LEGENDRE THEOREMS FOR SUBCLASSES OF OVERPARTITIONS

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Abstract. A. M. Legendre noted that Euler's pentagonal number theorem implies that the number of partitions of n into an even number of distinct parts almost always equals the number of partitions of n into an odd number of distinct parts (the exceptions occur when n is a pentagonal number). Subsequently other classes of partitions, including overpartitions, have yielded related Legendre theorems. In this paper, we examine four subclasses of overpartitions that have surprising Legendre theorems.

1. Introduction

In founding the theory of partitions [6], Euler proved that if $p(n)$ denotes the number of partitions of n, then

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \tag{1}$$

and he also proved the pentagonal number theorem for the reciprocal of (1):

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j-1)/2}. \tag{2}$$

Subsequently Legendre [16] noted that the following partition theorem is implicit in (2).

Legendre's Theorem. Let $pE(n)$ denote the number of partitions of n into an even number of distinct parts minus the number of partitions of n into an odd number of distinct parts. The E is for "excess." Then

$$pE(n) = \begin{cases} (-1)^j, & \text{if } n = j(3j-1)/2 \text{ for some } j \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Of course, the infinite product in (2) is one of the three classic theta constants studied by Gauss, Jacobi, and Ramanujan. The other two, in Ramanujan's notation are

$$\phi(-q) := \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1+q^n)} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \tag{4}$$

and

$$\psi(q) := \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^{2n-1})} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{5}$$

Corteel and Lovejoy in [5] and subsequent papers founded the study of overpartitions, where if $\overline{p}(n)$ is the number of overpartitions of n, then

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)}{(1-q^n)} = \frac{1}{\phi(-q)}. \tag{6}$$

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Overpartitions are ordinary partitions extended by allowing a possible overline designation on each part of the partitions. Thus $\overline{p}(3) = 8$ with the overpartitions in question being $3, \overline{3}, 2+1, 2+\overline{1}, \overline{2}+1, \overline{2}+\overline{1}, 1+1+1$, and $\overline{1}+1+1$.

One sees directly that (4) provides the Legendre theorem for overpartitions [1]. Namely the number of overpartitions of n into an even number of parts almost always equals the number of overpartitions of n into an odd number of parts (the exceptions occurring at the perfect squares). Thus one sees four overpartitions of 3 into an even number of distinct parts $(2+1, 2+\overline{1}, \overline{2}+1, \overline{2}+\overline{1})$ and four overpartitions of 3 into an odd number of parts $(3, \overline{3}, 1+1+1, \overline{1}+1+1)$.

The third theta constant occurs in the generating function for partitions with distinct odd parts [17],

$$\frac{1}{\psi(-q)} = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})}{(1-q^{2n})}.$$

In this case, the Legendre theorem, derived from (5), reveals that the number of partitions of n without repeated evens into an even number of parts almost always equals the number of partitions of n without repeated evens into an odd number of partitions (the exceptions occurring at the triangular numbers).

Each of these Legendre theorems has been proved bijectively. The bijective proof of the original Legendre theorem was given by F. Franklin [9]. His achievement was described by Hans Rademacher as the first major achievement of American mathematics. The Legendre theorems associated with (4) and (5) were first proved bijectively in [1].

Recently, F. Garvan and C. Jennings-Shaffer [13, 14, 15] have made a study of "smallest parts" functions associated with several Rogers-Ramanujan type identities in L. J. Slaters compendium [20]. The smallest parts function has been a great source for many studies since its introduction by the first author [2]. Here we list a few: [3, 8, 10, 11, 12, 18]. The companion of the smallest parts function for overpartitions was introduced and studied by Kathrin Bringmann, Jeremy Lovejoy and Robert Osburn [4].

Some of the instances considered in [13] have appealing corresponding interpretations in overpartitions. For example, let us define "top heavy" overpartitions (TH-overpartitions) as overpartitions in which there is both an overlined and a non-overlined largest part.

Now as we will see, it is a simple matter to show:

Theorem 1.1. *The number of TH-overpartitions of n equals one half the number of over partitions of n that have no 1's.*

For example, there are three top-heavy overpartitions of 5 (namely, $\overline{2}+2+1, \overline{2}+2+\overline{1}, \overline{1}+1+1+1+1$), and there are six overpartitions of 5 without 1's (namely, $5, \overline{5}, \overline{3}+\overline{2}, 3+\overline{2}, \overline{3}+2$, and $3+2$).

However, the related Legendre theorem for TH-overpartitions is a surprise.

