j \in \mathbb{N}_0).$$

Proof. Since $PS_n^{(j)} = 0$ for $n < j$, we see from the rational generating function $\varphi_j(t)$, given in (5.1), that

$$\begin{aligned}
\sum_{n=0}^{\infty} PS_n^{(j)} t^n &= \prod_{r=1}^{j} \frac{t}{1-r(r+1)t} \\
&= \frac{t}{1-j(j+1)t}\varphi_{j-1}(t) \\
&= \sum_{s=0}^{\infty}(j(j+1)^s t^{s+1} \sum_{m=0}^{\infty} PS_m^{(j-1)} t^m \qquad (5.14) \\
&= \sum_{s=0}^{\infty}\sum_{m=0}^{\infty} PS_m^{(j-1)}(j(j+1))^s t^{m+s+1} \\
&= \sum_{s=0}^{\infty}\sum_{\ell=0}^{\infty} PS_{\ell-1}^{(j-1)}(j(j+1))^s t^{\ell+s} \text{ where } m=\ell-1.
\end{aligned}$$

Comparing coefficients of t^n on both sides of (5.14), we see that

$$\begin{aligned}
PS_n^{(j)} &= \sum_{r=0}^{n-1} PS_{n-r-1}^{(j-1)}(j(j+1))^r \\
&= \sum_{r=0}^{n-j} PS_{n-r-1}^{(j-1)}(j(j+1))^r \text{ since } PS_{n-r-1}^{(j-1)} = 0 \text{ when } n-j<r \\
&= \sum_{r=0}^{n-j} PS_{j+r-1}^{(j-1)}(j(j+1))^{n-j-r} \text{ by reversing the order of summation} \\
&= \sum_{k=j}^{n} PS_{k-1}^{(j-1)}(j(j+1))^{n-k} \text{ where } k=j+r, \text{ as required.}
\end{aligned}$$

□

5.5. **A triangular recurrence relation.** The Stirling numbers of the second kind $\{S_n^{(j)}\}$ satisfy the triangular recurrence relation

$$\begin{aligned}
S_n^{(j)} &= S_{n-1}^{(j-1)} + jS_{n-1}^{(j)} \quad (n,j\in\mathbb{N}) \\
S_n^{(0)} &= S_0^{(j)} = 0;\; S_0^{(0)} = 1 \quad (n,j\in\mathbb{N}).
\end{aligned}$$

see [5, Page 208]. In [3, Lemma 3.1], the authors show that the Legendre Stirling numbers satisfy a similar type of triangular recurrence relation.

Theorem 5.3. *The Legendre-Stirling numbers of the second kind $\{PS_n^{(j)}\}$ satisfy the triangular recurrence relation*

$$PS_n^{(j)} = PS_{n-1}^{(j-1)} + j(j+1)PS_{n-1}^{(j)} \quad (n,j\in\mathbb{N}) \tag{5.15}$$

$$PS_n^{(0)} = PS_0^{(j)} = 0;\; PS_0^{(0)} = 1 \quad (n,j\in\mathbb{N}). \tag{5.16}$$

5.6. **A horizontal generating function and Legendre-Stirling numbers of the first kind.** The classical Stirling numbers of the second kind satisfy the following horizontal generating function

$$x^n = \sum_{j=0}^{n} S_n^{(j)}(x)_j \quad (n \in \mathbb{N}_0), \tag{5.17}$$

where

$$(x)_j := \prod_{r=0}^{j-1}(x-r) = x(x-1)\cdots(x-j+1).$$

is the usual Pochhammer or falling factorial symbol ($(x)_0 := 1$); see [5, page 207]. Indeed, many texts use the identity in (5.17) as the definition for the Stirling numbers of the second kind. We now show that the Legendre-Stirling numbers of the second kind satisfy a similar formula:

Theorem 5.4. *The Legendre-Stirling numbers* $\{PS_n^{(j)}\}$ *have the following horizontal generating function*

$$x^n = \sum_{j=0}^{n} PS_n^{(j)} \langle x \rangle_j, \tag{5.18}$$

where $\langle x \rangle_j$ *is the generalized falling factorial symbol defined by*

$$\langle x \rangle_j := \prod_{r=0}^{j-1}(x - r(r+1)) \quad (\langle x \rangle_0 := 1). \tag{5.19}$$

In particular, we see that $\{\langle x \rangle_n\}_{n=0}^{\infty}$ *forms a basis for the vector space* $\mathcal{P}$ *of all polynomials* $p(x)$ *under the usual operations of addition and scalar multiplication.*

