ARITHMETIC PROPERTIES OF m–ARY PARTITIONS WITHOUT GAPS

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Abstract. Motivated by recent work of Bessenrodt, Olsson, and Sellers on unique path partitions, we consider partitions of an integer n wherein the parts are all powers of a fixed integer $m \geq 2$ and there are no "gaps" in the parts; that is, if m^i is the largest part in a given partition, then m^j also appears as a part in the partition for each $0 \leq j < i$. Our ultimate goal is to prove an infinite family of congruences modulo powers of m which are satisfied by these functions.

1. Introduction

Motivated by the Murnaghan-Nakayama formula, Olsson [8] recently defined a special type of integer partition called a *unique path partition.* Subsequently, Bessenrodt, Olsson, and Sellers [3] characterized these unique path partitions and, in the process, considered integer partitions which they called *restricted binary partitions.* A binary partition is called *restricted* (for short, an *rb-partition*) if it satisfies the condition that, whenever 2^i is a part in the partition and $i \geq 1$ then 2^{i-1} is also a part. This means that there are no "missing" powers of 2 in the set of parts for each such partition; i.e., there are no "gaps" in the parts involved.

In this paper, we will denote the number of restricted binary partitions of the integer n by $c_2(n)$.

As an aside, we note that these restricted binary partitions are closely related to the binary partitions considered by Sloane and Sellers [12]. The binary partitions of Sloane and Sellers can be characterized by saying that if 2^i is the **largest** part of a given partition of n, then 2^{i-1} must also be a part. (This relationship is made even more clear when one considers the non–squashing partitions defined in [12], which can be naturally placed in bijection with the binary partitions defined by Sloane and Sellers, and the strictly decreasing partitions studied by Bessenrodt, Olsson, and Sellers which can be placed in bijection with the restricted binary partitions defined above.)

In [3], the authors prove numerous properties satisfied by $c_2(n)$. In this paper, we naturally generalize the restricted binary partitions to a family of restricted m–ary partitions for fixed

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$m \geq 2$. Namely, we will consider those m–ary partitions of n wherein if m^i is the largest part in a given partition, then m^j also appears as a part in the partition for each $0 \leq j < i$. We will denote the number of such partitions of n by $c_m(n)$.

In the next section of this paper, we prove properties of the functions $c_m(n)$ which generalize results related to $c_2(n)$ which appear in [3]. In the third section, we prove congruences satisfied by $c_m(n)$. This is an obvious consideration given the literature which already exists related to divisibility properties satisfied by m–ary partitions. Indeed, Churchhouse [4] initiated the study of congruence properties satisfied by the unrestricted binary partition function in the late 1960's. His work on binary partitions was substantially extended by Rødseth and Sellers [11] approximately 30 years later. Numerous authors have also considered such results for m–ary partitions; the interested reader is encouraged to see [1, 5, 6, 7, 9, 10] for more information.

2. Properties of $c_m(n)$

For a fixed $m \geq 2$, we see that the partitions enumerated by the function $c_m(n)$ have an ordinary generating function given by

$$C_m(q) = 1 + \frac{q^1}{1-q^1} + \frac{q^{1+m}}{(1-q^1)(1-q^m)} + \frac{q^{1+m+m^2}}{(1-q^1)(1-q^m)(1-q^{m^2})} + \dots$$

where the numerators of each fraction in this representation are present to guarantee that there are no gaps in the parts of the partitions. This means

$$C_m(q) := \sum_{n\geq 0} c_m(n)q^n = 1 + \sum_{n=0}^{\infty} \frac{q^{1+m+\cdots+m^n}}{(1-q)(1-q^m)\dots(1-q^{m^n})}. \tag{1}$$

Note that the $m = 2$ case of (1) is found in [3] where it is denoted $S(q)$.

The generating function result above is significant as it allows for rapid computation of numerous values of $c_m(n)$ via a computer algebra system. This is beneficial in the search for partition congruences which are the subject of the next section of the paper.