Theorem 1.2. *Let $THE(n)$ denote the number of TH-overpartitions with an even number of parts minus the number with an odd number of parts. Then*

$$THE(n) = \begin{cases} (-1)^n(2j-1), & \text{if } j^2 < n < (j+1)^2 \text{ for some } j \in \mathbb{Z}^+, \\ (-1)^n(2j-2), & \text{if } n = j^2 \text{ for some } j \in \mathbb{Z}^+. \end{cases}$$

Apart from top heavy overpartitions, there are three further classes of overpartitions suggested by the work of Garvan and Jennings-Shaffer. Namely, we shall consider (1) topped overpartitions where there must be an overlined largest part; (2) bottom heavy overpartitions where there must be both an overlined and a non-overlined smallest part, and (3) bottomed overpartitions where there must be an overlined smallest part.

This paper is organized as follows. In Section 2, we provide an analytic proof of each of Theorems 1.1 and 1.2 along with the Legendre theorem on topped overpartitions. Their combinatorial proofs with further combinatorial results will be given in Section 4. The Legendre theorems on bottom heavy overpartitions and bottomed overpartitions will be given in Section 3.

2. Topped/Top Heavy Overpartitions

We first define topped overpartitions and top-heavy overpartitions.

Definition 2.1. *An overpartition is called a "topped" overpartition (or T-overpartition) if there must be an overlined largest part. Also, an overpartition is called a "top-heavy" overpartition (or TH-overpartition) if there is both an overlined and a non-overlined largest part.*

Then we can easily see that the generating function for T-overpartitions is

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{(q;q)_n} &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{q^n(-1;q)_n}{(q;q)_n}-\frac{1}{2}\\ &= \frac{1}{2}\frac{(-q;q)_\infty}{(q;q)_\infty}-\frac{1}{2}.\end{aligned} \tag{7}$$

Here and in the sequel, we adopt the following q-series notation:

$$\begin{aligned}(a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}),\\ (a;q)_\infty &:= \lim_{n\to\infty}(a;q)_n.\end{aligned}$$

Also, the generating function for TH-overpartitions is

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^{2n}(-q;q)_{n-1}}{(q;q)_n} &= \sum_{n=1}^{\infty}\frac{q^n(-q;q)_n}{(q;q)_n}-\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{(q;q)_n}\\ &= -1+\frac{(-q^2;q)_\infty}{(q;q)_\infty}+\frac{1}{2}-\frac{1}{2}\frac{(-q;q)_\infty}{(q;q)_\infty}\\ &= \frac{1}{2}\frac{(-q^2;q)_\infty}{(q^2;q)_\infty}-\frac{1}{2}.\end{aligned} \tag{8}$$

Note that Theorem 1.1 is a direct implication of (8).

Now, recall the Rogers-Fine identity [7, p.15]:

$$\sum_{n=0}^{\infty}\frac{(\alpha;q)_n}{(\beta;q)_n}\tau^n=\sum_{n=0}^{\infty}\frac{(\alpha;q)_n(\alpha\tau q/\beta;q)_n\beta^n\tau^nq^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta;q)_n(\tau;q)_{n+1}}. \tag{9}$$

Theorem 2.1. *We have*

$$\sum_{n=1}^{\infty}\frac{q^n(q;q)_{n-1}}{(-q;q)_n}=\sum_{n=1}^{\infty}(-1)^{n-1}q^{n^2}. \tag{10}$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n(q;q)_{n-1}}{(-q;q)_n} &= -\frac{d}{dz}\bigg|_{z=1}\sum_{n=0}^{\infty}\frac{(z;q)_nq^n}{(-q;q)_n}\\ &= -\frac{d}{dz}\bigg|_{z=1}\sum_{n=0}^{\infty}\frac{(z;q)_n(-zq;q)_n(-1)^nq^{n^2+n}(1-zq^{2n+1})}{(-q;q)_n(q;q)_{n+1}}\\ &= \frac{q}{1-q}+\sum_{n=1}^{\infty}\frac{(q;q)_{n-1}(-q;q)_n(-1)^nq^{n^2+n}(1-q^{2n+1})}{(-q;q)_n(q;q)_{n+1}}\\ &= \frac{q}{1-q}+\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+n}(1-q^{2n+1})}{(1-q^n)(1-q^{n+1})}\\ &= \frac{q}{1-q}+\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+n}\big(1-q^{n+1}+q^{n+1}(1-q^n)\big)}{(1-q^n)(1-q^{n+1})}\\ &= \frac{q}{1-q}+\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+n}}{1-q^n}+\sum_{n=1}^{\infty}\frac{(-1)^nq^{(n+1)^2}}{(1-q^{n+1})}\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}}{1-q^n} - \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{1-q^n}$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} q^{n^2},$$

