THE ODD MOMENTS OF RANKS AND CRANKS

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Abstract. By modifying the definition of moments of ranks and cranks, we study the odd moments of ranks and cranks. In particular, we prove the inequality between the first crank moment $\overline{M}_1(n)$ and the first rank moment $\overline{N}_1(n)$:

$$\overline{M}_1(n) > \overline{N}_1(n).$$

We also study new counting function $\mathrm{ospt}(n)$ which is equal to $\overline{M}_1(n) - \overline{N}_1(n)$.

1. Introduction

Ramanujan's striking congruence properties of the partition function $p(n)$,

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11},$$

have motivated much research. Here, $p(n)$ denotes the number of partitions of n. In particular, toward a combinatorial explanation of the above congruences many partition statistics have been studied. Among them, the rank suggested by F. Dyson [6] and the crank suggested by the first author and F.G. Garvan [2] have proven successful and their own properties have been extensively studied. Here, the rank of partition λ is defined by $\lambda_1 - \ell(\lambda)$, where λ_1 is the largest part of λ and $\ell(\lambda)$ is the number of parts of λ, and the crank of partition λ, the crank $c(\lambda)$ of a partition is defined as

$$c(\lambda) := \begin{cases} \lambda_1, & \text{if } r = 0, \\ \omega(\lambda) - r, & \text{if } r \geq 1, \end{cases}$$

where r is the number of 1's in λ, $\omega(\lambda)$ is the number of parts in λ that are strictly larger than r. In this article, we study the rank and the crank moments which were introduced by A.O.L Atkin and Garvan [3]. Let $N(m,n)$ denote the number of partitions of n with rank m. Then the rank generating function $R(z,q)$ is given by

$$\begin{aligned} R(z,q) &= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) z^m q^n \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}. \end{aligned} \tag{1.1}$$

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Here and in the rest of the article, we will use the following standard q-series notation:

$$(a;q)_0 := 1,$$
$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),\ n \geq 1,$$

and

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n,\ |q|<1.$$

Let $M(m,n)$ denote the number of partitions of n with crank m. Then the crank generating function $C(z,q)$ is given by

$$\begin{aligned} C(z,q) &= \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)z^m q^n \\ &= \frac{(q)_\infty}{(zq)_\infty(z^{-1}q)_\infty}. \end{aligned} \tag{1.2}$$

The j-th moments of the rank and crank are defined by, respectively,

$$\begin{aligned} N_j(n) &= \sum_{k=-\infty}^{\infty} k^j N(k,n), \\ M_j(n) &= \sum_{k=-\infty}^{\infty} k^j M(k,n). \end{aligned} \tag{1.3}$$

From the symmetries $N(k,n) = N(-k,n)$ and $M(k,n) = M(-k,n)$, as can be immediately seen from their generating functions (1.1) and (1.2), respectively, $N_j(n)$ and $M_j(n)$ are zero whenever j is odd. Therefore, odd moments of ranks and cranks are never discussed. We propose the following modified rank and crank moments,

$$\begin{aligned} \overline{N}_j(n) &= \sum_{k=1}^{\infty} k^j N(k,n), \\ \overline{M}_j(n) &= \sum_{k=1}^{\infty} k^j M(k,n). \end{aligned} \tag{1.4}$$

The new odd rank and crank moments are now nontrivial.

Define the generating functions

$$C_k(q) = \sum_{n=1}^{\infty} \overline{M}_k(n)q^n,$$
$$R_k(q) = \sum_{n=1}^{\infty} \overline{N}_k(n)q^n.$$

In this article, we will focus on the first moment of rank and crank. Our first result is generating functions for these moments.

Theorem 1. *The following are true:*

$$C_1(q) = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)/2}}{1-q^n},$$

and

$$R_1(q) = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(3n+1)/2}}{1-q^n}.$$

Moreover, we can express the generating function for the first crank moment in terms of Eulerian series, which has very interesting combinatorial interpretation.