Using combinatorial arguments, we can also prove a set of recurrences satisfied by $c_m(n)$ which can be used to compute these values. We now state and prove this set of recurrences.

Theorem 2.1. *Let $m \geq 2$ be a fixed integer. Then $c_m(j) = 1$ for $1 \leq j \leq m$. Moreover, for $n \geq 1$, $c_m(mn+1) = c_m(mn) + c_m(n)$ and $c_m(mn+i) = c_m(mn+1)$ for $2 \leq i \leq m$.*

Proof. The initial conditions $c_m(j) = 1$ for $1 \leq j \leq m-1$ are clear as the only available part for such partitions is the number 1. Also, since no parts can be missing in each of these restricted m–ary partitions, it is the case that

$$\underbrace{1+1+\cdots+1}_{m \text{ times}}$$

is the only partition of m which is counted here. Also, the restricted m–ary partitions of $mn+i$ for $i \geq 2$ can clearly be put in one–to–one correspondence with the partitions of $mn+1$ by simply removing $i-1$ 1's from each of the partitions of $mn+i$. (Note that there will still be at least one part of size 1 in each of these partitions even after these removals take place.) So we now simply need to think about the partitions which are counted by $c_m(mn+1)$. Note that these come in two categories: either they contain exactly one 1 or they contain more than one 1. Those which contain more than one 1 can be put in one–to–one correspondence with the partitions counted by $c_m(mn)$ (by simply removing one of these 1's). Such partitions will still contain at least one copy of each power of m as a part, so this is fine. However, what do we do with the partitions counted by $c_m(mn+1)$ which only contain one 1? Well, we remove this single 1 and then divide all the remaining parts by m. Then these partitions are exactly those counted by $c_m(n)$ since $((mn+1)-1) \div m = n$ and this process does not remove all the copies of any particular part. Therefore, $c_m(mn+1) = c_m(mn) + c_m(n)$ and the theorem is proved. □

Clearly, Theorem 2.1 can also be used to compute numerous values of $c_m(n)$ with relatively ease.

3. An Infinite Family of Congruences

As has been noted in previous works on m–ary partition function congruences, it is often the case that very few (if any) divisibility properties exist for the $m = 2$ case of a family of functions. However, for $m \geq 3$, infinite families of congruences often exist for various m–ary partition functions. This is indeed the case with the family of functions $c_m(n)$.

With this in mind, we prove the following infinite family of congruences:

Theorem 3.1. *Let $m \geq 3$ be an odd prime and j any integer with $0 \leq j < m$. For all $n \geq 0$,*

$$c_m(m^{j+2}n + (m^{j+1} + m^j + \cdots + m^2)) \equiv 0 \pmod{m^j}.$$

In order to prove Theorem 3.1 we require a number of smaller results.

Lemma 3.2. *Let $m > j \geq 2$, and define $s_{j,i} = s_{j,i}(m)$ by*

$$\binom{km+1}{j} = \sum_{i=1}^{j} s_{j,i}\binom{k}{i}. \tag{2}$$

Then the $s_{j,i}$ are unique integers and $m^i \mid s_{j,i}$.

Remark 3.3. *The two omitted cases are*

$$\binom{km+1}{1} = s_{1,1}k + s_{1,0} = mk + 1 \tag{3}$$

and

$$\binom{km+1}{0} = s_{0,0} = 1. \tag{4}$$

When $j > 1$ *we see that* $s_{j,0} = 0$ *because* $\binom{0\cdot m+1}{j} = 0$.

Proof. (of Lemma 3.2) First the existence and uniqueness of the $s_{j,i}$ follows from the fact that the $\binom{k}{i}$ form a basis for the vector space of polynomials in k and $\binom{km+1}{j}$ is a polynomial in k of degree j. Furthermore, setting $k = 1$ yields

$$s_{j,1} = \binom{m+1}{j}, \tag{5}$$

and so $m \mid s_{j,1}$.