where the second equality follows from (9). □

Remark. We can prove Theorem 2.1 using (7). Since a minus sign is assigned to each part of T-overpartitions, we replace q by $-q$ in (7). Then

$$\frac{1}{2}\frac{(q;q)_\infty}{(-q;q)_\infty} - \frac{1}{2} = \frac{1}{2}\big(1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\big) - \frac{1}{2} = \sum_{n=1}^{\infty}(-1)^n q^{n^2},$$

where the first equality follows from

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}. \tag{11}$$

Theorem 2.2. *We have*

$$\sum_{n=1}^{\infty} \frac{q^{2n}(q;q)_{n-1}}{(-q;q)_n} = \sum_{n=1}^{\infty}(-1)^{n-1} q^{n^2} + \sum_{n=1}^{\infty}(-1)^n (2n-1) q^{n^2}\left(1 - q + q^2 - + \cdots + q^{2n}\right). \tag{12}$$

Note that Theorem 1.2 is a direct implication of the above theorem.

Proof. We have already proved that

$$\sum_{n=1}^{\infty} \frac{q^{n}(q;q)_{n-1}}{(-q;q)_n} = \sum_{n=1}^{\infty}(-1)^{n-1} q^{n^2}.$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{q^{2n}(q;q)_{n-1}}{(-q;q)_n} + \sum_{n=1}^{\infty}(-1)^n q^{n^2} &= \sum_{n=1}^{\infty} \frac{q^{2n}(q;q)_{n-1}}{(-q;q)_n} - \sum_{n=1}^{\infty} \frac{q^{n}(q;q)_{n-1}}{(-q;q)_n} \\
&= -\sum_{n=1}^{\infty} \frac{q^{n}(q;q)_{n}}{(-q;q)_n} \\
&= 1 - \sum_{n=0}^{\infty} \frac{q^{n}(q;q)_{n}}{(-q;q)_n} \\
&= 1 - \sum_{n=0}^{\infty} \frac{(q;q)_n(-q^2;q)_n(-1)^n q^{n^2+n}(1-q^{2n+2})}{(-q;q)_n(q;q)_{n+1}} \\
&= 1 - \frac{1}{1+q}\sum_{n=0}^{\infty}(1+q^{n+1})^2(-1)^n q^{n^2+n} \\
&= 1 - \frac{1}{1+q}\left(\sum_{n=0}^{\infty}(-1)^n q^{n^2+n} + 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1)^2} + \sum_{n=0}^{\infty}(-1)^n q^{n^2+3n+2}\right) \\
&= 1 - \frac{1}{1+q}\left(\sum_{n=0}^{\infty}(-1)^n q^{n^2+n} + 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1)^2} - \sum_{n=1}^{\infty}(-1)^n q^{n^2+n}\right) \\
&= \frac{q}{1+q} + \frac{2}{1+q}\sum_{n=1}^{\infty}(-1)^n q^{n^2},
\end{aligned} \tag{13}$$

where the fourth equality follows from (9).

On the other hand, the series we want is close to

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n(2n-1)q^{n^2}(1-q+q^2-q^3+\cdots+q^{2n}) \\
&=\sum_{n=1}^{\infty}(-1)^n(2n-1)q^{n^2}\frac{1+q^{2n+1}}{1+q} \\
&=\frac{1}{1+q}\sum_{n=1}^{\infty}(-1)^n(2n-1)q^{n^2}+\frac{1}{1+q}\sum_{n=2}^{\infty}(-1)^{n-1}(2n-3)q^{n^2} \\
&=\frac{2}{1+q}\sum_{n=1}^{\infty}(-1)^n q^{n^2}+\frac{q}{1+q}. \qquad (14)
\end{aligned}$$

Comparing (13) with (14), we see that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{q^{2n}(q;q)_{n-1}}{(-q;q)_n} &= -\sum_{n=1}^{\infty}(-1)^n q^{n^2}+\frac{q}{1+q}+\frac{2}{1+q}\sum_{n=1}^{\infty}(-1)^n q^{n^2} \\
&= -\sum_{n=1}^{\infty}(-1)^n q^{n^2}+\sum_{n=1}^{\infty}(-1)^n q^{n^2}(2n-1)\left(1-q+q^2-+\cdots+q^{2n}\right),
\end{aligned}$$

which proves the desired result. □

Remark. We can also prove Theorem 2.2 differentiating the Rogers-Fine identity in (9) as we did in the proof of Theorem 2.1, but we omit it.

3. Bottomed/Bottom Heavy Overpartitions

As companions of topped overpartitions and top-heavy overpartitions, we define bottomed overpartitions and bottom-heavy overpartitions.

