CONGRUENCES RELATED TO THE RAMANUJAN/WATSON MOCK THETA FUNCTIONS $\omega(q)$ AND $\nu(q)$

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ABSTRACT. Recently, Andrews, Dixit, and Yee introduced partition functions associated with the Ramanujan/Watson mock theta functions $\omega(q)$ and $\nu(q)$. Based on one of their results, mod 2 congruences for these mock theta functions are obtained. In addition, infinite families of mod 4 and mod 8 congruences are presented. Lastly, an elementary proof of the first explicit examples of congruences for $\omega(q)$ given by Waldherr is presented.

1. INTRODUCTION

In his last letter to Hardy in 1920, Ramanujan introduced the notion of a mock theta function along with a number of examples of order 3, 5, and 7. Since then, mock theta functions have been the subject of intense study.

Recently, the first and fourth authors with A. Dixit found a new partition function $p_\omega(n)$ that is associated with the third order mock theta function $\omega(q)$ [1]:

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2(n^2+n)}}{(q;q^2)_{n+1}^2},$$

where

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

One of the results in [1] yields a mod 2 congruence of the coefficients of $\omega(q)$, which led us to a further search for congruences. S. Garthwaite and D. Penniston [5] showed that the coefficients of $\omega(q)$ satisfy infinitely many congruences of the similar type as Ramanujan's partition congruences, and M. Waldherr [6] found the first explicit examples of congruences, suggested by some computations done by J. Lovejoy.

We define $p_\omega(n)$ by

$$\sum_{n=1}^{\infty} p_\omega(n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}, \tag{1}$$

where $(a;q)_\infty := \lim_{n\to\infty}(a;q)_n$. From its generating function definition, we see that $p_\omega(n)$ counts the number of partitions of n in which each odd part is less that twice the smallest part. In [1], it is shown that

$$\sum_{n=1}^{\infty} p_\omega(n)q^n = q\omega(q).$$

The main result of this paper is:

Theorem 1.1. *For nonnegative integers n and k,*

$$p_w\left(2^{2k+3}n + \frac{11 \cdot 2^{2k} + 1}{3}\right) \equiv 0 \pmod{4},$$

$$p_w\left(2^{2k+3}n + \frac{17 \cdot 2^{2k} + 1}{3}\right) \equiv 0 \pmod{8},$$

$$p_w\left(2^{2k+4}n + \frac{38 \cdot 2^{2k} + 1}{3}\right) \equiv 0 \pmod{4}.$$

In [1], another partition function $p_\nu(n)$ is defined. Namely,

$$\sum_{n=0}^{\infty} p_\nu(n)q^n = \sum_{n=0}^{\infty} q^n(-q^{n+1};q)_n(q^{2n+2};q)_\infty.$$

It is shown that

$$\sum_{n=0}^{\infty} p_\nu(n)q^n = \nu(-q),$$

where $\nu(q)$ is a third order mock theta function,

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}.$$

This mock theta function $\nu(q)$ is related to $\omega(q)$ as follows [4, p. 62, Equation (26.88)]:

$$\nu(-q) = q\omega(q^2) + (-q^2;q^2)_\infty\psi(q^2), \tag{2}$$

where

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

By (2), we can derive congruences of $p_\nu(n)$ from $p_\omega(n)$, which will be given in Section 4.

This paper is organized as follows. In Section 2, we present mod 2 congruences of $p_\omega(n)$ and $p_\nu(n)$. In Section 3, we prove Theorems 1.1. In Section 4, we prove congruences of $p_\nu(n)$. As noted earlier, Waldherr [6] provided the first explicit congruences for $\omega(q)$:

$$p_\omega(40n + 28) \equiv 0 \pmod{5},$$
$$p_\omega(40n + 36) \equiv 0 \pmod{5}.$$

In Section 5, we provide an elementary proof of the above congruences.

