The mth Largest and mth Smallest Parts of a Partition

George E. Andrews and Greg Simay

Abstract

The theory of overpartitions is applied to determine formulas for the number of partitions of n where (1) the mth largest part is k and (2) the mth smallest part is k.

1 Introduction

Corteel and Lovejoy [1] laid the foundations for the rich extensions of ordinary integer partitions to overpartitions. An overpartition of the integer n is a sum of positive integers adding to n in which the final occurrence of any given part may be overlined. For example, there are eight overpartitions of 3: 3, $\overline{3}$, $2+1$, $\overline{2}+1$, $2+\overline{1}$, $\overline{2}+\overline{1}$, $1+1+1$, $1+1+\overline{1}$.

We shall be interested in certain subclasses of overpartitions.

Namely, we define $\mathcal{G}_{j,k}(n)$ to be the number of overpartitions of n in which k is an overlined part and exactly j other parts (each larger than k) are overlined.

Similarly, we define $\mathcal{S}_{j,k}(n)$ to be the number of overpartitions of n in which k is an overlined part and exactly j other parts (each smaller than k) are overlined.

Now $\mathcal{G}_{j,k}(n)$ and $\mathcal{S}_{j,k}(n)$ play a central role in our main object. Namely, we wish to find efficient formulas for computing:

1. $g_m(n,k)$, the number of ordinary partitions of n in which k is the mth greatest summand (i.e. there are exactly $(m-1)$ *different* summands larger than k).

   For example, $g_2(6,1)=5$ with the partitions in question being $5+1$, $4+1+1$, $3+1+1+1$, $2+2+1+1$, $2+1+1+1+1$.

2. $s_m(n,k)$, the number of ordinary partitions of n in which k is the mth smallest summand (i.e. there are exactly $(m-1)$ *different* summands smaller than k).

   For example, $s_2(7,2)=5$ with the partitions in question being $4+2+1$, $3+2+1+1$, $2+2+2+1$, $2+2+1+1+1$, $2+1+1+1+1+1$.

Theorem 1.

$$g_m(n,k)=\sum_{j\geq 0}(-1)^{j+m-1}\binom{j}{m-1}\mathcal{G}_{j,k}(n). \tag{1.1}$$

Theorem 2.

$$s_m(n,k)=\sum_{j\geq 0}(-1)^{j+m-1}\binom{j}{m-1}\mathcal{S}_{j,k}(n). \tag{1.2}$$

As an example of (1.1), we have already seen that $g_2(6,1)=5$. We now note that $\mathcal{G}_{1,1}(6)=7$ because the overpartitions in question are $\overline{5}+\overline{1}$, $\overline{4}+1+\overline{1}$, $\overline{3}+2+\overline{1}$, $3+\overline{2}+\overline{1}$, $\overline{3}+1+1+\overline{1}$, $2+\overline{2}+1+\overline{1}$, and $\overline{2}+1+1+1+\overline{1}$. Also $\mathcal{G}_{2,1}(6)=1$ the overpartition counted being $\overline{3}+\overline{2}+\overline{1}$, and $\mathcal{G}_{j,1}(6)=0$ for $j>2$. Hence in the case $m=2$,$n=6$,$k=1$, (1.1) asserts

$$g_2(6,1)=\mathcal{G}_{1,1}(6)-\binom{2}{1}\mathcal{G}_{2,1}(6)+0$$

or

$$5=7-2\cdot 1.$$

As an example of (1.2), we see that $s_2(7,3)=3$ with the partitions in question being $3+3+1$, $3+1+1+1$, $3+2+2$. We see that $\mathcal{S}_{1,3}(7)=5$ because the overpartitions in question are $\overline{3}+3+\overline{1}$, $\overline{3}+2+\overline{2}$, $\overline{3}+\overline{2}+1+1$, $\overline{3}+2+1+\overline{1}$, $\overline{3}+1+1+1+\overline{1}$, and $\mathcal{S}_{2,3}(7)=1$ with the partition in question being $\overline{3}+\overline{2}+1+\overline{1}$. Hence in the case $m=2$, $n=7$, $k=3$, (1.2) asserts