Definition 3.1. *An overpartition is called a "bottomed" overpartition (or B-overpartition) if there must be an overlined smallest part. Also, an overpartition is called a "bottom-heavy" overpartition (or BH-overpartition) if there is both an overlined and a non-overlined largest part.*

Then one can easily see that the generating function for B-overpartitions is given as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{q^n(-q^{n+1};q)_\infty}{(q^n;q)_\infty} &= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n(q;q)_{n-1}}{(-q;q)_n} \\
&= -\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^n q^{n^2} \\
&= -\frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}\frac{(q;q)_\infty}{(-q;q)_\infty}-\frac{1}{2}\right) \\
&= \frac{1}{2}\frac{(-q;q)_\infty}{(q;q)_\infty}-\frac{1}{2}, \qquad (15)
\end{aligned}$$

where the second equality follows from (10) and the third equality follows from (11). Also, the generating function for BH-overpartitions is given as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{q^{2n}(-q^{n+1};q)_\infty}{(q^n;q)_\infty} &= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{2n}(q;q)_{n-1}}{(-q;q)_n} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty}\left(-\sum_{n=1}^{\infty}(-1)^n q^{n^2}+\sum_{n=1}^{\infty}(-1)^n(2n-1)q^{n^2}\left(1-q+q^2-\cdots+q^{2n}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-q;q)_\infty}{(q;q)_\infty}\left(-\sum_{n=1}^{\infty}(-1)^n q^{n^2} + \frac{1}{1+q}\sum_{n=1}^{\infty}(-1)^n(2n-1)q^{n^2}(1+q^{2n+1})\right)\\
&= \frac{(-q;q)_\infty}{(q;q)_\infty}\left(-\sum_{n=1}^{\infty}(-1)^n q^{n^2} + \frac{q}{1+q} + \frac{2}{1+q}\sum_{n=1}^{\infty}(-1)^n q^{n^2}\right)\\
&= \frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{q}{1+q} + \frac{1-q}{1+q}\sum_{n=1}^{\infty}(-1)^n q^{n^2}\right)\\
&= \frac{q(-q^2;q)_\infty}{(q;q)_\infty} + \frac{1-q}{1+q}\left(\frac{1}{2} - \frac{1}{2}\frac{(-q;q)_\infty}{(q;q)_\infty}\right)\\
&= \frac{q(-q^2;q)_\infty}{(q;q)_\infty} - \frac{1}{2}\frac{(-q^2;q)_\infty}{(q^2;q)_\infty} + \frac{1-q}{2(1+q)}, \qquad (16)
\end{aligned}
$$

where the second equality follows from (12) and the fifth equality follows from (11).

Theorem 3.1 (Legendre theorem for B-overpartitions)**.** *Let $BE(n)$ denote the number of B-overpartitions with an even number of parts minus the number with an odd number of parts. Then*

$$
BE(n) = \begin{cases} (-1)^k, & \text{if } n = k^2 \text{ for some } k \in \mathbb{Z}^+,\\ 0, & \text{otherwise.}\end{cases}
$$

Proof. We assign a minus sign to each part of B-overpartitions. Thus, by its generating function in (15),

$$
\sum_{n=1}^{\infty}\frac{-q^n(q^{n+1};q)_\infty}{(-q^n;q)_\infty} = \frac{1}{2}\frac{(q;q)_\infty}{(-q;q)_\infty} - \frac{1}{2} = \sum_{n=1}^{\infty}(-1)^n q^{n^2},
$$

where the last equality follows from (11). □

Theorem 3.2 (Legendre theorem for BH-overpartitions)**.** *Let $BHE(n)$ denote the number of BH-overpartitions with an even number of parts minus the number with an odd number of parts. Then*

$$
BHE(n) = \begin{cases} 0, & \text{if } n = k^2 \text{ for some } k \in \mathbb{Z}^+,\\ (-1)^n, & \text{otherwise.}\end{cases}
$$

Proof. We assign a minus sign to each part of BH-overpartitions. Then

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{q^{2n}(q^{n+1};q)_\infty}{(-q^n;q)_\infty} &= \frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{2n}(-q;q)_{n-1}}{(q;q)_n}\\
&= \frac{(q;q)_\infty}{(-q;q)_\infty}\left(\frac{1}{2}\frac{(-q^2;q)_\infty}{(q^2;q)_\infty} - \frac{1}{2}\right)\\
&= \frac{1}{2}\frac{(1-q)}{(1+q)} - \frac{1}{2}\frac{(q;q)_\infty}{(-q;q)_\infty}\\
&= -\frac{q}{1+q} - \sum_{n=1}^{\infty}(-1)^n q^{n^2},
\end{aligned}
$$

where the second equality follows from (8) and the fourth equality follows from (11). □

4. Combinatorial results

4.1. **Prelimiaries.** The Ferrers graph of a partition is a left-juxtaposed array of dots in which each row has as many dots as each part. Throughout this paper, for an overpartition, we will denote an overlined part by a row ending with a $*$ in its Ferrers graph. For instance, the Ferrers graph of the overpartition $\overline{5}+5+\overline{4}+\overline{3}+3+\overline{2}$ is shown in Frigure 1.