In addition, $s_{j,j} = m^j$ follows by comparing the coefficients of k^j on both sides of (2). Therefore,

$$\binom{km+1}{2} = m^2\binom{k}{2} + \binom{m+1}{2}\binom{k}{1}, \tag{6}$$

and we see that the assertions of the lemma are valid when $j = 2$.

Next, we prove by mathematical induction on i that each $s_{j,i}$ is integral. We already know this when $i = 1$ by (5). Now with $k = i_0$ in (2), we see that

$$s_{j,i_0} = \binom{i_0 m+1}{j} - \sum_{i=1}^{i_0-1} s_{j,i}\binom{i_0}{i}. \tag{7}$$

Hence, integrality follows by induction.

Finally, for $j \geq 3$,

$$j\binom{km+1}{j} = (km-j+2)\binom{km+1}{j-1}. \tag{8}$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{j} s_{j,i}\binom{k}{i} &= \frac{km-j+2}{j}\sum_{i=1}^{j-1} s_{j-1,i}\binom{k}{i} \\ &= \sum_{i=1}^{j-1} s_{j-1,i}\left(\frac{m(i+1)}{j}\cdot\frac{k-i}{i+1} + \frac{mi}{j} - \frac{j-2}{j}\right)\binom{k}{i} \\ &= \sum_{i=1}^{j-1} s_{j-1,i}\left(\frac{m(i+1)}{j}\binom{k}{i+1} + \frac{mi-j+2}{j}\binom{k}{i}\right), \end{aligned} \tag{9}$$

and so comparing coefficients of $\binom{k}{i}$ on both sides we find

$$s_{j,i} = \frac{mi}{j} s_{j-1,i-1} + \frac{mi-j+2}{j} s_{j-1,i}. \tag{10}$$

We have already shown that $m \mid s_{j,1}$ and that $m^i \mid s_{2,i}$ for each i. We now proceed by a double induction on j and i. The case $j = 2$ is proved for all i. For a given j, the case $i = 1$ is proved. The recursion (10) now concludes the induction. The two terms on the right of (10) might have denominators because of the j factor. However, m is relatively prime to j, and $m \cdot m^{i-1} = m^i$ divides the first term's numerator and m^i divides the second term's numerator. Hence, $m^i \mid s_{j,i}$. □

We now introduce further notation. We first highlight some important generating function information. As noted above,

$$C_m(q) := \sum_{n\geq 0} c_m(n)q^n = 1 + \sum_{n=0}^{\infty} \frac{q^{1+m+\cdots+m^n}}{(1-q)(1-q^m)\ldots(1-q^{m^n})}, \tag{11}$$

$$\begin{aligned} \overline{C_m}(q) \;&:=\; C_m(q) - 1 \\ &=\; \frac{q}{1-q} C_m(q^m) \\ &=\; \frac{q}{1-q} + \frac{q}{1-q}\overline{C_m}(q^m), \end{aligned} \tag{12}$$

(13)

and

$$h_j := \frac{q^j}{(1-q)^{j+1}} = \sum_{i=0}^{\infty} \binom{i}{j} q^i. \tag{14}$$

We also define the following two operators which allow us to complete generating function dissections in a straightforward way:

$$\eta_m^0 \sum_{n=0}^{\infty} A_n q^n = \sum_{n=0} A_{mn} q^n \tag{15}$$

and

$$\eta_m \sum_{n=0}^{\infty} A_n q^n = \sum_{n=0} A_{mn+1} q^n \tag{16}$$

With these in hand, we note a number of dissection results.