$$s_2(7,3)=\mathcal{S}_{1,1}(7)-\binom{2}{1}\mathcal{S}_{2,1}(7)+0$$

or

$$3=5-2\cdot 1$$

The reason we call these efficient formulas lies in the simple recurrences for $\mathcal{G}_{j,k}(n)$ and $\mathcal{S}_{j,k}(n)$ given in the following two results.

Theorem 3.

$$\mathcal{G}_{0,k}(n) = p(n-k), \tag{1.3}$$

and for $j > 0$*,*

$$\mathcal{G}_{j,k}(n) = \mathcal{G}_{j,k}(n-j) + \mathcal{G}_{j-1,k}(n-j-k), \tag{1.4}$$

where $p(n)$ *is the number of ordinary partitions of* n*.*

Theorem 4.

$$\mathcal{S}_{0,k}(n) = p(n-k), \tag{1.5}$$

and for $j > 0$*,*

$$\mathcal{S}_{j,k}(n) = \mathcal{S}_{j,k}(n-j) + \mathcal{S}_{j-1,k-1}(n-j-1) - \mathcal{S}_{j-1,k-1}(n-j-k). \tag{1.6}$$

Section 2 will be devoted to a short discussion of overpartitions and their "shadows", together with some elementary combinatorial observations. Section 3 will be devoted to Theorems 1 and 2, and Section 4 will treat Theorems 3 and 4.

2 Overpartitions

The *shadow* of an overpartition is the ordinary partition with the overlines removed. Thus $2+1$ is the shadow of each of $\overline{2}+\overline{1}$, $2+\overline{1}$, $\overline{2}+1$, and $2+1$. Hence each ordinary partition π with D different parts is the shadow of $\binom{D}{r}$ overpartitions in which exactly r parts are overlined. Note that there is exactly one overpartition corresponding to π with all D parts overlined.

In addition to this observation, we need the well-known fact that

$$\sum_{j\geq 0}(-1)^{j+R}\binom{j}{R}\binom{D}{j} = \begin{cases} 1 & \text{if } D = R, \\ 0 & \text{if } D \neq R. \end{cases} \tag{2.1}$$

In the case $D = R$,

$$\sum_{j\geq 0}(-1)^{j+R}\binom{j}{R}\binom{R}{j} = (-1)^{2R}\binom{R}{R}\binom{R}{R} = 1.$$

Note if $D < R$ then each summand is 0, and finally if $D > R$,

$$\begin{aligned}\sum_{j\geq 0}(-1)^{j+R}\binom{j}{R}\binom{D}{j} &= \frac{D!(-1)^R}{R!(D-R)!}\sum_{j\geq 0}(-1)^j\binom{D-R}{j-R} \\ &= \binom{D}{R}(1-1)^{D-R} = 0.\end{aligned}$$

3 Proofs of Theorems 1 and 2

First we prove Theorem 1. We proceed by examining the right side of (1.1).

Let π be an ordinary partition that is the shadow of some of the overpartitions enumerated by $\mathcal{G}_{j,k}(n)$. Thus k must be a part of π and there must be $D \geq j$ different parts of π that are larger than k.

What is the contribution to the sum

$$\sum_{j\geq 0}(-1)^{j+m-1}\binom{j}{m-1}\mathcal{G}_{j,k}(n)$$

of the overpartitions whose shadow is π?

From the first paragraph in Section 2, we see that this contribution is

$$\sum_{j\geq 0}(-1)^{j+m-1}\binom{j}{m-1}\binom{D}{j},$$

and by (2.1) this contribution is 0 unless $D = m-1$ in which case it is 1. But in the case $D = m-1 = j$, we see that there is exactly *one* overpartition counted by $\mathcal{G}_{j,k}(n)$, and (dropping the $j = m-1$ overlines plus the overline on k) we see that π was a partition in which the mth largest part is k.