For a nonnegative integer d, we also define the d-Durfee rectangle of an overpartition by the largest $(n+d)\times n$ rectangle that fits in the Ferrers graph. Here we allow no $*$'s in the Durfee rectangle. The

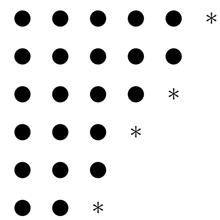

FIGURE 1. $\pi : \overline{5}+5+\overline{4}+\overline{3}+3+\overline{2}$.

0-Durfee rectangle becomes the Durfee square. For the overpartition shown in Figure 1, the Durfee suqare is the 3×3 square and the 1-Durfee rectangle is the 4×3 rectangle. The d-Durfee rectangle divides the Ferrers graph into three sections, namely the rectangle, the dots below the rectangle, and the dots to the right of the rectangle. For an overpartition π, we denote by π_B the overpartition consisting of the dots below the rectangle, and by π_R the conjugate of the overpartiton consisting of the dots to the right of the rectangle. If there is a $*$ in row $n+d+1$ and column $n+1$, then we assume that the $*$ belongs to π_B. For π in Figure 1, if we take the Durfee square, we obtain

$$\pi_B = \overline{3}+3+\overline{2}, \quad \pi_R = 3+\overline{2}+\overline{0}. \tag{17}$$

For the 1-Durfee rectangle,

$$\pi_B = 3+\overline{2}, \quad \pi_R = \overline{3}+\overline{2}+\overline{0}.$$

We now define four statistics on an overpartition π:

- $\overline{r}(\pi)$: smallest overlined part of π_B,
- $r(\pi)$: smallest non-overlined part of π_B.
- $\overline{c}(\pi)$: smallest overlined part of π_R,
- $c(\pi)$: smallest non-overlined part of π_R,

If there are no such parts, then we define these statistics to be infinity for convenience. From (17), we see that

$$\overline{r}(\pi) = \overline{2}, r(\pi) = 3, \overline{c}(\pi) = \overline{0}, c(\pi) = 3.$$

4.2. **Combinatorial proof of Theorem 2.1.** The series in Theorem 2.1 generates the excess of the number of T-overpartitions with an even number of parts over those with an odd number of parts. Call this number $TE(n)$.

Theorem 4.1 (Legendre theorem for T-partitions)**.** *We have*

$$\mathrm{TE}(n) = \begin{cases} (-1)^k, & \text{if } n = k^2 \text{ for some } k \in \mathbb{Z}^+, \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

For instance,

$n=1$	$\overline{1}$	TE(1)=1
$n=2$	$\overline{2}, \overline{1}+1$	TE(2) $=1-1=0$
$n=3$	$\overline{3}, \overline{1}+1+1$ $\overline{2}+1, \overline{2}+\overline{1}$	TE(3) $=2-2=0$
$n=4$	$\overline{4}, \overline{2}+1+1, \overline{2}+\overline{1}+1$ $\overline{3}+1, \overline{3}+\overline{1}, \overline{2}+2, \overline{1}+1+1+1$	TE(4) $=3-4=-1$
$n=5$	$\overline{5}, \overline{3}+1+1, \overline{3}+\overline{1}+1, \overline{2}+2+1, \overline{2}+2+\overline{1}, \overline{1}+1+1+1+1$ $\overline{4}+1, \overline{4}+\overline{1}, \overline{3}+2, \overline{3}+\overline{2}, \overline{2}+1+1+1, \overline{2}+\overline{1}+1+1$	TE(5) $=6-6=0$
$n=6$	$\overline{6}, \overline{4}+1+1, \overline{4}+\overline{1}+1, \overline{3}+2+1, \overline{3}+\overline{2}+1, \overline{3}+2+\overline{1}, \overline{3}+\overline{2}+\overline{1},$ $\overline{2}+2+2, \overline{2}+1+1+1+1, \overline{2}+\overline{1}+1+1+1$ $\overline{5}+1, \overline{5}+\overline{1}, \overline{4}+2, \overline{4}+\overline{2}, \overline{3}+3, \overline{3}+1+1+1,$ $\overline{3}+\overline{1}+1+1, \overline{2}+2+1+1, \overline{2}+2+\overline{1}+1, \overline{1}+1+1+1+1+1$	TE(6) $=10-10=0$

In this section, we will provide a combinatorial proof of (18).