2. Mod 2 congruences

We recall the following results from [1]:

$$\sum_{n=1}^{\infty} \frac{q^n}{(-q^n;q)_{n+1}(-q^{2n+2};q^2)_\infty} = \sum_{j=0}^{\infty}(-1)^j q^{6j^2+4j+1}(1 + q^{4j+2}), \tag{3}$$

and

$$\sum_{n=0}^{\infty} q^n(q^{n+1};q)_n(q^{2n+2};q^2)_\infty = \sum_{j=0}^{\infty}(-1)^j q^{j(3j+2)}(1 + q^{2j+1}). \tag{4}$$

Let $p_{\omega,o}(n)$ and $p_{\omega,e}(n)$ be the number of partitions of n counted by $p_\omega(n)$ into an odd number of parts and an even number of parts, respectively. Then it follows from (3) that

$$p_{\omega,o}(n) - p_{\omega,e}(n) = \begin{cases} (-1)^j, & \text{if } n = 6j^2 + 4j + 1 \text{ for some } j \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

This yields the following obvious result.

Theorem 2.1. *We have*

$$p_\omega(n) \equiv \begin{cases} 1 \pmod{2}, & \text{if } n = 6j^2 + 4j + 1 \text{ for some } j \in \mathbb{Z}, \\ 0 \pmod{2}, & \text{otherwise.} \end{cases}$$

Corollary 2.2. *Let $p \geq 5$ be prime, and let r be such that $0 \leq r \leq p-1$, and $(3r-1)(p+1)/2$ is a quadratic nonresidue mod p. Then, for all $n \geq 0$,*

$$p_\omega(pn + r) \equiv 0 \pmod{2}.$$

Proof. Assume that there exists an integer j such that

$$pn + r = 6j^2 + 4j + 1.$$

Then

$$\begin{aligned} 3(pn + r) - 1 &= 3(6j^2 + 4j + 1) - 1 \\ &= 18j^2 + 12j + 2 \\ &= 2(9j^2 + 6j + 1) \\ &= 2(3j + 1)^2. \end{aligned}$$

This yields

$$3r - 1 \equiv 2(3j + 1)^2 \pmod{p}$$

or

$$(3r - 1)\left(\frac{p+1}{2}\right) \equiv (3j + 1)^2 \pmod{p}.$$

But, we have chosen r such that $(3r - 1)\left(\frac{p+1}{2}\right)$ is a quadratic nonresidue mod p, so it cannot be congruent to a square. $\square$

Similarly, (4) yields the following mod 2 result.

Theorem 2.3. *We have*

$$p_\nu(n) \equiv \begin{cases} 1 \pmod{2}, & \text{if } n = 3j^2 + 2j \text{ for some } j \in \mathbb{Z}, \\ 0 \pmod{2}, & \text{otherwise.} \end{cases}$$

Corollary 2.4. *For all $n \geq 0$ and $r = 2, 3$,*

$$p_\nu(4n + r) \equiv 0 \pmod{2}.$$

Proof. If j is even, then $3j^2 + 2j \equiv 0 \pmod{4}$. If j is odd, then $3j^2 + 2j \equiv 1 \pmod{4}$. So $4n+2$ and $4n + 3$ can never be represented as $3j^2 + 2j$. $\square$

Corollary 2.5. *Let $p \geq 5$ be prime, and let r be such that $0 \leq r \leq p - 1$, and $(3r + 1)$ is a quadratic nonresidue mod p. Then, for all $n \geq 0$,*

$$p_\nu(pn + r) \equiv 0 \pmod{2}.$$

Proof. For some j, let

$$n = 3j^2 + 2j,$$

which is equivalent to

$$3n + 1 = (3j + 1)^2$$

Replacing n by $pn + r$ yields

$$3(pn + r) + 1 = (3j + 1)^2,$$

and this gives

$$3r + 1 \equiv (3j + 1)^2 \pmod{p}.$$

But, we have chosen r such that $3r+1$ is a quadratic nonresidue mod p, so it cannot be congruent to a square. $\square$

3. PROOF OF THEOREM 1.1

We start with a formula from page 63 of [7]:

$$f(q^8) + 2q\omega(q) + 2q^3\omega(-q^4) = \frac{\phi(q)\phi(q^2)^2}{(q^4; q^4)_\infty^2} =: F(q), \tag{5}$$

where

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty, \tag{6}$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}. \tag{7}$$