Hence in

$$\sum_{j\geq 0}(-1)^{j+m-1}\binom{j}{m-1}\mathcal{G}_{k,j}(n)$$

the only contributions (of exactly 1) come from overpartitions in one-to-one correspondence with the partitions in which k is the mth largest part.

Therefore

$$g_m(n,k) = \sum_{j\geq 0}(-1)^{j+m-1}\binom{j}{m-1}\mathcal{G}_{k,j}(n).$$

Now we prove Theorem 2. We proceed by examining the right side of (1.2). The reasoning here is exactly the same except that now the other overlined .parts apart from k are all smaller than k rather than larger than k. Hence

$$s_m(n,k) = \sum_{j\geq 0}(-1)^{j+m-1}\binom{j}{m-1}\mathcal{S}_{k,j}(n).$$

4 Proof of Theorem 3 and 4.

We define

$$\gamma_{k,j}(q) = \sum_{n\geq 0} \mathcal{G}_{k,j}(n)q^n, \tag{4.1}$$

and

$$\sigma_{k,j}(q) = \sum_{n\geq 0} \mathcal{S}_{k,j}(n)q^n. \tag{4.2}$$

In the following

$$(q)_j = (1-q)(1-q^2)\cdots(1-q^j).$$

Now the generating function of partitions into distinct parts with k as smallest part and j parts each $> k$, is clearly

$$q^k \frac{q^{(k+1)+(k+2)+\cdots+(k+j)}}{(q)_j} = \frac{q^{k+kj+\binom{j+1}{2}}}{(q)_j}.$$

Hence

$$\gamma_{k,j}(q) = \frac{q^{k(j+1)+\binom{j+1}{2}}}{(q)_j(q)_\infty} \tag{4.3}$$

Therefore

$$\gamma_{k,0}(q) = \frac{q^k}{(q)_\infty} = \sum_{n\geq 0} p(n-k)q^n, \tag{4.4}$$

and for $j > 0$

$$(1-q^j)\gamma_{k,j}(q) = q^{k+j}\gamma_{k-1,j-1}(q). \tag{4.5}$$

Now comparing coefficients of q^n on both sides of (4.4) and (4.5), we deduce (1.3) and (1.4).

Next we recall the generating function for partitions into distinct parts with k as largest part and j parts each $< k$, is clearly [2, p. 59]

$$q^k q^{\binom{j+1}{2}} \begin{bmatrix} k-1 \\ j \end{bmatrix},$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} 0, & \text{if } B > A \text{ or } B < 0, \\ \frac{(q)_A}{(q)_B(q)_{A-B}}, & 0 \leq B \leq A \end{cases}$$

Hence

$$\sigma_{k,j}(q) = q^{k+\binom{j+1}{2}} \begin{bmatrix} k-1 \\ j \end{bmatrix} \frac{1}{(q)_\infty}. \tag{4.6}$$

Therefore

$$\sigma_{k,0}(q) = \frac{q^k}{(q)_\infty} = \sum_{n\geq 0} p(n-k)q^n, \tag{4.7}$$

and for $j > 0$

$$(1-q^j)\sigma_{k,j}(q) = (1-q^{k-1})q^{j+1}\sigma_{k-1,j-1}(q) \tag{4.8}$$

We now compare coefficients of q^n on both sides of (4.7) and (4.8) to deduce (1.5) and (1.6).

References

[1] S. Corteel and J. Lovejoy, *Overpartitions*, Trans. Amer. Math. Soc., **356**(2004), 1623-1635.

[2] P.A. MacMahon, *Combinatory Analysis, Vol. II*, Cambridge University Press, Cambridge, 1918 (Reprinted: AMS Chelsea, 2001).

THE PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PA 16802
gea1@psu.edu