Theorem 2. *The following is true.*

$$C_1(q) = \sum_{k=1}^{\infty} \frac{k q^{k^2}}{(q)_k^2}. \tag{1.5}$$

Very interestingly, there is an inequality between the first crank moment and the first rank moment.

Theorem 3. *For all positive integers n,*

$$\overline{M}_1(n) > \overline{N}_1(n).$$

In [1], the first author introduced the number of smallest part function. Let $\mathrm{spt}(n)$ denote the number of smallest parts in the partitions of n. For example, the partitions of 5 are

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1,$$

and so $\mathrm{spt}(5) = 14$. Surprisingly, there is a relation between $\mathrm{spt}(n)$ and moments as follows:

$$\mathrm{spt}(n) = \overline{M}_2(n) - \overline{N}_2(n). \tag{1.6}$$

In light of Theorem 3 and (1.6), it is natural to define $\mathrm{ospt}(n)$ as

$$\mathrm{ospt}(n) = \overline{M}_1(n) - \overline{N}_1(n). \tag{1.7}$$

Before stating what $\mathrm{ospt}(n)$ counts, we need to introduce some notation. In the partition λ, we define an even string in the partition λ as the consecutive parts starting from some even number $2k+2$ of which length is an odd number greater than or equal to $2k+1$ such that $2k+1$ and $2k+2$ plus the length of the string (the number of consecutive parts) do not appear as a part. We also define an odd string in the partition λ as the consecutive numbers starting from some odd number $2k+1$ of which length is greater than or equal to $2k+1$ such that the part $2k+1$ appears only one time and $2k+2$ plus the length of string does not appear as a part. Then, we have the following theorem.

Theorem 4. *For all positive integers n,*

$$\mathrm{ospt}(n) = \sum_{\lambda \vdash n} \mathrm{ST}(\lambda),$$

where $\mathrm{ST}(\lambda)$ is the number of even and odd strings in the partition λ.

Partitions of 6	The number of even strings	The number of odd strings
6	0	0
5+1	0	1 (1 is the odd string.)
4+2	1 (2 is the even string.)	0
4+1+1	0	0
3+3	0	0
3+2+1	0	0
3+1+1+1	0	1 (1is the odd string.)
2+2+2	1 (2 is the even string.)	0
2+2+1+1	0	0
2+1+1+1+1	0	0
1+1+1+1+1+1	0	0

Table 1. The number of strings in the partitions of 6.

Here, we give two examples. Since $\overline{M}_1(6) = 16$ and $\overline{N}_1(6) = 12$, we have $\text{ospt}(6) = 4$. On the other hand, in the table 1, we can see the total number of strings in the partitions of 6 is 4.

Since $\overline{M}_1(9) = 52$ and $\overline{N}_1(9) = 42$, we have $\text{ospt}(9) = 10$. Here we list the partitions of 9 which have even or odd strings in Table 2.

Partitions of 9	The number of even strings	The number of odd strings
8+1	0	1
7+2	1	0
6+2+1	1	1
5+3+1	0	1
5+2+2	1	0
4+4+1	0	1
4+3+2	1	0
3+3+2+1	0	1
2+2+2+2+1	0	1

Table 2. The number of strings in the partitions of 9.

This paper is organized as follows. In Section 2, we prove Theorems 1 and 2, and we discuss their combinatorial implications. In Section 3, we prove Theorem 3. In Section 4, we prove Theorem 4 and give combinatorial identities derived from the theorems. We conclude the paper with a suggestion for further research.

2. The first moments of rank and crank

In this section, we will prove Theorems 1 and 2. After proving the theorems, we will focus on combinatorial implications of the theorems. We start by proving Theorem 1.