By [3, p.40, Entry 25 (v), (vi)], we have

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2. \tag{8}$$

Lemma 3.1. *Let*

$$F(q) = F_0(q^4) + qF_1(q^4) + q^2F_2(q^4) + q^3F_3(q^4).$$

Then

$$F_0(q) = \frac{\phi(q)^3}{(q; q)_\infty^2}, \tag{9}$$

$$F_1(q) = \frac{2\psi(q^2)\phi(q)^2}{(q; q)_\infty^2}, \tag{10}$$

$$F_2(q) = \frac{4\phi(q)\phi(q^2)\psi(q^4)}{(q; q)_\infty^2}, \tag{11}$$

$$F_3(q) = \frac{8\psi(q^2)\phi(q^2)\psi(q^4)}{(q; q)_\infty^2}. \tag{12}$$

Proof. Separating even n's and odd n's, we obtain

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{4n^2} + \sum_{n=-\infty}^{\infty} q^{4n^2+4n+1}.$$

Thus

$$F(q) = \frac{1}{(q^4;q^4)_\infty^2} \left(\sum_{n=-\infty}^\infty q^{4n^2} + q \sum_{n=-\infty}^\infty q^{4n^2+4n} \right) \left(\sum_{n=-\infty}^\infty q^{8n^2} + q^2 \sum_{n=-\infty}^\infty q^{8n^2+8n} \right)^2$$

$$= \frac{1}{(q^4;q^4)_\infty^2} \left(\sum_{n=-\infty}^\infty q^{4n^2} + q \sum_{n=-\infty}^\infty q^{4n^2+4n} \right)$$

$$\times \left(\left(\sum_{n=-\infty}^\infty q^{8n^2} \right)^2 + 2q^2 \sum_{n=-\infty}^\infty q^{8n^2} \sum_{n=-\infty}^\infty q^{8n^2+8n} + q^4 \left(\sum_{n=-\infty}^\infty q^{8n^2+8n} \right)^2 \right)$$

Consequently extracting the portion for $F_0(q^4)$, we have

$$F_0(q^4) = \frac{1}{(q^4;q^4)_\infty^2} \sum_{n=-\infty}^\infty q^{4n^2} \left(\left(\sum_{n=-\infty}^\infty q^{8n^2} \right)^2 + q^4 \left(\sum_{n=-\infty}^\infty q^{8n^2+8n} \right)^2 \right)$$

$$= \frac{\phi(q^4)}{(q^4;q^4)_\infty^2} \left(\phi(q^8)^2 + 4q^4 \psi(q^{16})^2 \right),$$

where the last equality follows from

$$\sum_{n=-\infty}^\infty q^{n^2+n} = 2 \sum_{n=0}^\infty q^{n^2+n} = 2\psi(q^2).$$

Finally, by (8), we obtain the identity (9).

Similarly, the other identities can be proved. We omit the details. $\square$

The following lemmas can easily be proved by the binomial theorem and induction so we state without proof.

Lemma 3.2. *For any positive integer n,*

$$(1+x)^{2^n} \equiv (1+x^2)^{2^{n-1}} \pmod{2^n}.$$

Lemma 3.3. *For any prime p,*

$$(1+x)^p \equiv (1+x^p) \pmod{p}.$$

We first prove the case $k=0$ in Theorem 3.4 in a separate theorem.

Theorem 3.4. *For any nonnegative integer n,*

$$p_\omega(8n+4) \equiv 0 \pmod 4,$$
$$p_\omega(8n+6) \equiv 0 \pmod 8,$$
$$p_\omega(16n+13) \equiv 0 \pmod 4.$$

Proof. Examining the left hand side of (5), we see that the only terms producing q^n for $n \not\equiv 0,3,7 \pmod 8$ come from $2q\omega(q)$. Consequently, if we can prove that in $F(q)$ the terms for $q^{8n+4}, q^{8n+6}, q^{16n+13}$, respectively, have coefficients divisible by 8, we are done.

