The Truncated Pentagonal Number Theorem

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Abstract

A new expansion is given for partial sums of Euler's pentagonal number series. As a corollary we derive an infinite family of inequalities for the partition function, $p(n)$.

1 Introduction

In [3], the second author produced the fastest known algorithm for the generation of the partitions of n. The work required a proof of the following inequality: For $n > 0$

$$p(n) - p(n-1) - p(n-2) + p(n-5) \leqq 0, \tag{1.1}$$

where $p(n)$ is the number of partitions of n [2].

Upon reflection, one expects that there might be an infinite family of such inequalities where (1.1) is the second entry, and the trivial inequality

$$p(n) - p(n-1) \geqq 0 \tag{1.2}$$

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is the first.

In this paper, we shall prove:

Theorem 1. *For $k \geqq 1$,*

$$\frac{1}{(q;q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(3j+1)/2}(1-q^{2j+1}) = 1+(-1)^{k-1}\sum_{n=1}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \tag{1.3}$$

where

$$\begin{aligned}(A;q)_n &= \prod_{j=0}^{\infty}\frac{(1-Aq^j)}{(1-Aq^{j+n})} \\ &= \big((1-A)(1-Aq)\cdots(1-Aq^{n-1}) \quad \textit{if } n \textit{ is a positive integer}\big)\end{aligned}$$

and

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} 0, & \textit{if } B<0 \textit{ or } B>A \\ \frac{(q;q)_A}{(q;q)_B(q;q)_{A-B}}, & \textit{otherwise.}\end{cases}$$

Corollary 1. *For $n > 0$, $k \geqq 1$*

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\left(p(n-j(3j+1)/2)-p(n-j(3j+5)/2-1)\right) \geqq 0 \tag{1.4}$$

with strict inequality if $n \geqq k(3k+1)/2$.

We note that (1.1) is the case $k = 2$ and (1.2) is the case $k = 1$. In the final section of the paper, we note the relationship of this result to D. Shanks's formula for the truncated pentagonal number series [4].

2 Proof of Theorem 1

Denote the left side of (1.3) by L_k and the right side by R_k.

Clearly

$$L_1 = \frac{1-q}{(q;q)_\infty} = \frac{1}{(q^2;q)_\infty} = \sum_{n=0}^{\infty}\frac{q^{2n}}{(q;q)_n} = R_1, \tag{2.1}$$

where we have invoked [2, p. 19, eq. (2.25)]. Thus Theorem 1 is true when $k = 1$.

It is immediate from the definition that

$$L_{k+1} - L_k = \frac{(-1)^k q^{k(3k+1)/2}(1 - q^{2k+1})}{(q;q)_\infty}. \tag{2.2}$$

On the other hand, for $k > 1$, we see by [2, p.35 eq.(3.3.4)], that

$$\begin{aligned}
R_k &= 1 + (-1)^{k-1} \sum_{n=1}^{\infty} \frac{q^{\binom{k}{2}+(k-1)n}}{(q;q)_n} \left(\begin{bmatrix} n \\ k \end{bmatrix} - q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} \right) \\
&= 1 + (-1)^{k-1} \sum_{n=1}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_k (q;q)_{n-k}} + (-1)^k \sum_{n=1}^{\infty} \frac{q^{\binom{k}{2}+(k+2)n-n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k \end{bmatrix} \\
&= 1 + \frac{(-1)^{k-1} q^{\binom{k}{2}}}{(q;q)_k} \sum_{n=0}^{\infty} \frac{q^{(k+1)(n+k)}}{(q;q)_n} + (-1)^k \sum_{n=1}^{\infty} \frac{q^{\binom{k+1}{2}+(k+2)n}}{(q;q)_n} \left((-1 + q^{-n} + 1) \right) \begin{bmatrix} n-1 \\ k \end{bmatrix} \\
&= 1 + \frac{(-1)^{k-1} q^{k(3k+1)/2}}{(q;q)_k (q^{k+1};q)_\infty} + (-1)^k \sum_{n=1}^{\infty} \frac{q^{\binom{k+1}{2}+(k+1)n}}{(q;q)_{n-1}} \begin{bmatrix} n-1 \\ k \end{bmatrix} + R_{k+1} - 1 \\
&= \frac{(-1)^{k-1} q^{k(3k+1)/2}}{(q;q)_\infty} + R_{k+1} + (-1)^k \sum_{n=1}^{\infty} \frac{q^{\binom{k+1}{2}+(k+1)n}}{(q;q)_k (q;q)_{n-k-1}} \\
&= \frac{(-1)^{k-1} q^{k(3k+1)/2}}{(q;q)_\infty} + R_{k+1} + \frac{(-1)^k q^{\binom{k+1}{2}+(k+1)^2}}{(q;q)_k} \sum_{n=0}^{\infty} \frac{q^{(k+1)n}}{(q;q)_n} \\
&= \frac{(-1)^{k-1} q^{k(3k+1)/2}}{(q;q)_\infty} + R_{k+1} + \frac{(-1)^k q^{k(3k+1)/2+2k+1}}{(q;q)_k (q^{k+1};q)_\infty} \\
&= R_{k+1} - \frac{(-1)^k q^{k(3k+1)/2}(1 - q^{2k+1})}{(q;q)_\infty}
\end{aligned} \tag{2.3}$$

We may rewrite (2.3) as

$$R_{k+1} - R_k = \frac{(-1)^k q^{k(3k+1)/2}(1 - q^{2k+1})}{(q;q)_\infty} \tag{2.4}$$

Thus $L_1 = R_1$ and both sequences satisfy the same first order recurrence. So for $k \geqq 1$,

$$L_k = R_k$$

and Theorem 1 is proved.