Proof. We prove (5.18) by induction on $n \in \mathbb{N}_0$. Since $\langle x \rangle_0 = 1$, we see that the result holds for $n = 0$. Assume (5.18) holds for $n \geq 1$. Then

$$x^n = x \cdot x^{n-1} = \sum_{j=0}^{n-1} PS_{n-1}^{(j)} x \cdot \langle x \rangle_j. \tag{5.20}$$

Since

$$x \cdot \langle x \rangle_j = (x - j(j+1) + j(j+1)) \cdot \langle x \rangle_j = \langle x \rangle_{j+1} + j(j+1)\langle x \rangle_j,$$

we see that (5.20) becomes

$$\begin{aligned}
x^n &= \sum_{j=0}^{n-1} PS_{n-1}^{(j)} \langle x\rangle_{j+1} + \sum_{j=0}^{n-1} PS_{n-1}^{(j)} j(j+1)\langle x\rangle_j \\
&= \sum_{j=1}^{n} PS_{n-1}^{(j-1)} \langle x\rangle_{j} + \sum_{j=0}^{n-1} PS_{n-1}^{(j)} j(j+1)\langle x\rangle_j \\
&= \langle x\rangle_n + \sum_{j=1}^{n-1} \left(PS_{n-1}^{(j-1)} + j(j+1)PS_{n-1}^{(j)}\right)\langle x\rangle_j \quad (PS_{n-1}^{(n-1)} = 1) \\
&= \langle x\rangle_n + \sum_{j=1}^{n-1} PS_{n}^{(j)}\langle x\rangle_j \text{ by (5.15)} \\
&= \sum_{j=0}^{n} PS_{n}^{(j)}\langle x\rangle_j \text{ since } PS_n^{(0)} = 0 \text{ and } PS_n^{(n)} = 1.
\end{aligned}$$

This completes the proof of this theorem. □

It is well-known that the formula in (5.17) can be inverted to yield

$$(x)_n = \sum_{k=0}^{n} s_n^{(k)} x^k, \tag{5.21}$$

where $s_n^{(k)}$ is the Stirling number of the first kind. We note that a similar inversion formula holds for (5.18). Indeed, for each $n \in \mathbb{N}$, consider the lower triangular $n \times n$ matrix

$$P_n := \left(PS_k^{(j)}\right)_{0\le k,j\le n-1}.$$

Since $\det(P_n) = 1$ for all $n \in \mathbb{N}$, we see that P_n^{-1} exists and is an lower triangular matrix with integer coefficients. If we write

$$P_n^{-1} = \left(PS_k^{(j)}\right)_{0\le j,k\le n-1}^{-1} := (Ps_k^{(j)})_{0\le j,k\le n-1},$$

we call the number $Ps_k^{(j)}$ the *Legendre-Stirling number of the first kind.* In general, by applying P_n^{-1} to the identity in (5.18), we obtain the following counterpart to the classical identity in (5.21).

Theorem 5.5. *The Legendre-Stirling numbers of the first kind* $\{Ps_n^{(j)}\}$ *satisfy the horizontal generating function*

$$\langle x\rangle_n = \sum_{j=0}^{n} Ps_n^{(j)} x^j. \tag{5.22}$$

We list a few Legendre-Stirling numbers of the first kind:

j/n	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$j=0$	1	0	0	0	0	0	0	0	0
$j=1$	0	1	-2	12	-144	2880	-86400	3628800	-203212800
$j=2$	0	0	1	-8	108	-2304	72000	-3110400	177811200
$j=3$	0	0	0	1	-20	508	-17544	808848	-48405888
$j=4$	0	0	0	0	1	-40	1708	-89280	5808528
$j=5$	0	0	0	0	0	1	-70	4648	-349568
$j=6$	0	0	0	0	0	0	1	-112	10920
$j=7$	0	0	0	0	0	0	0	1	-168
$j=8$	0	0	0	0	0	0	0	0	1

Table 2: Legendre-Stirling numbers of the first kind ($Ps_4^{(3)} = -20$, $Ps_8^{(4)} = 5808528$)

As an example to illustrate Theorem 5.5, we read down the column $n = 4$ to see that

$$\langle x\rangle_4 = x(x-2)(x-6)(x-12) = -144x + 108x^2 - 20x^3 + x^4.$$

Theorem 5.6. *The Legendre-Stirling numbers of the first kind $\{Ps_n^{(j)}\}$ satisfy the following triangular recurrence relation*

$$Ps_n^{(j)} = Ps_{n-1}^{(j-1)} - n(n-1)Ps_{n-1}^{(j)}$$
$$Ps_n^{(0)} = Ps_0^{(j)} = 0 \textit{ except } Ps_0^{(0)} = 1.$$