Lemma 3.4.

$$\eta_m h_0 = h_0, \tag{17}$$

$$\eta_m h_1 = s_{1,1} h_1 + h_0, \tag{18}$$

$$\eta_m q h_1 = s_{1,1} h_1, \tag{19}$$

and for $j > 1$,

$$\eta_m h_j = \sum_{i=1}^{j} s_{j,i} h_i. \tag{20}$$

Proof. Equation (17) is immediate, and noting $s_{1,1} = m$, we see that

$$\eta_m h_1 = \sum_{n=0}^{\infty} (mn+1) q^n = m h_1 + h_0 = s_{1,1} h_1 + h_0,$$

and

$$\eta_m q h_1 = \sum_{n=0}^{\infty} mnq^n = m h_1 = s_{1,1} h_1.$$

Finally, for $j > 1$,

$$\begin{aligned} \eta_m h_j &= \sum_{k=0}^{\infty} \binom{km+1}{j} q^k \\ &= \sum_{k=0}^{\infty} \sum_{i=1}^{j} s_{j,i} \binom{k}{i} q^k \\ &= \sum_{i=1}^{j} s_{j,i} h_i. \end{aligned}$$

□

Lemma 3.5.

$$\eta_m^0 q h_1 = s_{1,1} h_1 - q h_0 \tag{21}$$

and

$$\eta_m^0 q h_0 = q h_0. \tag{22}$$

Proof. Note that

$$\begin{aligned}\eta_m^0 q h_1 &= \eta_m^0 \sum_{k=0}^{\infty} k q^{k+1} \\ &= \sum_{k=1}^{\infty} (km-1) q^k \\ &= m h_1 - q h_0 \\ &= s_{1,1} h_1 - q h_0\end{aligned}$$

and

$$\eta_m^0 h_0 = \eta_m^0 \sum_{k=0}^{\infty} q^k = \sum_{k=0}^{\infty} q^k = \frac{1}{1-q} = h_0.$$

□

Lemma 3.6.

$$\sum_{n=1}^{\infty} c_m(m^2 n) q^n = s_{1,1} h_1 + (s_{1,1} h_1 - q h_0) \overline{C_m}(q)$$

Proof. We know from [2] that

$$\sum_{n=0}^{\infty} c_m(mn) q^n = 1 + \frac{q}{1-q} C_m(q). \tag{23}$$

Hence by (23) and (12),

$$\begin{aligned}\sum_{n=1}^{\infty} c_m(mn) q^n &= \frac{q}{1-q} C_m(q) \\ &= \frac{q}{1-q} + \frac{q}{1-q} \overline{C_m}(q) \\ &= \frac{q}{1-q} + \frac{q^2}{(1-q)^2} + \frac{q^2}{(1-q)^2} \overline{C_m}(q^m).\end{aligned}$$

Therefore, by Lemma 3.5,

$$\begin{aligned}\sum_{n=1}^{\infty} c_m(m^2 n) q^n &= \eta_m^0 \sum_{n=1}^{\infty} c_m(mn) q^n \\ &= \eta_m^0 (q h_0 + q h_1 + q h_1 \overline{C_m}(q^m)) \\ &= q h_0 + s_{1,1} h_1 - q h_0 + (s_{1,1} h_1 - q h_0) \overline{C_m}(q) \\ &= s_{1,1} h_1 + (s_{1,1} h_1 - q h_0) \overline{C_m}(q).\end{aligned}$$

□

Lemma 3.7.

