PARTITIONS WITH FIXED DIFFERENCES BETWEEN LARGEST AND SMALLEST PARTS

GEORGE E. ANDREWS, MATTHIAS BECK, AND NEVILLE ROBBINS

Abstract. We study the number $p(n,t)$ of partitions of n with difference t between largest and smallest parts. Our main result is an explicit formula for the generating function $P_t(q) := \sum_{n\geq 1} p(n,t)\, q^n$. Somewhat surprisingly, $P_t(q)$ is a rational function for $t > 1$; equivalently, $p(n,t)$ is a quasipolynomial in n for fixed $t > 1$. Our result generalizes to partitions with an arbitrary number of specified distances.

Enumeration results on integer partitions form a classic body of mathematics going back to at least Euler, including numerous applications throughout mathematics and some areas of physics; see, e.g., [2]. A *partition* of a positive integer n is, as usual, an integer k-tuple $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$, for some k, such that

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k \,.$$

The integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the *parts* of the partition. We are interested in the counting function

$$p(n,t) := \#\text{partitions of } n \text{ with difference } t \text{ between largest and smallest parts.}$$

It is immediate that

$$p(n,0) = d(n)$$

where $d(n)$ denotes the number of divisors of n. Charmingly, $p(n,1)$ equals the number of *non*divisors of n:

$$p(n,1) = n - d(n)\,,$$

which can be explained bijectively by the fact that the partitions counted by $p(n,0)+p(n,1)$ contain exactly one sample with k parts, for each $k = 1, 2, \ldots, n$ [1, Sequence A049820], or by the generating function identity

$$\sum_{n\geq 1} p(n,1)\, q^n = \sum_{m\geq 1} \frac{q^m}{1-q^m}\, \frac{q^{m+1}}{1-q^{m+1}} = \frac{q}{(1-q)^2} - \sum_{m\geq 1} \frac{q^m}{1-q^m}\,.$$

(The last equation follows from a few elementary operations on rational functions). An even less obvious instance of our partition counting function is

$$p(n,2) = \binom{\lfloor \frac{n}{2} \rfloor}{2}\,, \tag{1}$$

as observed by Reinhard Zumkeller in 2004 [1, Sequence A008805]. (It is not clear to us where in the literature this formula first appeared, though specific values of $p(n,k)$ are well represented in

Date: 14 July 2014.
2010 *Mathematics Subject Classification.* Primary 11P84; Secondary 05A17.
Key words and phrases. Integer partition, fixed difference between largest and smallest parts, rational generating function, quasipolynomial.
We thank an anonymous referee for numerous helpful suggestions. M. Beck's research was partially supported by the US National Science Foundation (DMS-1162638).

[1], where Sequences A000005, A049820, A008805, A128508, and A218567–A218573 give the first values of $p(n,k)$ for fixed $k = 0, 1, \ldots, 10$, and Sequence A097364 paints a general picture of $p(n,t)$.)

We remark that $p(n,2)$ is arithmetically quite different from $p(n,0)$ and $p(n,1)$: namely, $p(n,2)$ is a *quasipolynomial*, i.e., a function that evaluates to a polynomial when n is restricted to a fixed residue class modulo some (minimal) positive integer, the *period* of the quasipolynomial. (For $p(n,2)$ this period is 2.) Equivalently, the accompanying generating function evaluates to a rational function all of whose poles are rational roots of unity. (See, e.g., [4, Chapter 4] for more on quasipolynomials and their rational generating functions.) Our goal is to prove closed formulas for these generating functions

$$P_t(q) := \sum_{n \geq 1} p(n,t)\, q^n .$$

Theorem 1. *For $t > 1$,*

$$P_t(q) = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})} - \frac{q^{t-1}}{(1-q^t)^2(1-q^{t-1})^2(1-q^{t-2})\cdots(1-q^2)} + \frac{q^t}{(1-q^t)(1-q^{t-1})^2(1-q^{t-2})\cdots(1-q)} .$$

Written in terms of the usual shorthand $(q)_m := (1-q)(1-q^2)\cdots(1-q^m)$, Theorem 1 says

$$P_t(q) = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})} - \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t} + \frac{q^t}{(1-q^{t-1})(q)_t} .$$