Proof. From (5.22), we see that

$$\sum_{k=0}^{n} Ps_n^{(k)}x^k = \langle x\rangle_n = \langle x\rangle_{n-1}(x - n(n-1)) = \sum_{k=0}^{n-1} Ps_{n-1}^{(k)}x^{k+1} - \sum_{k=0}^{n-1} n(n-1)Ps_{n-1}^{(k)}x^k; \tag{5.23}$$

the theorem follows by comparison of coefficients on both sides of (5.23). □

If we replace x by t^{-1} in (5.22), multiply both sides by t^n, we obtain the identity

$$\prod_{j=0}^{n-1}(1 - j(j+1)t) = \sum_{j=0}^{n} Ps_n^{(j)}t^{n-j}. \tag{5.24}$$

Compare this formula in (5.24) to the rational generating function for the Legendre-Stirling numbers $\{PS_n^{(j)}\}$ given in (5.1); in this way, we see the remarkable duality between the two sets of numbers $\{PS_n^{(j)}\}$ and $\{Ps_n^{(j)}\}$.

Replacing t by $-t$ in (5.24), we obtain

$$\prod_{j=0}^{n-1}(1 + j(j+1)t) = \sum_{j=0}^{n}(-1)^{n+j}Ps_n^{(j)}t^{n-j}. \tag{5.25}$$

Comparison of coefficients on both sides of this identity shows that $(-1)^{n+j}Ps_n^{(j)} > 0$ for $j = 1, 2, \ldots, n$. Consequently, the numbers

$$P\mathfrak{s}_n^{(j)} := (-1)^{n+j}Ps_n^{(j)} = \left|Ps_n^{(j)}\right| \geq 0 \quad (n, j \in \mathbb{N}_0); \tag{5.26}$$

we call these numbers $\{P\mathfrak{s}_n^{(j)}\}$ the *unsigned* or *absolute Legendre-Stirling numbers of the first kind.*

We note the duality between the Legendre-Stirling numbers and the Legendre-Stirling numbers of the first kind has recently been considered in depth by E. S. Egge in [6]. Indeed, several new results in this direction have been obtained; in particular, see [6, Theorems 2.2 and 2.3]. Moreover, the author also obtains an interesting combinatorial interpretation of the Legendre-Stirling numbers of the first kind in terms of pairs of permutations (see [6, Theorem 2.5]).

5.7. **Unimodality of the Legendre-Stirling numbers.** Define

$$A_n(x) := \sum_{j=0}^{n} PS_n^{(j)} x^j \quad (n \in \mathbb{N}_0); \tag{5.27}$$

since $PS_n^{(n)} > 0$, we see that $A_n(x)$ is a real polynomial of degree exactly n. Since $PS_n^{(0)} = 1$, we see that $x = 0$ is a root of $A_n(x)$ for $n \geq 1$. Notice from (1.2) that $A_0(x) = 1$.

From (5.15), we see that

$$\begin{aligned} A_n(x) &= \sum_{j=0}^{n} PS_{n-1}^{(j-1)} x^j + \sum_{j=0}^{n} j(j+1) PS_{n-1}^{(j)} x^j \\ &= xA_{n-1}(x) + x(xA_n(x))'' \\ &= x\left(A_{n-1}(x) + 2A'_{n-1}(x) + xA''_{n-1}(x)\right). \end{aligned} \tag{5.28}$$

We list the first few of these polynomials when $n = 1, 2, 3$:

$$A_1(x) = x, \quad A_2(x) = x(x+2), \quad A_3(x) = x(x^2 + 8x + 4). \tag{5.29}$$

Observe that the roots of these polynomials in (5.29) are simple, distinct, and non-positive real numbers. In fact,

Theorem 5.7. *Let $n \in \mathbb{N}$. The zeros of $A_n(x)$, defined in (5.27), are real, simple, distinct, and non-positive. Moreover, in this case, $A_n(0) = 0$.*

Proof. It is clear that the theorem is true for $n = 1, 2, 3$. We assume the theorem is valid for some $n - 1$, where $n \geq 4$. We now prove the assertion, using induction, for n. From our induction hypothesis, $A_{n-1}(x)$ has $n - 1$ simple, distinct, real non-positive roots; consequently, $A_{n-1}(x)$ has $n - 2$ relative extreme points, say $\{y_{n-1,\nu}\}_{\nu=1}^{n-2}$ satisfying

$$y_{n-1,n-2} < y_{n-1,n-1} < \ldots < y_{n-1,1} < 0$$

satisfying

$$\operatorname{sign}\left[A_{n-1}(y_{n-1,\nu})\right] = (-1)^{\nu}, \quad \operatorname{sign}\left[A''_{n-1}(y_{n-1,\nu})\right] = (-1)^{\nu+1} \quad (\nu = 1, 2, \ldots, n-2). \tag{5.30}$$