$$\begin{aligned}\sum_{n=0}^{\infty} c_m(m^3n+m^2)q^n &= s_{1,1}s_{2,2}h_2 + (s_{2,1}s_{1,1} + s_{1,1}s_{1,1} - s_{1,1})h_1 \\ &\quad + s_{1,1}h_0 + (s_{1,1}s_{2,2}h_2 + (s_{1,1}s_{2,1} - s_{1,1})h_1)\overline{C_m}(q)\end{aligned}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} c_m(m^3n+m^2)q^n &= \eta_m \sum_{n=1}^{\infty} c_m(m^2n)q^n \\ &= \eta_m(s_{1,1}h_1 + (s_{1,1}h_1 - qh_0)\overline{C_m}(q)) \\ &\qquad \text{by Lemma 3.6} \\ &= \eta_m\left(s_{1,1}h_1 + (s_{1,1}h_1 - qh_0)\left(\frac{q}{1-q} + \frac{q}{1-q}\overline{C_m}(q^m)\right)\right) \\ &\qquad \text{by (12)} \\ &= \eta_m(s_{1,1}h_1 + s_{1,1}h_2 - qh_1 + (s_{1,1}h_2 - qh_1)\overline{C_m}(q^m)) \\ &= s_{1,1}(s_{1,1}h_1 + h_0) + s_{1,1}(s_{2,2}h_2 + s_{2,1}h_1) \\ &\quad - s_{1,1}h_1 + (s_{1,1}(s_{2,2}h_2 + s_{2,1}h_1) - s_{1,1}h_1)\overline{C_m}(q) \\ &= s_{1,1}s_{2,2}h_2 + (s_{1,1}s_{2,1} + s_{1,1}s_{1,1} - s_{1,1})h_1 \\ &\quad + s_{1,1}h_0 + (s_{1,1}s_{2,2}h_2 + (s_{1,1}s_{2,1} - s_{1,1})h_1)\overline{C_m}(q)\end{aligned}$$

□

Lemma 3.8.

$$\begin{aligned}&\sum_{n=0}^{\infty} c_m(m^4n+m^3+m^2)q^n \\ =\ & s_{1,1}s_{2,2}s_{3,3}h_3 + (s_{1,1}s_{2,2}s_{3,2} + s_{1,1}s_{2,1}s_{2,2} + s_{1,1}s_{2,2}s_{2,2} - s_{1,1}s_{2,2})h_2 \\ & + (s_{1,1}s_{2,2}s_{3,1} + s_{1,1}s_{2,1}s_{2,1} + s_{1,1}s_{2,2}s_{2,1} + s_{1,1}s_{2,1}s_{1,1} + s_{1,1}s_{1,1}s_{1,1} - s_{1,1}s_{2,1} - s_{1,1}s_{1,1})h_1 \\ & + (s_{1,1}s_{2,1} + s_{1,1}s_{1,1})h_0 \\ & + \{s_{1,1}s_{2,2}s_{3,3}h_3 + (s_{1,1}s_{2,2}s_{3,2} + s_{1,1}s_{2,1}s_{2,2} - s_{1,1}s_{22})h_2 \\ & + (s_{1,1}s_{2,2}s_{3,1} + s_{1,1}s_{2,1}s_{2,1} - s_{1,1}s_{2,1})h_1)\}\overline{C_m}(q).\end{aligned}$$

Proof.

$$\begin{aligned}&\sum_{n=0}^{\infty} c_m(m^4n+m^3+m^2)q^n \\ =\ & \eta_m \sum_{n=0}^{\infty} c_m(m^3n+m^2)q^n\end{aligned}$$