Proof of Theorem 1. First, we derive the generalized Lambert series representation for $C_1(q)$. We begin the the generalized Lambert series representation of the crank generating function

[4],

$$\frac{(q)_\infty}{(zq)_\infty(q/z)_\infty} = \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1-z)(-1)^n q^{n(n+1)/2}}{1-zq^n}.$$

Applying the differential operator $z\frac{d}{dz}$ on both sides, we obtain

$$\begin{aligned}
&z\frac{d}{dz}\left(\frac{(q)_\infty}{(zq)_\infty(q/z)_\infty}\right)\\
&=\frac{1}{(q)_\infty}z\frac{d}{dz}\sum_{n=-\infty}^{\infty}\frac{(1-z)(-1)^n q^{n(n+1)/2}}{1-zq^n}\\
&=\frac{z}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}(1-q^n)}{(1-zq^n)^2}\\
&=\frac{z}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}(1-q^n)}{(1-zq^n)^2}+\frac{1}{z(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}(1-q^n)}{(1-q^n/z)^2}.
\end{aligned}$$

Therefore, by the symmetry of z and z^{-1},

$$C_1(q)=\lim_{z\to 1}\frac{z}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}(1-q^n)}{(1-zq^n)^2}=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}}{1-q^n}.$$

Next, we derive the generalized Lambert series representation for $R_1(q)$. We begin the the generalized Lambert series representation of the rank generating function [7, Eq.(7.11)]

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(q/z)_n}=\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1-z)(-1)^n q^{n(3n+1)/2}}{1-zq^n}.$$

Applying the differential operator $z\frac{d}{dz}$ on both sides, we obtain

$$\begin{aligned}
z\frac{d}{dz}\left(\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(q/z)_n}\right)&=\frac{1}{(q)_\infty}z\frac{d}{dz}\sum_{n=-\infty}^{\infty}\frac{(1-z)(-1)^n q^{n(3n+1)/2}}{1-zq^n}\\
&=\frac{z}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}q^{n(3n+1)/2}(1-q^n)}{(1-zq^n)^2}.
\end{aligned}$$

Similarly, by the symmetry of z and z^{-1},

$$R_1(q)=\lim_{z\to 1}\frac{z}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(3n+1)/2}(1-q^n)}{(1-zq^n)^2}=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(3n+1)/2}}{1-q^n}.$$

□

A proof of Theorem 2 can be also obtained from differentiating a proper q-series identity.

Proof of Theorem 2. From [5, Eq. (5.14)],

$$(aq)_\infty\sum_{n=0}^{\infty}\frac{b^n q^{n^2}}{(q)_n(aq)_n}=\sum_{n=0}^{\infty}\frac{(-1)^n(b/a)_n a^n q^{n(n+1)/2}}{(q)_n},$$

differentiating with respect to b gives

$$(aq)_\infty \sum_{n=0}^{\infty} \frac{nb^{n-1}q^{n^2}}{(q)_n(aq)_n} = \sum_{n=1}^{\infty} \frac{(-1)^n(b/a)_n a^n q^{n(n+1)/2}}{(q)_n} \sum_{k=0}^{n} \frac{-q^k/a}{1-bq^k/a}.$$

Letting $b \to a$, we find that gives

$$(aq)_\infty \sum_{n=0}^{\infty} \frac{na^{n-1}q^{n^2}}{(q)_n(aq)_n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}a^{n-1}q^{n(n+1)/2}}{1-q^n}.$$

Substituting $a = 1$ and dividing both sides by $(q)_\infty$, we arrive at (1.5). □

Remark. E. Deutsch authored the sequence for $C_1(q)$ and gave the generating function in Theorem 2 in the Online Encyclopedia of Integer Sequences (A115995). V. Jovovic also contributed to the sequence, in particular, he gave the generating function in Theorem 1.

Now we focus on the combinatorial interpretation of the first crank moment. By Theorem 1, we see that

$$\begin{aligned} C_1(q) &= \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}q^{(n^2+n)/2}}{1-q^n} \\ &= \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \sum_{k=0}^{\infty} q^{\binom{n+1}{2}+kn} \end{aligned} \tag{2.1}$$

We can think of the right side as a weighted count of partitions as follows.