Thus $P_t(q)$ is rational for $t > 1$, and so $p(n,t)$ is a quasipolynomial in n, of degree t and period $\operatorname{lcm}(1,2,\ldots,t)$. For example, for $t = 2$, Theorem 1 gives

$$P_2(q) = \frac{q^4}{(1-q)^3(1+q)^2}$$

which confirms (1). The rational generating function given by Theorem 1 in the case $t = 3$ simplifies to

$$P_3(q) = \frac{q^5 + q^6 + q^7 - q^8}{(1-q^2)^2(1-q^3)^2}$$

which (by way of a computer algebra system or a straightforward binomial expansion) translates to the partition counting function

$$p(n,3) = \frac{1}{108} \times \begin{cases} n^3 - 18n & \text{if } n \equiv 0 \bmod 6, \\ n^3 - 3n + 2 & \text{if } n \equiv 1 \bmod 6, \\ n^3 - 30n + 52 & \text{if } n \equiv 2 \bmod 6, \\ n^3 + 9n - 54 & \text{if } n \equiv 3 \bmod 6, \\ n^3 - 30n + 56 & \text{if } n \equiv 4 \bmod 6, \\ n^3 - 3n - 2 & \text{if } n \equiv 5 \bmod 6 \end{cases}$$

$$= \begin{cases} m(2m^2-1) & \text{if } n = 6m, \\ m(2m^2+1) & \text{if } n = 6m+1, \\ m(2m^2+2m-1) & \text{if } n = 6m+2, \\ m(2m^2+3m+2) & \text{if } n = 6m+3, \\ (m-1)(2m^2-1) & \text{if } n = 6m-2, \\ m^2(2m-1) & \text{if } n = 6m-1. \end{cases}$$

Using this explicit form of $p(n,3)$, one easily affirms a conjecture about the recursive structure of $p(n,3)$ given in [1, Sequence A128508] in the positive.

Proof of Theorem 1. We will use the usual shorthand

$$(A)_m := (1-A)(1-A\,q)\cdots(1-A\,q^{m-1})$$

as well as *Heine's transformation* (see, e.g., [2, p. 38])

$$\sum_{m\geq 0} \frac{(a)_m (b)_m\, z^m}{(q)_m (c)_m} = \frac{(\frac{c}{b})_\infty (bz)_\infty}{(c)_\infty (z)_\infty} \sum_{j\geq 0} \frac{(\frac{abz}{c})_j (b)_j (\frac{c}{b})^j}{(q)_j (bz)_j}\,. \tag{2}$$

Now we construct the generating function for $p(n,t)$. A partition of n with difference t between smallest and largest part starts with some part m, ends with the part $m+t$, and could include any of the numbers $m+1, m+2, \ldots, m+t-1$ as parts. Translated into geometric series, this gives

$$\begin{aligned} P_t(q) &= \sum_{m\geq 1} \frac{q^m}{1-q^m} \frac{1}{1-q^{m+1}} \cdots \frac{1}{1-q^{m+t-1}} \frac{q^{m+t}}{1-q^{m+t}} = q^t \sum_{m\geq 1} \frac{q^{2m}(q)_{m-1}}{(q)_{m+t}} = q^{t+2} \sum_{m\geq 0} \frac{q^{2m}(q)_m}{(q)_{m+t+1}} \\ &= \frac{q^{t+2}}{(q)_{t+1}} \sum_{m\geq 0} \frac{(q)_m (q)_m\, q^{2m}}{(q)_m (q^{t+2})_m} \overset{(2)}{=} \frac{q^{t+2}(q^{t+1})_\infty (q^3)_\infty}{(q)_{t+1}(q^{t+2})_\infty (q^2)_\infty} \sum_{j\geq 0} \frac{(q^{-t+2})_j (q)_j\, q^{j(t+1)}}{(q)_j (q^3)_j} \\ &= \frac{q^{t+2}}{(q)_t} \sum_{j=0}^{t-2} \frac{(q^{-t+2})_j\, q^{j(t+1)}}{(q^2)_{j+1}} = \frac{q^{t+2}}{(q)_t} \sum_{j=0}^{t-2} \frac{(1-q^{t-2})(1-q^{t-3})\cdots(1-q^{t-j-1})(-1)^j q^{2j+\binom{j+1}{2}}}{(q^2)_{j+1}} \\ &= \frac{q^{t+2}(1-q)}{(1-q^t)(1-q^{t-1})} \sum_{j=0}^{t-2} \frac{(-1)^j q^{2j+\binom{j+1}{2}}}{(q)_{j+2}(q)_{t-j-2}} = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t} \sum_{j=0}^{t-2} \begin{bmatrix} t \\ j+2 \end{bmatrix} (-1)^j q^{\binom{j+3}{2}}\,. \end{aligned}$$

Thus, by the q-binomial theorem (see, e.g., [2, p. 36])

$$\begin{aligned} P_t(q) &= \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t} \sum_{j=2}^{t} \begin{bmatrix} t \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}} = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t} \left((q)_t - 1 + q \begin{bmatrix} t \\ 1 \end{bmatrix} \right) \\ &= \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})} - \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t} + \frac{q^t}{(1-q^{t-1})(q)_t}\,. \end{aligned}$$

□

A natural question concerns the growth behavior of $p(n,t)$. We see in the above example that the quasipolynomial $p(n,3)$ has a constant leading coefficient, which of course determines the asymptotic growth of $p(n,3)$. Something similar can be said in general.