For T-overpartitions, we stick with the 1-Durfee rectangle. Since the largest part of a T-overpartition is overlined, the first row (i.e, last column) always ends with a $*$, and we exclude this column from π_R. For π in Figure 2, we have

$$\pi_B = \overline{6}+4+4+\overline{3}, \quad \pi_R = 7+5+3+3+\overline{2},$$

and

$$\overline{r}(\pi) = \overline{3}, r(\pi) = 4, \overline{c}(\pi) = \overline{2}, c(\pi) = 3.$$

Let the 1-Durfee rectangle be of size $(n+1)\times n$. Since the last column ending with $*$ is removed from π_R, we see that

$$\begin{aligned}\overline{1} \leq& \overline{r}(\pi) \leq \overline{n},\\ 1 \leq& r(\pi) \leq n,\\ \overline{1} \leq& \overline{c}(\pi) \leq \overline{n},\\ 1 \leq& c(\pi) \leq n+1,\end{aligned}$$

if these are finite. For a T-overpartition π of n, we define its weight as $wt(\pi) = (-1)^{\ell-1}q^n$, where ℓ is the number of parts.

We now define two involutions.

- First involution ι_1: If both $r(\pi)$ and $\overline{c}(\pi)$ are greater than n, then we do nothing. If $r(\pi) < \overline{c}(\pi)$, then we move the row of length $r(\pi)$ to π_R creating a new column, and then add a $*$ under the last dot of the new column. If $\overline{c}(\pi) \leq r(\pi)$, then we delete the $*$ in the column of length $\overline{c}(\pi)$ and move the column to π_B creating a new row. Clearly, this mapping is a weight-preserving involution since the number of dots remains the same. In addition, the number of parts is increased or decreased by 1 after the column or row is moved. Thus it is a sign-reversing involution in this case.
- Second involution ι_2: If both $\overline{r}(\pi)$ and $c(\pi)$ are greater than n, then we do nothing. If $\overline{r}(\pi) \leq c(\pi)$, then we delete the $*$ and move the row of length $\overline{r}(\pi)$ to π_R creating a new column. If $c(\pi) < \overline{r}(\pi)$, then we move the column of length $c(\pi)$ to π_B creating a new row, and then add a $*$ at the end. In this case, there is one exception, namely if $c(\pi) = n$ and there is no part of length $n+1$ in π_R, then this map is not admissible because the resulting partition cannot be an overpartition. Thus in this case, we do nothing. Clearly, this mapping is a weight-preserving involution since

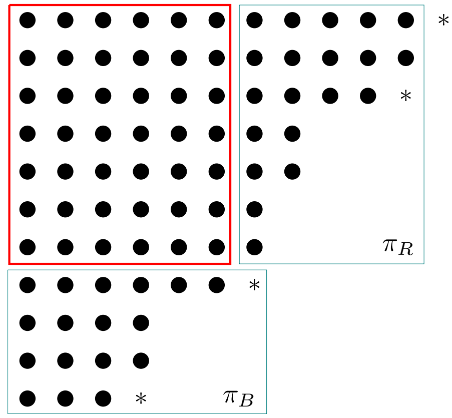

FIGURE 2. $\pi : \overline{11} + 11 + \overline{10} + 8 + 8 + 7 + 7 + \overline{6} + 4 + 4 + \overline{3}$.

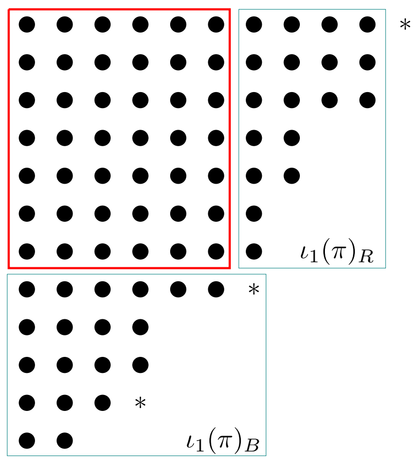

FIGURE 3. $\iota_1(\pi) : \overline{10} + 10 + 10 + 8 + 8 + 7 + 7 + \overline{6} + 4 + 4 + \overline{3} + 2$.

the number of dots remains the same. In addition, the number of parts is increased or decreased by 1 after the column or row is moved. Thus it is a sign-reversing involution.