For the first congruence, it suffices to prove that the odd powers in $F_0(q)$ vanish mod 8, which implies that the coefficient of q^{8n+4} in $F(q)$ is divisible by 8. By (9) in Lemma 3.1,

$$
\begin{aligned}
F_0(q) &= \frac{\phi(q)^3}{(q;q)_\infty^2} \\
&= \frac{(q^2;q^2)_\infty^3(-q;q^2)_\infty^6}{(q;q)_\infty^2} \\
&= \frac{(q^2;q^2)_\infty^3(-q;q^2)_\infty^8}{(q^2;q^2)_\infty^2(q;q^2)_\infty^2(-q;q^2)_\infty^2} \\
&\equiv \frac{(q^2;q^2)_\infty^3(-q^2;q^4)_\infty^4}{(q^2;q^2)_\infty^2(q^2;q^4)_\infty^2} \pmod{8},
\end{aligned}
$$

where the last congruence follows from Lemma 3.2. Thus mod 8 $F_0(q)$ is an even function, and returning to the observations at the beginning of this section, we see that $p_w(8n+4)$ is divisible by 4.

We now prove the second congruence, which is equivalent to that the odd powers in $F_2(q)$ vanish mod 16, which implies that the coefficient of q^{8n+6} in $F(q)$ is divisible by 16. By (11) in Lemma 3.1,

$$
\begin{aligned}
\frac{1}{4}F_2(q) &= \frac{1}{(q;q)_\infty^2}\phi(q)\phi(q^2)\psi(q^4) \\
&= \frac{1}{(q;q)_\infty^2}\psi(q)^2\psi(q^2) \\
&= \frac{1}{(q;q)_\infty^2}\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty} \\
&= \frac{1}{(q;q^2)_\infty^4}\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty} \\
&\equiv \frac{1}{(q^2;q^4)_\infty^2}\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty} \pmod{4}.
\end{aligned}
$$

By the analysis at the beginning of this section, we obtain that $p_w(8n+6)$ is divisible by 8.

Lastly, for $p_w(16n+13)$, we begin by considering the powers q^{16n+13} in $F(q)$ which still only come from the term $2q\omega(q)$. By looking at the dissection of $F(q)$ in Lemma 3.1, the terms q^{16n+13} in $F(q)$ must come from $qF_1(q^4)$. Next, from $16n+13 = 4(4n+3)+1$, if we can proof that the

powers of the form q^{4n+3} in $\frac{1}{2}F_1(q)$ vanish mod 4, we would be done. By (10) in Lemma 3.1,

$$
\begin{aligned}
\frac{1}{2}F_1(q) &= \frac{\psi(q^2)\phi(q)^2}{(q;q)_\infty^2} \\
&= \frac{\psi(q^2)(q^2;q^2)_\infty^2(-q;q^2)_\infty^4}{(q;q)_\infty^2} \\
&= \frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty}(-q;q)_\infty^2(-q;q^2)_\infty^4 \\
&= \frac{(q^4;q^4)_\infty(-q;q)_\infty^2(-q;q^2)_\infty^4}{(q;q^2)_\infty(-q;q^2)_\infty} \\
&= \frac{(q^4;q^4)_\infty(-q;q^2)_\infty^4}{(q;q^2)_\infty^3(-q;q^2)_\infty} \\
&\equiv \frac{(q^4;q^4)_\infty(q;q^2)_\infty^4}{(q;q^2)_\infty^3(-q;q^2)_\infty} \quad (\mathrm{mod}\ 4) \\
&= \frac{(q^4;q^4)_\infty(q;q^2)_\infty^2}{(q^2;q^4)_\infty} \\
&= \frac{(q^4;q^4)_\infty\phi(-q)}{(q^2;q^4)_\infty(q^2;q^2)_\infty} \\
&= \frac{(-q^2;q^2)_\infty(q^2;q^2)_\infty}{(q^2;q^4)_\infty(q^2;q^2)_\infty}\phi(-q) \\
&= \phi(-q)(-q^2;q^2)_\infty^2 \\
&= \left(1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\right)(-q^2;q^2)_\infty^2 \\
&= (-q^2;q^2)_\infty^2 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}(-q^2;q^2)_\infty^2.
\end{aligned}
$$