3 Proof of Corollary 2

We see by Theorem 1 that the generating function for

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\left(p(n-j(3j+1)/2)-p(n-j(3j+5)/2-1)\right) \tag{3.1}$$

is

$$\begin{aligned}(-1)^{k-1}L_k &= (-1)^{k-1}R_k\\ &= (-1)^{k-1}+\sum_{n=1}^{\infty}\frac{q^{\binom{k}{2}+(k+1)/n}}{(q;q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix},\end{aligned} \tag{3.2}$$

and since $\begin{bmatrix}n-1\\k-1\end{bmatrix}=0$ for $n<k$, we see that the expression in (3.1) is identically 0 for $0<n<k(3k+1)/2$. Furthermore the terms in the series in (3.2) all have non-negative coefficients. the first non-zero term occurs for $n=k$ and is

$$\frac{q^{k(3k+1)/2}}{(q;q)_k}$$

which has positive coefficients of q^n for $n\geqq k(3k+1)/2$. Thus Corollary 2 is proved.

4 Shanks's Formula

In [4], D. Shanks proved that

$$1+\sum_{j=1}^{k}(-1)^j\left(q^{j(3j-1)/2}+q^{j(3j+1)/2}\right)=\sum_{j=0}^{k}\frac{(-1)^j(q;q)_k q^{jk+\binom{j+1}{2}}}{(q;q)_j} \tag{4.1}$$

We note that the left-hand side of (4.1) has $(2k+1)$ terms of the pentagonal number series while the numerator of L_{k+1} has $2k$ terms. As we will see, it is possible to deduce from Theorem 1 a companion to (4.1) treating the case with an even number of terms.

Theorem 2.

$$\sum_{j=0}^{k}(-1)^j q^{j(3j+1)/2}(1-q^{2j+1})=\sum_{j=0}^{k}\frac{(-1)^j(q;q)_{k+1}q^{(k+2)j+\binom{j}{2}}}{(q;q)_j} \tag{4.2}$$

Proof. By Theorem 1 (with k replaced by $k+1$),

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^j q^{j(3j+1)/2}(1-q^{2j+1})\\
&= (q;q)_\infty \left(1+(-1)^k \sum_{n=1}^{\infty} \frac{q^{\binom{k+1}{2}+(k+2)n}(q^{n-k};q)_k}{(q;q)_n (q;q)_k}\right)\\
&= (q;q)_\infty (-1)^k \sum_{n=0}^{\infty} \frac{q^{\binom{k+1}{2}+(k+2)n}}{(q;q)_n} \sum_{j=0}^{k} \frac{(-1)^j q^{\binom{j}{2}+(n-k)j}}{(q;q)_j (q;q)_{k-j}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by [2, p.36, eq.(3.3.6)])}\\
&= (q;q)_\infty (-1)^k q^{\binom{k+1}{2}} \sum_{j=0}^{k} \frac{(-1)^j q^{\binom{j}{2}-kj}}{(q;q)_j (q;q)_{k-j}} \frac{1}{(q^{j+k+2};q)_\infty}\\
&= \sum_{j=0}^{k} \frac{(-1)^{k-j} q^{\binom{j-k}{2}} (q;q)_{j+k+1}}{(q;q)_j (q;q)_{k-j}}\\
&= \frac{1}{(q;q)_n} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{j-k}{2}} (q;q)_{j+k+1}\\
&= (-1)^k (1-q^{k+1}) \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j q^{\binom{j-2}{2)}} (q^{k+2};q)_j\\
&= (-1)^k (1-q^{k+1}) q^{\binom{k+1}{2}} \sum_{j=0}^{k} \frac{(q^{-k};q)_j (q^{k+2};q)_j}{(q;q)_j}\\
&= (q;q)_{k+1} \sum_{k=0}^{k} \frac{(-1)^j q^{\binom{j}{2}+(k+2)j}}{(q;q)_j},
\end{aligned}$$

where the last line follows from [2, p.38, next to last line with $b = q^{-k}$, then $t = 1$ and $c \to 0$]. Thus Theorem 3 is proved. □

It is an easy exercise to deduce (4.1) from Theorem 3 and vice versa. Consequently we could prove Theorem 1 by starting with (4.1), then deducing Theorem 3, and then reversing the proof of Theorem 3 to obtain Theorem 1. We chose this way of proceeding because of the natural motivation provided by knowing Corollary 2 in the cases $k = 1, 2$.

Finally we note that (4.1) and truncated identities like it arose in important ways in [1]. Thus it is possible that there are extensions of our Theorem 1 that might have applications to mock theta functions.

References

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