For instance, let π be the T-overpartition given in Figure 2. For the first involution ι_1 we compare $r(\pi)$ with $\overline{c}(\pi)$, i.e, $\overline{c}(\pi) = 2 \leq 4 = r(\pi)$, and we obtain the T-overpartition shown in Figure 3. For the second involution ι_2, we compare $c(\pi)$ with $\overline{r}(\pi)$, i.e, $c(\pi) = 3 \geq 3 = \overline{r}(\pi)$, and we obtain the T-overpartition shown in Figure 4.

We are now ready to prove (18).

- Step 1: Apply ι_1 on the set of T-overpartitions. Since ι_1 is sign-reversing on its non-fixed points, the non-fixed points are cancelled out.
- Step 2: Then apply ι_2 on the fixed point set under ι_1. Since ι_2 is also sign-reversing on its non-fixed points, all the non-fixed points are cancelled out.

Let us identify the fixed points. First, the fixed points of ι_1 are T-overpartitions π in which π_R has no overlined parts (i.e., $\overline{c}(\pi) = \infty$) and π_B has no non-overlined parts (i.e., $r(\pi) = \infty$).

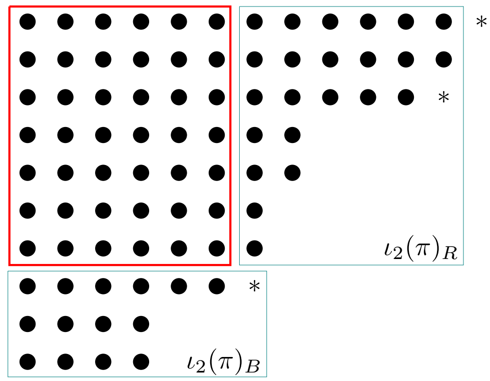

FIGURE 4. $\iota_2(\pi) : \overline{12} + 12 + \overline{11} + 8 + 8 + 7 + 7 + \overline{6} + 4 + 4$.

Note that if π_B has an overlined part of length n, then there must be a part of length $n+1$ in π_R. Thus, among those T-overpartitions fixed under ι_1, fixed under ι_2 are the T-overpartitions in which the parts of π_R are of length either $n+1$ or n, but not both, (i.e., $\overline{c}(\pi) = \infty$ and $c(\pi) = n, n+1, \infty$), and π_B has no parts (i.e., $r(\pi) = \infty$ and $\overline{r}(\pi) = \infty$). Thus, after applying ι_1 and then ι_2 successively, we obtain the following identity:

$$\sum_{n=1}^{\infty} TE(n) q^n = \frac{-q}{1-q} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n^2+n}}{1-q^n} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n^2+2n+1}}{1-q^{n+1}}, \tag{19}$$

where $q/(1-q)$ accounts for T-overpartitions with only one part (i.e., the 1-Durfee rectangle is of size 1×0), and the first summation accounts for the fixed T-overpartitions π with π_B being the empty partition and π_R having no parts of length $n+1$, and the second summation accounts for the fixed T-overpartitions π with π_B being empty and π_R having at least one part of length $n+1$.

For further cancellation, we do the following steps:

- Step 3: Among the remaining T-overpartitions after Step 2, namely the T-overpartitions generated by the right hand side of (19), we consider T-overpartitions with at least one $(n+1)$ in π_R, namely T-overpartitions generated by the second summation in (19). We consider the T-overpartitions with at least two $(n+1)$'s that are generated by the second sum. We move one of the $(n+1)$'s to the empty partition π_B. Since there are at least two $(n+1)$'s, this operation is well defined resulting in a T-overpartition with its 1-Durfee rectangle being of size $(n+2) \times (n+1)$ and the columns to the right of the rectangle being of length $n+1$ if any. Since the number of parts is increased by 1, it cancels T-overpartitions generated by the first summation except the ones with the 1-Durfee rectangle of size 2×1. Figure 5 shows this cancellation ocurring in this step. Thus we have

$$\sum_{n=1}^{\infty} TE(n) q^n = \frac{-q}{1-q} + \frac{q^2}{1-q} + \sum_{n=2}^{\infty} (-1)^{n+1} q^{(n+1)^2}, \tag{20}$$

  where $q^2/(1-q)$ accounts for the T-overpartitions with the 2×1 rectangle, and the summation accounts for the T-overpartitions with exactly one $(n+1)$ in π_R.
- Step 4: Among the remaining T-overpartitions, if a T-overpartition has the 2×1 rectangle (i.e., generated by $q^2/(1-q)$), then we can move the dot in the second row to the first row, which cancels the T-overpartition with only one part greater than 1. Thus the final remaining T-overpartitions are the ones with an $(n+1) \times n$ rectangle and exactly one part of length $(n+1)$ in π_R. Thus we have