$$\begin{aligned}
&= \eta_m \Bigg(s_{1,1}s_{2,2}h_2 + (s_{1,1}s_{2,1} + s_{1,1}s_{1,1} - s_{1,1})h_1 + s_{1,1}h_0 + \\
&\quad +(s_{1,1}s_{2,2}h_2 + (s_{1,1}s_{2,1} - s_{1,1})h_1) \left(\frac{q}{1-q} + \frac{q}{1-q}\overline{C_m}(q^m) \right) \Bigg) \\
&= \eta_m \, (s_{1,1}s_{2,2}h_3 + (s_{1,1}s_{2,2} + s_{1,1}s_{2,1} - s_{1,1})h_2 \\
&\quad +(s_{1,1}s_{2,1} + s_{1,1}s_{1,1} - s_{1,1})h_1 + s_{1,1}h_0 \\
&\quad +(s_{1,1}s_{2,2}h_3 + (s_{1,1}s_{2,1} - s_{1,1})h_2)\overline{C_m}(q^m)) \\
&= s_{1,1}s_{2,2}(s_{3,3}h_3 + s_{3,2}h_2 + s_{3,1}h_1) \\
&\quad +(s_{1,1}s_{2,2} + s_{1,1}s_{2,1} - s_{1,1})(s_{2,2}h_2 + s_{2,1}h_1) \\
&\quad +(s_{1,1}s_{2,1} + s_{1,1}s_{1,1} - s_{1,1})(s_{1,1}h_1 + h_0) + s_{1,1}h_0 \\
&\quad + \{s_{1,1}s_{2,2}(s_{3,3}h_3 + s_{3,2}h_2 + s_{3,1}h_1) \\
&\quad +(s_{1,1}s_{2,1} - s_{1,1})(s_{2,2}h_2 + s_{2,1}h_1)\} \, \overline{C_m}(q) \\
&= s_{1,1}s_{2,2}s_{3,3}h_3 \\
&\quad +(s_{1,1}s_{2,2}s_{3,2} + s_{1,1}s_{2,2}s_{2,2} + s_{1,1}s_{2,1}s_{2,2} - s_{1,1}s_{2,2})h_2 \\
&\quad +(s_{1,1}s_{2,2}s_{3,1} + s_{1,1}s_{2,2}s_{2,1} + s_{1,1}s_{2,1}s_{2,1} + s_{1,1}s_{2,1}s_{1,1} \\
&\quad +s_{1,1}s_{1,1}s_{1,1} - s_{1,1}s_{2,1} - s_{1,1}s_{1,1})h_1 + (s_{1,1}s_{2,1} + s_{1,1}s_{1,1})h_0 \\
&\quad +\{s_{1,1}s_{2,2}s_{3,3}h_3 + (s_{1,1}s_{2,2}s_{3,2} + s_{1,1}s_{2,1}s_{2,2} - s_{1,1}s_{2,2})h_2 \\
&\quad +(s_{1,1}s_{2,2}s_{3,1} + s_{1,1}s_{2,1}s_{2,1} - s_{1,1}s_{2,1})h_1\}\overline{C_m}(q)
\end{aligned}$$

□

With these initial cases complete, we now develop tools necessary to prove the more general result stated in Theorem 3.1.

Lemma 3.9.

$$\sum_{n=0}^{\infty} c_m(m^{j+2}n + m^{j+1} + \cdots + m^2)q^n \;=\; \sum_{i=0}^{j+1}(P_{j,i} - Q_{j,i})h_i + \overline{C_m}(q)\sum_{i=1}^{j+1}(R_{j,i} - T_{j,i})h_i$$

where, for $i > 0$, $P_{j,i}$ and $R_{j,i}$ are sums of monomials in the $s_{u,v}$ of degree $j+1$ while $Q_{j,i}$ and $T_{j,i}$ are of degree j. Moreover, $Q_{j,j+1} = Q_{j,0} = 0$ and $P_{j,0}$ is of degree j in the $s_{u,v}$. Finally, for $t > 0$, we have the following:

$$P_{j+1,t} = \sum_{i=t}^{j+2}(P_{j,i} + R_{j,i-1})s_{i,t} \tag{24}$$

$$Q_{j+1,t} = \sum_{i=t}^{j+2}(Q_{j,i} + T_{j,i-1})s_{i,t} \tag{25}$$

$$R_{j+1,t} = \sum_{i=t}^{j+2} R_{j,i-1} s_{i,t} \tag{26}$$

$$T_{j+1,t} = \sum_{i=t}^{j+2} T_{j,i-1} s_{i,t} \tag{27}$$

$$P_{j+1,0} = P_{j,1} - Q_{j,1} + P_{j,0} - Q_{j,0} \tag{28}$$

Proof. We begin with the recurrences.