Since $(-q^2;q^2)_\infty^2$ does not contain any term of the form q^{4n+3}, it suffices to show that the coefficient of q^{4n+3} vanishes mod 2 in $\sum_{n=1}^{\infty}(-1)^n q^{n^2}(-q^2;q^2)_\infty^2$. Now

$$
\sum_{n=1}^{\infty}(-1)^n q^{n^2}(-q^2;q^2)_\infty^2 \equiv \sum_{n=1}^{\infty}(-1)^n q^{n^2}(q^4;q^4)_\infty \quad (\mathrm{mod}\ 2).
$$

But 3 is not a quadratic residue modulo 4, so there is no term of q^{4n+3} in this sum. And so we are done. $\qquad\square$

We now prove Theorem 1.1 by induction on k. For $k = 0$, the congruences are given in Theorem 3.4. To prove the congruence for any positive integer k, we consider the following sequence g_k defined by

$$
g_k = 4g_{k-1} - 1, \tag{13}
$$

which has the solution

$$
g_k = 2^{2k}g_0 - \frac{2^{2k} - 1}{3}.
$$

In particular, for any nonnegative integer n,

$$
g_k = \begin{cases} 2^{2k+3}n + \frac{11 \cdot 2^{2k}+1}{3} & \text{if } g_0 = 8n+4, \\ 2^{2k+3}n + \frac{17 \cdot 2^{2k}+1}{3} & \text{if } g_0 = 8n+6, \\ 2^{2k+4}n + \frac{38 \cdot 2^{2k}+1}{3} & \text{if } g_0 = 16n+13. \end{cases} \tag{14}
$$

From (13), we see that $g_k \equiv 3 \pmod{4}$ for $k > 0$. Thus, on the left hand side of (5), the coefficient of q^{g_k} comes from $2q\omega(q) + 2q^3\omega(-q^4)$. By the definition of $p_w(n)$, we have

$$
2q\omega(q) + q^3\omega(-q^4) = 2q\omega(q) - 2q^{-1}\big(-q^4\omega(-q^4)\big)
$$
$$
= 2\left(\sum_{n=1}^{\infty} p_w(n)q^n - \sum_{n=1}^{\infty} p_w(n)(-1)^n q^{4n-1}\right),
$$

and the coefficient of q^{g_k} in $2q\omega(q) + 2q^3\omega(-q^4)$ is

$$
2\big(p_\omega(g_k) - (-1)^{g_{k-1}} p_\omega(g_{k-1})\big).
$$

By the induction hypothesis, we know that $p_\omega(g_{k-1}) \equiv 0 \pmod{4}$. Therefore, to show that $p_\omega(g_k) \equiv 0 \pmod{4}$, it suffices to show that the coefficient of q^{g_k} in $F(q)$ in (5) vanishes mod 8, which is indeed true as seen in (12) in Lemma 3.1.

4. CONGRUENCES FOR $p_\nu(n)$

The formula (2) in the Introduction is the key to the results in this section.

Theorem 4.1. *For any nonnegative integer n,*

$$
p_\nu(8n+6) \equiv 0 \pmod{4}.
$$

Proof. Note that

$$
(-q^2; q^2)_\infty \psi(q^2) = \frac{(-q^2; q^2)_\infty (q^4; q^4)_\infty}{(q^2; q^4)_\infty}
$$
$$
= (-q^2; q^2)_\infty^2 (q^4; q^4)_\infty
$$
$$
= (-q^4; q^4)_\infty^2 (-q^2; q^4)_\infty^2 (q^4; q^4)_\infty
$$
$$
\equiv (q^4; q^4)_\infty^2 (-q^2; q^4)_\infty^2 (q^4; q^4)_\infty \pmod{4}
$$
$$
= (q^8; q^8)_\infty^2 (q^4; q^8)_\infty^2 (-q^2; q^4)_\infty^2 (q^4; q^4)_\infty
$$
$$
\equiv (q^8; q^8)_\infty^2 (-q^4; q^8)_\infty^2 (-q^2; q^4)_\infty^2 (q^4; q^4)_\infty \pmod{4}
$$
$$
= (q^8; q^8)_\infty \big[(q^8; q^8)_\infty (-q^4; q^8)_\infty^2\big] \big[(-q^2; q^4)_\infty^2 (q^4; q^4)_\infty\big]
$$
$$
= (q^8; q^8)_\infty \left[1 + 2\sum_{n=1}^{\infty} q^{4n^2}\right] \left[1 + 2\sum_{m=1}^{\infty} q^{2m^2}\right]
$$
$$
\equiv (q^8; q^8)_\infty \left[1 + 2\sum_{n=1}^{\infty} q^{4n^2} + 2\sum_{m=1}^{\infty} q^{2m^2}\right] \pmod{4}.
$$