Since $A'_{n-1}(y_{n-1,\nu}) = 0$ for $\nu = 1, 2, \ldots, n-2$, we see from (5.28) that

$$A_n(y_{n-1,\nu}) = y_{n-1,\nu}\left(A_{n-1}(y_{n-1,\nu}) + y_{n-1,\nu}A''_{n-1}(y_{n-1,\nu})\right)$$

and hence

$$\operatorname{sign}\left[A_n(y_{n-1,\nu})\right] = (-1)^{\nu+1} \quad (\nu = 1, 2, \ldots, n-2). \tag{5.31}$$

Consequently, from (5.31) and the Intermediate Value Theorem, there are $n-3$ zeros $x_{n,\nu}$ ($\nu = 3, 4, \ldots n-1$) of $A_n(x)$ such that

$$y_{n-1,n-2} < x_{n,n-1} < y_{n-1,n-3} < x_{n,n-2} < \ldots < y_{n-1,2} < x_{n,3} < y_{n-1,1} < 0.$$

Since $x = 0$ is also a root of $A_n(x)$, we have shown so far that $A_n(x)$ has $n-2$ roots. We now argue that $A_n(x)$ must have two more roots. Indeed, by definition, $A_n'(0) = PS_n^{(1)} = 2^{n-1}$ so $\operatorname{sign}(A_n'(0)) = 1$; consequently, it follows that $A_n(x) < 0$ for $x < 0$ and sufficiently close to 0. Moreover, since $\operatorname{sign}[A_n(y_{n-1,1})] = 1$ (see (5.31)), we see that $A_n(x)$ has another zero $x_{n,2}$ strictly between $y_{n-1,1}$ and 0. Since 0 is also a root of $A_n(x)$, we have so far determined that $A_n(x)$ has $n-1$ zeros. Finally, by (5.31), we see that

$$\operatorname{sign}[A_n(y_{n-1,n-2})] = (-1)^{n-1},$$

and since $\operatorname{sign}[A_n(x)] = (-1)^n$ for $x \to -\infty$, we see that there is still another zero $x_{n,n} < y_{n-1,n-2}$ of $A_n(x)$. This completes the proof of the theorem. □

We recall (see [5, Chapter VII]) that a real sequence $\{a_k\}_{k=0}^{\infty}$ is called unimodal if there exists two integers $N_1 \leq N_2$ such that

(i) $k \leq N_1 - 2 \Longrightarrow$ that $a_k \leq a_{k+1}$,
(ii) $a_{N_1-1} < a_{N_1} = a_{N_1+1} = \ldots = a_{N_2} > a_{N_2+1}$
(iii) $k \geq N_2 + 1 \Longrightarrow a_k \geq a_{k+1}$.

In this case, if $N_1 = N_2$ we say that $\{a_k\}_{k=0}^{\infty}$ is unimodal with a peak; otherwise we say that $\{a_k\}_{k=0}^{\infty}$ is unimodal with a plateau of $N_2 - N_1$ points.

The following theorem is established in [5, Theorem B, Chapter VII].

Theorem 5.8. *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of non-negative real numbers. If, for each $N \in \mathbb{N}$, the polynomial*

$$q_N(x) := \sum_{k=0}^{N} a_k x^k \quad (a_p \neq 0),$$

has only real roots, then $\{a_k\}_{k=0}^{\infty}$ is unimodal with either a peak or a plateau of 2 points.

It is well known, see [5, Chapter VII], that the Stirling numbers of the second kind are unimodal. As a consequence of Theorems 5.7 and 5.8, we have the following result on Legendre-Stirling numbers.

Theorem 5.9. *The Legendre-Stirling numbers are unimodal with either a peak or a plateau of 2 points.*

Moreover, from Theorem 5.8 and the identity in (5.25), we immediately obtain the following result.