$$\begin{aligned}
& \sum_{i=0}^{j+2} (P_{j+1,i} - Q_{j+1,i}) h_i + \overline{C_m}(q) \sum_{i=1}^{j+2} (R_{j+1,i} - T_{j+1,i}) h_i \\
= \; & \eta_m \sum_{n=0}^{\infty} c_m (m^{j+2} n + m^{j+1} + \cdots + m^2) q^n \\
= \; & \eta_m \left(\sum_{i=0}^{j+1} (P_{j,i} - Q_{j,i}) h_i + \left(\frac{q}{1-q} + \frac{q}{1-q} \overline{C_m}(q^m) \right) \sum_{i=1}^{j+1} (R_{j,i} - T_{j,i}) h_i \right) \\
= \; & \eta_m \left(\sum_{i=0}^{j+2} ((P_{j,i} + R_{j,i-1}) - (Q_{j,i} + T_{j,i-1})) h_i + (P_{j,0} - Q_{j,0}) h_0 \right. \\
& \left. + \overline{C_m}(q^m) \sum_{i=0}^{j+2} (R_{j,i-1} - T_{j,i-1}) h_i \right) \\
= \; & \sum_{i=0}^{j+2} ((P_{j,i} + R_{j,i-1}) - (Q_{j,i} + T_{j,i-1})) \sum_{t=1}^{i} s_{i,t} h_t \\
& + (P_{j,1} - Q_{j,1}) h_0 + (P_{j,0} - Q_{j,0}) h_0 \\
& + \overline{C_m}(q) \sum_{i=0}^{j+2} (R_{j,i-1} - T_{j,i-1}) \sum_{t=1}^{i} s_{i,t} h_t
\end{aligned}$$

Comparing coefficients of h_t in the above string of equations confirms (24) – (28). □

We note that these recurrences directly establish by mathematical induction (with Lemma 3.7 as the base case) that

$$P_{j,j+1} = s_{1,1} s_{2,2} \ldots s_{j,j} s_{j+1,j+1} = R_{j,j+1},$$

$$R_{j,0} = T_{j,0} = 0;$$

the latter fact is used tacitly in the above.

Lemma 3.10. *For $i \geq 1$, $P_{j,i}$ and $R_{j,i}$ are sums of monomials (all with coefficients equal to 1) of degree $j+1$ in the $s_{u,v}$ while $Q_{j,i}$ and $T_{j,i}$ are sums of monomials (all with coefficients equal to 1) of degree j in the $s_{u,v}$. Furthermore, the terms in $P_{j,i}$, $R_{j,i}$, $Q_{j,i}$, $T_{j,i}$ are constructed according to the following:*

First we note that in every monomial we have

$$s_{1,1}s_{i,h}s_{k,l}\dots s_{m,i},$$

namely the final subscript is i and the first two subscripts are $1,1$. This follows from the recurrences and the base case of Lemma 3.7.

- *Rule for $R_{j,i}$ terms:*
$$s_{1,1}s_{i_2,k_2}\dots s_{i_j,k_j}s_{i_{j+1},i}$$
each $i_m = k_{m-1}+1$, $j+1 \geq i_m \geq k_m$, $1 \leq m \leq j+1$.
- *Rule for $T_{j,i}$ terms:*
$$s_{1,1}s_{i_2,k_2}\dots s_{i_{j-1},k_{j-1}}s_{i_j,i}$$
each $i_m = k_{m-1}+1$, $j \geq i_m \geq k_m$, $1 \leq m \leq j$.
- *Rule for $P_{j,i}$ terms:*
$$s_{1,1}s_{i_2,k_2}\dots s_{i_j,k_j}s_{i_{j+1},i}$$
each $i_m = k_{m-1}$ or $i_m = k_{m-1}+1$, $j+1 \geq i_m \geq k_m$, $1 \leq m \leq j+1$, with the added condition that if $i_m = k_{m-1}+1$, then $i_r = k_{r-1}+1$ for each $r < m$.
- *Rule for $Q_{j,i}$ terms:*
$$s_{1,1}s_{i_2,k_2}\dots s_{i_{j-1},k_{j-1}}s_{i_j,i}$$
each $i_m = k_{m-1}$ or $i_m = k_{m-1}+1$, $j \geq i_m \geq k_m$, $1 \leq m \leq j$, with the added condition that if $i_m = k_{m-1}+1$, then $i_r = k_{r-1}+1$ for each $r < m$.