Theorem 5. *For all positive integers n,*

$$\overline{M}_1(n) = \sum_{\lambda \vdash n} \sum_{j \geq 1} (-1)^{j+1} w_j,$$

where w_j is defined by

$$w_j = \begin{cases} \lambda_j - \lambda_{j+1}, & \text{if } \lambda_1 > \lambda_2 > \cdots > \lambda_j > \lambda_{j+1} \\ 0, & \text{otherwise,} \end{cases}$$

for $j \geq 1$.

Proof of Theorem 5. In (2.1),

$$q^{\binom{n+1}{2}+kn}$$

generates the partition $\pi = (n+k,\, n+k-1,\, \cdots,\, 1+k)$. We append this partition into λ generated by $\frac{1}{(q;q)_\infty}$ from the largest parts. For example, if $\lambda = (2,2,1)$ and $\pi = (6,5,4,3)$, then the resulting partition is (8,7,5,3). In this way, we find a surjection

$$\phi : \cup_{n,k \in \mathbb{N}} \mathcal{P}_{n,k} \times \mathcal{P},$$

where $\mathcal{P}_{n,k}$ is the set of partitions of the form $(n+k,\, n+k-1,\, \cdots,\, 1+k)$ and $\mathcal{P}$ is the set of ordinary partitions. Now, we want to find the preimage $\phi^{-1}(\lambda)$ of a fixed $\lambda \in \mathcal{P}$. For this purpose, we define $\ell_s(\lambda)$ as the largest positive integer j satisfying $\lambda_1 > \lambda_2 > \cdots > \lambda_j > \lambda_{j+1}$. (For convenience, if the number of parts in λ is ℓ, we define $\lambda_{\ell+1} = 0$.) If there is no such j, we define $\ell_s(\lambda)$ to be zero. (This is the case $\lambda_1 = \lambda_2$.) Suppose that $\ell_s(\lambda) = 0$. Then, clearly, $\phi^{-1}(\lambda) = \emptyset$ since if $\pi \in \mathcal{P}_{n,k}$ is appended, then the first n parts of the resulting partition should be distinct. Now suppose that $\ell_s(\lambda) > 0$. Then, for $i \leq \ell_s(\lambda)$, there are $\lambda_i - \lambda_{i+1}$

preimages in $\cup_{k=0}^{\lambda_i-\lambda_{i+1}-1}\mathcal{P}_{i,k}$. Finally, if $i>\ell_s(\lambda)$, then there is no preimage in $\cup_{k\in\mathbb{N}}\mathcal{P}_{i,k}$. By taking the sign into the consideration, this completes the proof. □

Remark. In [8], the third author introduced the subpartitions with gap d. Then $w_j\neq 0$ only if λ has the subpartition with gap 1 of length $\geq j$.

From Theorem 2, we have the following interesting partition identity.

Theorem 6.

$$\overline{M}_1(n)=\sum_{\lambda\vdash n}\sum_{j\geq 0}(-1)^{j+1}w_j=\sum_{\lambda\vdash n}d(\lambda),$$

where $d(\lambda)$ is the size of Durfee square of λ.

This is very curious combinatorial identity since even the positivity of $\sum_{\lambda\vdash n}\sum_{j\geq 0}(-1)^{j+1}w_j$ is not clear at all, nor is the relationship of the sum of part size differences to cranks or Durfee squares.