Theorem 5.10. *The unsigned Legendre-Stirling numbers of the first kind $\{P\mathfrak{s}_n^{(j)}\}$, defined in (5.26), are unimodal with either a peak or a plateau of 2 points.*

In the following table, we summarize the properties of the Legendre-Stirling numbers discussed in this section and compare these properties to their 'classical cousins', the Stirling numbers of the first and second kind:

Property	**Stirling Numbers 2nd Kind**	**Legendre-Stirling Numbers**
Vertical RR	$S_n^{(j)} = \sum_{r=j}^n S_{r-1}^{(j-1)} j^{n-r}$	$PS_n^{(j)} = \sum_{r=j}^n PS_{r-1}^{(j-1)} (j(j+1))^{n-r}$
Rational GF	$\prod_{r=1}^{j} \frac{1}{1-rx} = \sum_{n=0}^\infty S_n^{(j)} x^{n-j}$	$\prod_{r=0}^{j} \frac{1}{1-r(r+1)x} = \sum_{n=0}^\infty PS_n^{(j)} x^{n-j}$
Triangular RR	$S_n^{(j)} = S_{n-1}^{(j-1)} + jS_{n-1}^{(j)}$ $S_n^{(0)} = S_0^{(j)} = 0$; $S_0^{(0)} = 1$	$PS_n^{(j)} = PS_{n-1}^{(j-1)} + j(j+1)PS_{n-1}^{(j)}$ $PS_n^{(0)} = PS_0^{(j)} = 0$; $PS_0^{(0)} = 1$
Horizontal GF	$x^n = \sum_{j=0}^n S_n^{(j)} (x)_j$ where $(x)_j = x(x-1)\ldots(x-j+1)$	$x^n = \sum_{j=0}^n PS_n^{(j)} \langle x\rangle_j$ where $\langle x\rangle_j = x(x-2)\ldots(x-(j-1)j)$
1st Kind Numbers	$(x)_n = \sum_{j=0}^n s_n^{(j)} x^j$	$\langle x\rangle_n = \sum_{j=0}^n ps_n^{(j)} x^j$

Table 3: A comparison of properties of Stirling numbers of the second kind and Legendre-Stirling numbers

6. Divisibility Properties of the Legendre-Stirling numbers

The Legendre-Stirling numbers have appealing divisibility properties as outlined in Theorem 6.1 below. In order to prove this theorem, we collect some results and observations. We note that the classical Stirling numbers of the second kind also satisfy some interesting divisibility properties; see [5, Chapter 5, Section 8].

Wilson's Theorem [2, p. 38] asserts that for any prime p,

$$(p-1)! \equiv -1 \pmod{p}. \tag{6.1}$$

Fermat's Little Theorem [2, p. 36] says that, for any prime p and integer n,

$$n^p \equiv n \pmod{p}; \tag{6.2}$$

of course, this means that $(n^p - n)/p$ is an integer. Furthermore

$$\frac{(n+kp)^k - n}{p} \equiv \frac{n^p - n}{p} \pmod{p} \tag{6.3}$$

because the binomial theorem reveals that the difference between the right side and the left side of (6.3) is a multiple of p.

It should be noted that fractions will occur in some of the congruences appearing in the proof of Theorem 6.1. This is perfectly acceptable for congruences modulo p as long as the denominators are relatively prime to p. All of this is legitimate since the integers modulo p form the field $\mathbb{Z}_p$.

Next, we note that the n^{th} difference of a polynomial $f(x)$ of degree $< n$ is zero [17, p. 3, eq. I-9]. In particular,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} f(j) = 0, \tag{6.4}$$

or, after shifting the index,

$$\sum_{j=-k}^{n-k} (-1)^j \binom{n}{k+j} f(j) = 0 \quad (k \in \mathbb{N}). \tag{6.5}$$

We are now ready to prove:

Theorem 6.1. *If p is an odd prime, then*

$$PS_p^{(j)} \equiv \begin{cases} 1 & \text{if } j = 1 \text{ or } j = p \\ 2 & \text{if } j = (p+1)/2 \\ 0 & \text{if otherwise} \end{cases} \pmod{p}. \tag{6.6}$$

Proof. The top line of (6.6) is immediate since $PS_n^{(n)} = 1$ for all n; furthermore, from the definition of $PS_p^{(1)}$ and (6.2), we see that

$$PS_p^{(1)} = 2^{p-1} \equiv 1 \pmod{p}$$

from (6.2). To finish the proof, we consider three cases.

Case 1: $1 < j < (p-1)/2$

We know that $PS_1^{(j)} = 0$ and so

$$\begin{aligned} PS_p^{(j)} &= PS_p^{(j)} - PS_1^{(j)} \\ &= \sum_{m=0}^{j} (-1)^{m+j} \frac{(2m+1)}{(m+j+1)!(j-m)!} \left((m^2+m)^p - (m^2+m) \right). \end{aligned}$$