Proof. These rules for construction of the $P_{j,i}$, $R_{j,i}$, $Q_{j,i}$, $T_{j,i}$ follow by induction on j. This is most easily seen by noting that Lemma 3.7 plus the recurrences (26) and (27) establish the rules for $R_{j,i}$ and $T_{j,i}$ immediately. One then uses these facts about $R_{j,i}$ and $T_{j,i}$ to see that the rules for $P_{j,i}$ and $Q_{j,i}$ now follow directly from (24) and (25) with Lemma 3.7 as the base case. □

Lemma 3.11. *For $j \geq 1$, $P_{j+1,0} = P_{j,1}$ and $Q_{j,0} = 0$.*

Proof. When $j = 1$ this follows from Lemmas 3.7 and 3.8. Now inspecting the proof of Lemma 3.9, we see that

$$P_{j+1,0} - Q_{j+1,0} = P_{j,1} - Q_{j,1} + P_{j,0} - Q_{j,0}.$$

We proceed by induction on j.

By the induction hypothesis, $Q_{j,0} = 0$. So to prove the $j+1$ case we need only prove that

$$Q_{j,1} = P_{j,0},$$

or by the induction hypothesis,

$$Q_{j,1} = P_{j-1,1},$$

but this is immediate from the rules for $Q_{j,i}$ and $P_{j,i}$. Hence the lemma is true. □

We are now in a position to prove Theorem 3.1.

Proof. By Lemma 3.9,

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_m(m^{j+2}n + m^{j+1} + \cdots + m^2)q^n \\
= &\sum_{i=1}^{j+1}(P_{j,i} - Q_{j,i})h_i + P_{j,0}h_0 + \overline{C_m}(q)\sum_{i=1}^{j+1}(R_{j,i} - T_{j,i})h_i \\
= &\sum_{i=1}^{j+1}(P_{j,i} - Q_{j,i})h_i + P_{j-1,1}h_0 + \overline{C_m}(q)\sum_{i=1}^{j+1}(R_{j,i} - T_{j,i})h_i.
\end{aligned}$$

Now each $P_{j,i}$ and $R_{j,i}$ is made up of monomials in $s_{u,v}$ of degree $j+1$. So by Lemma 3.2, $m^{j+1} \mid P_{j,i}$ and $m^{j+1} \mid R_{j,i}$. Similarly, each $Q_{j,i}$ and $T_{j,i}$ is made up of monomials in $s_{u,v}$ of degree j. So by Lemma 3.2, $m^j \mid Q_{j,i}$ and $m^j \mid T_{j,i}$. Finally, $m^j \mid P_{j-1,1}$. So all the coefficients above have m^j as a factor. Therefore,

$$m^j \mid c_m(m^{j+2}n + m^{j+1} + \cdots + m^2)$$

for all $j \geq 1$ and all $n \geq 0$. □

4. Closing Thoughts

We close by noting that Theorem 3.1, while extremely satisfying, does not appear to be best possible (based on numerical evidence). That is to say, it appears that a more general result is true although the authors have yet to prove this. So we close with the following conjecture which we leave as future work.

Conjecture 4.1. *For a fixed $m \geq 3$ and for all $n \geq 0$,*

$$c_m(m^{j+2}n + (m^{j+1} + m^j + \cdots + m^2)) \equiv 0 \pmod{\frac{m^j}{c_j}}$$

where $c_j = 1$ if m is odd and $c_j = 2^{j-1}$ if m is even.

It is worth noting that such a result is reminiscent of the results obtained in [10] as well as those found by Rødseth [9], Andrews [1], and Gupta [7] in their work on unrestricted m–ary partition function congruences.

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