3. Proof of Theorem 3

By Theorem 1, we see that

$$C_1(q)-R_1(q)=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}q^{\binom{n+1}{2}}\frac{(1-q^{n^2})}{1-q^n}$$

We begin by noting that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{\binom{n+1}{2}}(1-q^{n^2})}{1-q^n}&=\sum_{n=1}^{\infty}(-1)^{n+1}q^{\binom{n+1}{2}}\sum_{j=0}^{n-1}q^{jn}\\&=\sum_{n=1}^{\infty}(-1)^{n+1}q^{\binom{n+1}{2}}\sum_{j=1}^{n-1}q^{(j-1)n}\\&=\sum_{j=1}^{\infty}f_j(q),\end{aligned}$$

where

$$f_j(q)=\sum_{n=j}^{\infty}(-1)^{n+1}q^{\binom{n}{2}+jn}=\sum_{n=0}^{\infty}(-1)^{n+j+1}q^{\binom{n+j}{2}+j(n+j)}.$$

Theorem 7. *For $i\geq 0$,*

$$\begin{aligned}f_{2i+1}(q)+f_{2i+2}(q)&=q^{6i^2+5i+1}(1-q^{4i+2})(1-q^{6i+4})\\&\quad+\sum_{j=1}^{\infty}q^{6i^2+8ij+2j^2+7i+5j+2}(1-q^{4i+2})(1-q^{4i+2j+3})\\&\quad+\sum_{j=1}^{\infty}q^{6i^2+8ij+2j^2+5i+3j+1}(1-q^{2i+1})(1-q^{4i+2j+2}).\end{aligned}\tag{3.1}$$

Proof of Theorem 7. The right side of (3.1) multiplied out is

$$\begin{aligned}
&q^{6i^2+5i+1} - q^{6i^2+9i+3} - q^{6i^2+11i+5} + q^{6i^2+15i+7}\\
&\quad + \sum_{j=1}^{\infty} q^{6i^2+8ij+2j^2+7i+5j+2}\left(1 - q^{4i+2} - q^{4i+2j+3} + q^{8i+2j+5}\right)\\
&\quad + \sum_{j=1}^{\infty} q^{6i^2+8ij+2j^2+5i+3j+1}\left(1 - q^{2i+1} - q^{4i+2j+2} + q^{8i+2j+5}\right)\\
&= q^{6i^2+5i+1} - q^{6i^2+9i+3} - q^{6i^2+11i+5} + q^{6i^2+15i+7}\\
&\quad + T_1 - T_2 - T_3 + T_4\\
&\quad + S_1 - S_2 - S_3 + S_4.
\end{aligned}$$

Inspection immediately reveals that

$$S_4 = T_2.$$

Furthermore,

$$\begin{aligned}
q^{6i^2+15i+7} + T_4 - S_2 &= \sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+15i+7j+7} - \sum_{j=1}^{\infty} q^{6i^2+8ij+2j^2+7i+5j+2}\\
&= 0,
\end{aligned}$$

which follows from the fact that the second sum is seen to be identified with the first once we replace j by $j+1$ in the second sum. Hence, the right hand side of (3.1) is equal to

$$\begin{aligned}
&q^{6i^2+5i+1} - q^{6i^2+9i+3} - q^{6i^2+11i+5} + T_1 - T_3 + S_1 - S_3\\
&= \sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+15i+9j+9} - \sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+11i+7j+5}\\
&\quad + \sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+5i+3j+1} - \sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+9i+5j+3}\\
&= \sum_{j=0}^{\infty} q^{\binom{2j+2i+3}{2}+(2i+2)(2j+2i+3)} - \sum_{j=0}^{\infty} q^{\binom{2j+2i+2}{2}+(2i+2)(2j+2i+2)}\\
&\quad - \sum_{j=0}^{\infty} q^{\binom{2j+2i+1}{2}+(2i+1)(2j+2i+1)} - \sum_{j=0}^{\infty} q^{\binom{2j+2i+2}{2}+(2i+1)(2j+2i+2)}\\
&= f_{2i+2}(q) + f_{2i+1}(q)
\end{aligned}$$

where we have replaced j by $j+1$ in T_1 for the first equality. □

Corollary 8. $\frac{1}{(q;q)_\infty}\left(f_{2i+1}(q) + f_{2i+2}(q)\right)$ *has non-negative power series coefficients.*