Now, since $j < (p-1)/2$, we see that $m + j + 1 \leq 2j + 1 < p$ for each $0 \leq m \leq j$. Consequently, none of the denominators in this sum is divisible by p, while (6.2) tells us that the numerator of each term is divisible by p. Thus

$$PS_p^{(j)} \equiv 0 \pmod{p}.$$

Case 2: $j = (p-1)/2$

In this case,

$$\begin{aligned} PS_p^{(j)} &= PS_p^{(j)} - PS_1^{(j)} \\ &= \sum_{m=0}^{j-1} (-1)^{m+j} \frac{(2m+1)}{(m+j+1)!(j-m)!} \left((m^2+m)^p - (m^2+m) \right) \\ &+ \frac{1}{(2j)!} \left((m^2+m)^p - (m^2+m) \right). \end{aligned}$$

Again, no denominator has any prime factors larger than $p-1$, while (6.2) reveals that each numerator is divisible by p. Hence

$$PS_p^{(p-1)/2} \equiv 0 \pmod{p}.$$

Case 3: $(p+1)/2 \leq j < p$.

This is the problematic case. It is foreshadowed by Case 2. The last term in the sum in Case 2 appeared to have $p = 2j + 1$ in the denominator. However, the term $2m + 1$ (at $m = j$) cancelled the p in the denominator and the rest followed. In this third case, we actually have p in the denominator in some terms. So now we suppose

$$p = 2j - 2k + 1$$

where $1 \leq k < j$. The terms with a factor p in the denominator are

$$m = j-2k, j-2k+1, \ldots, j-k-1, j-k+1, \ldots, j.$$

Note that when $m = j-k$, we have that $2m+1 = p$ cancelling the p in the denominator. Thus, as in Cases 1 and 2, the $m = j-k$ term vanishes modulo p. We propose to combine the terms $j-k-a$ and $j-k+a$ for $1 \leq a \leq k$. First we note that

$$(j-k\pm a)^2 + (j-k\pm a) = \frac{1}{4}(2j-2k+1)^2 \pm a(2j-2k+1) + a^2 - \frac{1}{4}. \tag{6.7}$$

Thus the term at $m = j-k-a$ is

$$\begin{aligned}
&\frac{(-1)^{a+k}(2(j-k-a)+1)}{(k+a)!(2j-k-a+1)(2j-k-a)\cdots(2j-2k+2)(2j-2k)!} \\
&\text{x}\left(\frac{((j-k-a)^2+(j-k-a))^{2j-2k+1}-(j-k-a)^2-(j-k-a)}{2j-2k+1}\right) \\
&\equiv \left(\frac{((\frac{2j-2k+1}{2})^2 - a(2j-2k+1) + a^2 - \frac{1}{4})^{2j-2k-1} - ((\frac{2j-2k+1}{2})^2 - a(2j-2k+1) + a^2 - \frac{1}{4})}{2j-2k+1}\right) \\
&\text{x}\ \frac{(-1)^{a+k}(-2a)}{(k+a)!(k-a)!(-1)} \pmod p \qquad (\text{by } (6.1) \text{ and } (6.7)) \\
&\equiv \frac{(-1)^{a+k}2a}{(k+a)!(k-a)!}\left(\frac{(a^2-\frac{1}{4})^k - (a^2-\frac{1}{4})}{p} + a\right) \pmod p \text{ by (6.3) and since } p = 2j-2k+1.
\end{aligned}$$

On the other hand, the term at $m = j-k+a$ is

$$\begin{aligned}
&\left(\frac{((\frac{2j-2k+1}{2})^2 + a(2j-2k+1) + a^2 - \frac{1}{4})^{2j-2k-1} - ((\frac{2j-2k+1}{2})^2 + a(2j-2k+1) + a^2 - \frac{1}{4})}{2j-2k+1}\right) \\
&\text{x}\ \frac{(-1)^{a+k}(2(j-k+a)+1)}{(k-a)!(2j-k+a+1)(2j-k+a)\cdots(2j-2k+2)(2j-2k)!} \\
&\equiv \frac{(-1)^{a+k}2a}{(k-a)!(k+a)!(-1)}\left(\frac{(a^2-\frac{1}{4})^k - (a^2-\frac{1}{4})}{p} - a\right) \pmod p.
\end{aligned}$$

Adding these two terms together, we see that the expressions with p in the denominator cancel each other. Consequently all that remains is

$$(-1)^{a+k}\frac{4a^2}{(k-a)!(k+a)!}$$

and thus, in summing a from 1 to k, we find that

$$\begin{aligned} PS_p^{(j)} &= PS_p^{(j)} - PS_1^{(j)} \\ &\equiv \sum_{a=1}^{k} (-1)^{a+k} \frac{4a^2}{(k-a)!(k+a)!} \pmod p \\ &= (-1)^k \frac{2}{(2k)!} \sum_{a=-k}^{k} (-1)^a \binom{2k}{k-a} a^2. \end{aligned}$$