in which the terms for q^{8n+6} vanish. This completes the proof.   □

Again, by (2), we see that $p_\nu(2n-1) = p_\omega(n)$. Thus the following congruences immediately follow from Theorem 1.1.

Theorem 4.2. *For any nonnegative integers n and k,*

$$p_v\left(2^{2k+4}n + \frac{11 \cdot 2^{2k+1} - 1}{3}\right) \equiv 0 \pmod{4},$$

$$p_v\left(2^{2k+4}n + \frac{17 \cdot 2^{2k+1} - 1}{3}\right) \equiv 0 \pmod{4},$$

$$p_v\left(2^{2k+5}n + \frac{38 \cdot 2^{2k+1} - 1}{3}\right) \equiv 0 \pmod{4}.$$

5. WALDHERR'S CONGRUENCES

In this section, we provide an elementary proof of the first explicit examples of congruences for $p_\omega(n)$.

Theorem 5.1 (Waldherr [6]).

$$p_\omega(40n + 28) \equiv 0 \pmod{5},$$
$$p_\omega(40n + 36) \equiv 0 \pmod{5}.$$

Proof. First note that the terms for q^{40n+28} and q^{40n+36} in $F(q)$ in (5) come from $2q\omega(q)$ only. Also, since $40n + 28 = 4(10n + 7)$ and $40n + 36 = 4(10n + 9)$, it suffices to show that in $F_0(q)$ the terms for q^{10n+7} and q^{10n+9} vanish mod 5.

Since

$$\phi(q) = (-q; q^2)_\infty^2 (q^2; q^2)_\infty = \frac{(q^2; q^2)_\infty^5}{(q; q)_\infty^2 (q^4; q^4)_\infty^2},$$

by Lemma 3.1, we have

$$F_0(q) = \frac{\phi(q)^3}{(q; q)_\infty^2} = \frac{(q^2; q^2)_\infty^{15}}{(q; q)_\infty^8 (q^4; q^4)_\infty^6}.$$

From [2, Equation (5.11)],

$$\frac{1}{(q; q)_\infty^8} = \left(\frac{(q^4; q^4)_\infty^{14}}{(q^2; q^2)_\infty^{14}(q^8; q^8)_\infty^4} + 4q\frac{(q^4; q^4)_\infty^2 (q^8; q^8)_\infty^4}{(q^2; q^2)_\infty^{10}(q^8; q^8)_\infty^4}\right)^2$$

Since $10n + 7$ and $10n + 9$ are odd, we consider odd powers of q only in $F_0(q)$, which appear in

$$8q\frac{(q^4; q^4)_\infty^{16}}{(q^2; q^2)_\infty^{24}}\frac{(q^2; q^2)_\infty^{15}}{(q^4; q^4)_\infty^6} = 8q\frac{(q^4; q^4)_\infty^{10}(q^2; q^2)_\infty}{(q^2; q^2)_\infty^{10}}$$

$$\equiv 8q\frac{(q^{20}; q^{20})_\infty^2}{(q^{10}; q^{10})_\infty^2}\sum_{n=-\infty}^{\infty} q^{n(3n-1)} \pmod{5},$$

where the congruence follows from Lemma 3.3 and the Euler's pentagonal number theorem.

Now we can easily check that $n(3n - 1) \equiv 0, 2, 4 \mod 10$, from which it follows that the coefficients of q^{10n+7} and q^{10n+9} are divisible by 5. This completes the proof. $\qquad\square$

Acknowledgements

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