Proof of Corollary 8. Clearly, if $a \neq b$, then

$$\frac{(1-q^a)(1-q^b)}{(q;q)_\infty} = \prod_{\substack{n=1\\ n\neq a,b}}^{\infty} \frac{1}{1-q^n}$$

has non-negative power series coefficient, and this fact shows that every term in the right side of (3.1) has non-negative coefficients. □

Finally, we arrive at

$$\begin{aligned}C_1(q)-R_1(q) &= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{\binom{n+1}{2}}(1-q^{n^2})}{1-q^n}\\ &= \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty} f_j(q)\\ &= \frac{1}{(q;q)_\infty}\sum_{j=0}^{\infty}\left(f_{2j+1}(q)+f_{2j+2}(q)\right),\end{aligned}$$

and Corollary 8 implies the non-negativity of the coefficients in the last expression, which finishes the proof of Theorem 3.

Remark. From the proof of Corollary 8, we can actually have positivity of $C_1 - R_1$ as $f_0(q) + f_1(q)$ has the term

$$\frac{q}{(q;q)_\infty}\left((1-q^2)(1-q^4)\right)$$

which has positive coefficients as 1 is available as a part.

4. The ospt(n) Function

In this section, we will investigate combinatorial implications of the result in the previous section, which leads us to define ospt(n). To prove Theorem 4, we start from restating (3.1).

$$\begin{aligned}f_{2i+1}(q)+f_{2i+2}(q) &= \sum_{j=0}^{\infty} q^{(2i+2)+(2i+3)+\cdots+(4i+2j+2)}(1-q^{2i+1})(1-q^{4i+2j+3})\\ &\quad+\sum_{j=0}^{\infty} q^{(2i+1)+(2i+2)+\cdots+(4i+2j+1)}(1-q^{2i+1})(1-q^{4i+2j+2})\\ &\quad+\sum_{j=0}^{\infty} q^{(2i+1)+(2i+2)+\cdots+(4i+2j+2)}(1-q^{2i+1})(1-q^{4i+2j+3})\\ &= \sum_{j=0}^{\infty} q^{(2i+2)+(2i+3)+\cdots+(4i+2j+2)}(1-q^{2i+1})(1-q^{4i+2j+3})\\ &\quad+\sum_{j=1}^{\infty} q^{(2i+1)+(2i+2)+\cdots+(4i+j)}(1-q^{2i+1})(1-q^{4i+j+1})\\ &=: \mathrm{ST}_{2i}(q)+\mathrm{ST}_{2i+1}(q).\end{aligned}$$

Then, we can see that

$$\frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}\mathrm{ST}_{2k}(q)$$

is the generating function for the number of even strings in the partitions of n. Similarly, we can think of

$$\frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}\mathrm{ST}_{2k+1}(q)$$

as the generating function for the number of odd strings in the partitions of n, which completes the proof of Theorem 4.

Here is another partition theoretic interpretation for $\mathrm{ospt}(n)$. By Theorem 3, we see that

$$\begin{aligned} C_1(q) - R_1(q) &= \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{(n^2+n)/2}(1-q^{n^2})}{1-q^n} \\ &= \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} q^{(n^2+n)/2}(1+q^n+q^{2n}+\cdots+q^{n^2-n}) \end{aligned}$$

has positive power series coefficients. If we define that $ospt(n) = \overline{M}_1(n) - \overline{N}_1(n)$, then we can think this function as a weighted count of partition.

Theorem 9. *For all positive integers n,*

$$ospt(n) = \sum_{\lambda \vdash n} \sum_{j \geq 1} (-1)^{j+1} f_j'(\lambda),$$

where $w_j'(\lambda) = min\{w_j(\lambda), j\}$.

As the proof is very similar to Theorem 5, we omit the proof here.

5. Concluding Remark

It would be very interesting to find bijective proofs for the results in this paper. In particular, it would be nice if one can find a bijection for

$$\mathrm{ospt}(n) = \overline{M}_1(n) - \overline{N}_1(n) = \sum_{\lambda \vdash n} \mathrm{ST}(\lambda).$$

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