By (6.5), this last term is zero if $k > 1$. If $k = 1$, this term is

$$\frac{2(-1)}{2}\left(-2\binom{2}{0}1^2\right) = 2;$$

that is to say,

$$PS_p^{(p+1)/2} \equiv 2 \pmod p$$

and

$$PS_p^{(j)} \equiv 0 \pmod p$$

for $(p+1)/2 < j < p$. This concludes all the possibilities and proves the theorem. $\square$

We show in the next section that Theorem 6.1 is, remarkably, equivalent to the important Euler's Criterion (see [2, p. 116, Theorem 9-1]):

Theorem 6.2. (*Euler's Criterion*) *Let p be an odd prime. The number a is a quadratic residue modulo p if and only if*

$$a^{(p-1)/2} \equiv 1 \pmod p.$$

7. Implications of Theorem

We begin with what Theorem 6.1 implies for polynomials (mod p), that is, polynomials in $\mathbb{Z}_p[x]$.

Corollary 7.1. *For each odd prime p,*

$$x^p - x \equiv \langle x\rangle_p + 2\langle x\rangle_{(p+1)/2} \pmod p,$$

where, as previously defined in (5.19),

$$\langle x\rangle_n = x(x-2)(x-6)\ldots(x-(n-1)n).$$

Proof. This follows immediately from Theorem 6.1 and the fact that

$$x^p = \sum_{j=0}^{p} PS_p^{(j)} \langle x\rangle_j.$$

$\square$

Let us recall that numbers that are the product of two consecutive integers are called *pronic numbers*. If a number n is congruent to a pronic number modulo p (a prime), it is called a *pronic residue modulo p*. If not, it is called a *pronic non-residue modulo p*. For example, when $p = 11$, the pronic residue classes are $0, 1, 2, 6, 8, 9$; the pronic non-residues classes are $3, 4, 5, 7, 10$. We note that, in general, the pronic residue classes of any prime p are precisely the zeros $(\bmod\, p)$ of the polynomial $\langle n\rangle_{(p+1)/2}$.

We note that since

$$j(j+1) \equiv (p-j-1)(p-j) \pmod{p},$$

the pronic residues symmetrically repeat. For example, when $p = 11$, as j runs from 0 to 10, $j(j+1)$ yields $0, 2, 6, 1, 9, 8, 9, 1, 6, 2, 0$. Consequently,

$$\begin{aligned} \langle x\rangle_p &\equiv \langle x\rangle^2_{(p-1)/2}(x-(p-1)(p+1)/4) \\ &\equiv \langle x\rangle_{(p+1)/2}\langle x\rangle_{(p-1)/2} \pmod{p}. \end{aligned} \tag{7.1}$$

To illustrate this, notice that

$$\begin{aligned} \langle x\rangle_{11} &= x(x-2)(x-6)(x-12)(x-20)(x-30)(x-42)(x-56)(x-72)(x-90)(x-110) \\ &\equiv x(x-2)(x-6)(x-1)(x-9)(x-8)(x-9)(x-1)(x-6)(x-2)x \pmod{11} \\ &= x^2(x-2)^2(x-6)^2(x-1)^2(x-9)^2(x-8) \pmod{11} \\ &\equiv \langle x\rangle^2_5(x-30) \equiv .\langle x\rangle^2_5(x-8) \pmod{11} \\ &\equiv \langle x\rangle_6\langle x\rangle_5 \pmod{11}. \end{aligned}$$

Corollary 7.2. *If n is a pronic non-residue mod p, and $r_0, r_1, \ldots, r_{(p-1)/2}$ is a complete set of pronic residues excluding $(p^2-1)/4$, then*

$$\prod_{j=0}^{(p-1)/2} (n - r_j) \equiv -2 \pmod{p}. \tag{7.2}$$

Proof. By (6.2), Corollary 7.1, and (7.1) ,

$$\begin{aligned} 0 \equiv n^p - n &\equiv \langle n\rangle_p + 2\langle n\rangle_{(p+1)/2} \\ &\equiv \langle n\rangle_{(p+1)/2}(\langle n\rangle_{(p-1)/2} + 2) \pmod{p}. \end{aligned}$$