Corollary 2. *If $t > 1$ then $p(n,t) = \dfrac{n^t}{t\,(t!)^2} + O(n^{t-1})$ as $n \to \infty$.*

Proof. It is well known that the first-order asymptotics of a quasipolynomial stems from the highest-order poles of its rational generating function. (This follows from first principles, essentially partial-fraction decomposition; see [3] for far-reaching generalizations.) In our case, $P_t(q)$ has a unique highest-order pole at $q = 1$ of order t. Thus the leading coefficient of $p(n,t)$ equals $\frac{1}{t!}$ times the lowest coefficient of the Laurent series of $P_t(q)$ at $q = 1$ which is

$$\lim_{q\to 1} \frac{(1-q)^{t+1}(2q^t - q^{2t} - q^{t-1})}{(1-q^t)^2(1-q^{t-1})^2(1-q^{t-2})\cdots(1-q)} = \frac{1}{t\cdot t!}\,. \qquad \square$$

Next we shall generalize Theorem 1 by considering *partitions with specified distances*. Let $p(n, t_1, t_2, \ldots, t_k)$ be the number of partitions of n such that, if σ is the smallest part then $\sigma + t_1 + t_2 + \cdots + t_k$ is the largest part and each of $\sigma + t_1$, $\sigma + t_1 + t_2$, $\ldots$, $\sigma + t_1 + t_2 + \cdots + t_{k-1}$ appear as parts. We consider the related generating function

$$P_{t_1,\ldots,t_k}(q) := \sum_{n\geq 1} p(n, t_1, t_2, \ldots, t_k)\, q^n.$$

We note that when $k = 1$ this is simply $P_t(q)$ from above.

Theorem 3. *For $t := t_1 + t_2 + \cdots + t_k > k$,*

$$P_{t_1,\ldots,t_k}(q) = \frac{(-1)^k q^{T-\binom{k+1}{2}} \left(\sum_{j=0}^{k} \begin{bmatrix} t \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}} - (q)_t\right)}{\begin{bmatrix} t-1 \\ k \end{bmatrix} (1-q^t)(q)_t}\,,$$

where $T := kt_1 + (k-1)t_2 + \cdots + 2t_{k-1} + t_k$ and $\begin{bmatrix} A \\ B \end{bmatrix} := \frac{(q)_A}{(q)_B (q)_{A-B}}$.

For example, for $k = 2$ and $t_1 = t_2 = 2$, we have $p(11, 2, 2) = 2$ since $1 + 1 + 1 + 3 + 5$ and $1+2+3+5$ are the unique two partitions of 11 that contain three parts whose consecutive distances are 2. Theorem 3 says in this case

$$P_{2,2}(q) = \frac{q^9 + q^{10} + q^{11} + q^{12} - q^{13}}{(1-q^2)(1-q^3)^2(1-q^4)^2}$$

which translates to

$$p(n,2,2) = \frac{1}{6912}\begin{cases} 3n^4 - 20n^3 - 24n^2 288n & \text{if } n \equiv 0 \bmod 12, \\ 3n^4 - 20n^3 - 78n^2 + 492n - 397 & \text{if } n \equiv 1 \bmod 12, \\ 3n^4 - 20n^3 - 24n^2 - 48n + 304 & \text{if } n \equiv 2 \bmod 12, \\ 3n^4 - 20n^3 - 78n^2 + 1260n - 2781 & \text{if } n \equiv 3 \bmod 12, \\ 3n^4 - 20n^3 - 24n^2 - 480n + 2816 & \text{if } n \equiv 4 \bmod 12, \\ 3n^4 - 20n^3 - 78n^2 + 492n + 155 & \text{if } n \equiv 5 \bmod 12, \\ 3n^4 - 20n^3 - 24n^2 + 720n - 3024 & \text{if } n \equiv 6 \bmod 12, \\ 3n^4 - 20n^3 - 78n^2 + 492n + 35 & \text{if } n \equiv 7 \bmod 12, \\ 3n^4 - 20n^3 - 24n^2 - 480n + 3328 & \text{if } n \equiv 8 \bmod 12, \\ 3n^4 - 20n^3 - 78n^2 + 1260n - 3213 & \text{if } n \equiv 9 \bmod 12, \\ 3n^4 - 20n^3 - 24n^2 - 48n - 208 & \text{if } n \equiv 10 \bmod 12, \\ 3n^4 - 20n^3 - 78n^2 + 492n + 547 & \text{if } n \equiv 11 \bmod 12. \end{cases}$$