But $\mathbb{Z}_p$ is a field and $\langle n\rangle_{(p+1)/2}$ cannot be 0 mod p because n is not a pronic residue. Therefore

$$\langle n\rangle_{(p-1)/2} \equiv -2 \pmod{p},$$

which is the desired result. □

Example 7.1. $p = 11$, $n = 5$,

$$\begin{aligned} &5\cdot(5-1)\cdot(5-2)\cdot(5-6)\cdot(5-9) \\ &= 5\cdot 4\cdot 3\cdot(-1)\cdot(-4) = 240 \equiv -2 \pmod{11}, \end{aligned}$$

and $p = 11$, $n = 3$,

$$\begin{aligned} &3\cdot(3-1)\cdot(3-2)\cdot(3-6)\cdot(3-9) \\ &= 3\cdot 2\cdot 1\cdot(-3)\cdot(-6) = 108 \equiv -2 \pmod{11}. \end{aligned}$$

Remark 7.1. *It is important to point out that while Corollary 7.2 appears to be somewhat esoteric, it is actually equivalent to Theorem 6.1. This is because if Corollary 7.2 is true, the polynomial*

$$\langle x\rangle_p + 2\langle x\rangle_{(p+1)/2} \equiv \langle x\rangle_{(p+1)/2}(\langle x\rangle_{(p-1)/2} + 2) \pmod{p}$$

is monic of degree p *and each of* $0, 1, 2, \ldots, p-1$ *is a root. However, by (6.2),*

$$x^p - x$$

is also a monic polynomial whose roots are $0, 1, 2, \ldots, p-1$*. Therefore the difference between these polynomials is of degree* $< p$ *with* p *distinct roots. Hence they are identical* $(\bmod\, p)$*. Theorem 6.1 now follows by coefficient comparisons* $(\bmod\, p)$*. Consequently, a purely number-theoretic proof of Corollary 7.2 would provide an alternative proof of Theorem 6.1.*

Remark 7.2. *In light of the fact that quadratic residues are much more familiar than pronic residues, we should note that Corollary 7.2 implies an equivalent result for quadratic residues. Recall that* 0 *is classed as neither a quadratic residue or non-residue* $(\bmod\, p)$*.*

Corollary 7.3. *Let* p *be an odd prime and* m *be a quadratic non-residue* $(\bmod\, p)$*. Suppose*

$$\{s_1, s_2, \ldots, s_{(p-1)/2}\}$$

is a complete set of quadratic residues $(\bmod\, p)$*. Then*

$$\prod_{j=1}^{(p-1)/2} (m - s_j) \equiv -2 \pmod{p}.$$

Proof. Note that if r is a pronic residue $\bmod\, p$, then there exists j such that

$$r \equiv j^2 - j \pmod{p} \tag{7.3}$$

$$4r + 1 \equiv (2j-1)^2 \pmod{p}. \tag{7.4}$$

Conversely, if $4r+1$ is a quadratic residue $\bmod\, p$ then it is congruent to $h^2 \bmod p$ and since one of h or $p-h$ is odd, (7.4) is true. Hence (7.3) is true. Thus, returning to Corollary 7.2, if n is a pronic non-residue $(\bmod\, p)$, then

$$\begin{aligned}
-2 &\equiv \prod_{j=1}^{(p-1)/2} (n - j(j-1)) \pmod{p} \\
&\equiv 2^{p-1} \prod_{j=1}^{(p-1)/2} (n - j(j-1)) \pmod{p} \text{ by (6.2)} \\
&= 2^{p-3} \prod_{j=1}^{(p-1)/2} (4n + 1 - (2j-1)^2) \\
&\equiv \prod_{j=1}^{(p-1)/2} (m - s_j) \pmod{p}.
\end{aligned}$$

□

Remark 7.3. *Remarkably, we can show that Corollary 7.3 is equivalent to Euler's Criterion (Theorem 6.2) for quadratic reciprocity; that is to say,* a *is a quadratic residue modulo* p *if and only if*

$$a^{(p-1)/2} \equiv 1 \pmod{p}. \tag{7.5}$$

Hence over $\mathbb{Z}_p$, we see that

$$a^{(p-1)/2} - 1 \equiv \prod_{j=1}^{(p-1)/2} (a - s_j).$$

Furthermore, Euler's Criterion also tells us that

$$a^{(p-1)/2} \equiv -1 \pmod p \tag{7.6}$$

if a is a quadratic non-residue. Consequently, for any non-residue m,

$$\prod_{j=1}^{(p-1)/2} (m - s_j) \equiv m^{(p-1)/2} - 1 \equiv -2 \pmod p \tag{7.7}$$

which is Corollary 7.3.

In conclusion, we see that Theorem 6.1 implies Euler's Criterion (7.5) and (7.6). Conversely, one can start with (7.7) and deduce Corollary 7.3 which is equivalent to Corollary 7.2 which, in turn, is equivalent to Theorem 6.1. In other words, Theorem 6.1 and Euler's Criterion (Theorem 6.2) are equivalent.

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