Proof of Theorem 3. Again we start with the natural generating function

$$\begin{aligned}
P_{t_1,\dots,t_k}(q) &= \sum_{m\ge 1} \frac{q^m\, q^{m+t_1}\, q^{m+t_1+t_2}\cdots q^{m+t_1+t_2+\cdots+t_k}}{(1-q^m)(1-q^{m+1})\cdots(1-q^{m+t_1+t_2+\cdots+t_k})} = \sum_{m\ge 1}\frac{q^{(k+1)m+T}}{(q^m)_{t+1}} \\
&= \sum_{m\ge 1}\frac{q^{(k+1)m+T}(q)_{m-1}}{(q)_{m+t}} = q^{T+k+1}\sum_{m\ge 0}\frac{q^{(k+1)m}(q)_m}{(q)_{m+t+1}} = \frac{q^{T+k+1}}{(q)_{t+1}}\sum_{m\ge 0}\frac{(q)_m(q)_m\, q^{(k+1)m}}{(q)_m(q^{t+2})_m} \\
&\overset{(2)}{=} \frac{q^{T+k+1}(q^{t+1})_\infty(q^{k+2})_\infty}{(q)_{t+1}(q^{k+1})_\infty(q^{t+2})_\infty}\sum_{j\ge 0}\frac{(q^{k+1-t})_j(q)_j q^{(t+1)j}}{(q)_j(q^{k+2})_j} \\
&= \frac{q^{T+k+1}(q)_k}{(q)_t}\sum_{j=0}^{t-k-1}\frac{(q^{-(t-k+1)})_j q^{(t+1)j}}{(q)_{j+k+1}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{q^{T+k+1}(q)_k}{(q)_t}\sum_{j=0}^{t-k-1}\frac{(1-q^{t-k-1})(1-q^{t-k-2})\cdots(1-q^{t-k-j})(-1)^j q^{\binom{j}{2}-j(t-k-1)+(t+1)j}}{(q)_{j+k+1}} \\
&= \frac{q^{T+k+1}(q)_k}{(q)_t}\sum_{j=0}^{t-k-1}\frac{(q)_{t-k-1}(-1)^j q^{\binom{j+1}{2}+j(k+1)}}{(q)_{j+k+1}(q)_{j-k-j-1}} \\
&= \frac{q^{T+k+1}(q)_k(q)_{t-k-1}}{(q)_t(q)_t}\sum_{j=0}^{t-k-1}\begin{bmatrix} t \\ j+k+1\end{bmatrix}(-1)^j q^{\binom{j+k+2}{2}-\binom{k+2}{2}} \\
&= \frac{q^{T+k+1}(q)_k}{\left[{t-1 \atop k}\right](1-q^t)(q)_t}\sum_{j=0}^{t-k-1}\begin{bmatrix} t \\ j+k+1\end{bmatrix}(-1)^j q^{\binom{j+k+2}{2}-\binom{k+2}{2}} \\
&= \frac{q^{T-\binom{k+1}{2}}(-1)^{k+1}}{\left[{t-1 \atop k}\right](1-q^t)(q)_t}\sum_{j=k+1}^{t}\begin{bmatrix} t \\ j\end{bmatrix}(-1)^j q^{\binom{j+1}{2}} \\
&= \frac{q^{T-\binom{k+1}{2}}(-1)^{k}}{\left[{t-1 \atop k}\right](1-q^t)(q)_t}\left(\sum_{j=0}^{k}\begin{bmatrix} t \\ j\end{bmatrix}(-1)^j q^{\binom{j+1}{2}} - (q)_t\right).
\end{aligned}$$

□

References

1. *The On-Line Encyclopedia of Integer Sequences*, published electronically at `http://oeis.org`, 2014.
2. George E. Andrews, *The Theory of Partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1976 original.
3. Philippe Flajolet and Robert Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.
4. Richard P. Stanley, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
E-mail address: `andrews@math.psu.edu`

Department of Mathematics, San Francisco State University, San Francisco, CA 94132, USA
E-mail address: `[mattbeck,nrobbins]@sfsu.edu`