$$\sum_{n=1}^{\infty} TE(n) q^n = \sum_{n=1}^{\infty} (-1)^n q^{n^2},$$

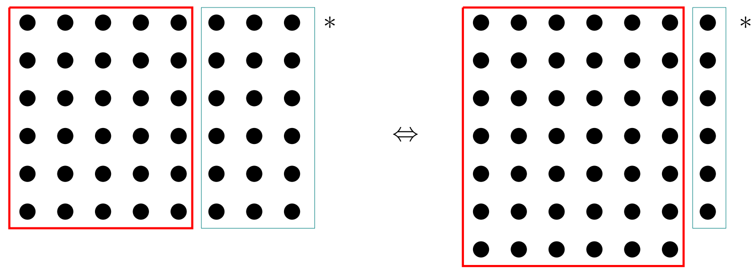

FIGURE 5. Step 4

which proves (18) as desired.

4.3. **Combinatorial proof of Theorem 2.2.** The series in Theorem 2.2 generates the excess of the number of TH-overpartitions with an even number of parts over those with an odd number of parts. Call this number $THE(n)$.

Theorem 4.2 (Legendre theorem for TH-overpartitions)**.**

$$THE(n) = \begin{cases} (-1)^n(2k-2), & \text{if } n = k^2, \\ (-1)^n(2k-1), & \text{if } k^2 < n < (k+1)^2 \end{cases} \tag{21}$$

for some $k \in \mathbb{Z}^+$.

For instance,

$n=1$	No partitions	THE(1)=0
$n=2$	$\bar{1}+1$	THE(2) $=1$
$n=3$	$\bar{1}+1+1$	THE(3) $=-1$
$n=4$	$\bar{2}+2, \bar{1}+1+1+1$	THE(4) $=2$
$n=5$	$\bar{2}+2+1, \bar{2}+2+\bar{1}, \bar{1}+1+1+1+1$	THE(5) $=-3$
$n=6$	$\bar{3}+3, \bar{2}+2+1+1, \bar{2}+2+\bar{1}+1, \bar{1}+1+1+1+1+1$ $\bar{2}+2+2$	THE(6) $=4-1=3$
$n=7$	$\bar{2}+2+2+1, \bar{2}+2+2+\bar{1}$ $\bar{3}+3+1, \bar{3}+3+\bar{1}$ $\bar{2}+2+1+1+1, \bar{2}+2+\bar{1}+1+1, \bar{1}+1+1+1+1+1+1$	THE(7) $=2-5=-3$

We now prove (21) combinatorially. The proof is very similar to the proof of (18). The involutions ι_1 and ι_2 will be applied first and further cancellations will follow. Hence we will present the outline highlighting the differences.

First, for a TH-overpartition π of n, we define its weight as $wt(\pi) = (-1)^\ell q^n$, where ℓ is the number of parts and we will stick with the 2-Durfee rectangle. Let the 2-Durfee rectangle of π be of size $(n+2)\times n$.

Then, by the definitions of π_B and π_R, we see that

$$\begin{aligned}\overline{1} \leq &\overline{r}(\pi) \leq \overline{n},\\ 1 \leq &r(\pi) \leq n,\\ \overline{2} \leq &\overline{c}(\pi) \leq \overline{n+1},\\ 2 \leq &c(\pi) \leq n+2.\end{aligned}$$

if they are finite. In order for the involutions ι_1 and ι_2 to be defined well on the set of TH-overpartitions, we require $\overline{r}(\pi) \geq \overline{2}$ and $r(\pi) \geq 2$. We apply the involutions ι_1 and ι_2 successively as follows.

- Step 1: Apply ι_1 on TH-overpartitions.
- Step 2: Then apply ι_2 on the fixed point set under ι_1.

First, TH-overpartitions with $2 \leq r(\pi), \overline{c}(\pi) \leq n$ are cancelled under ι_1, and then TH-overpartitions with $2 \leq \overline{r}(\pi), c(\pi) \leq n-1$ are cancelled under ι_2. If $\overline{r}(\pi) = n$, then there must be a non-overlined part of length $n+2$ in π_R, so these TH-overpartitions with $\overline{r}(\pi) = n$ cancel TH-overpartitions with at least one $n+2$ and one n in π_R. Thus the fixed points are TH-overpartitions π such that

D1. the 2-Durfee rectangle is of size 2×0, or
D2. the 2-Durfee rectangle is of size 3×1, or
D3. the 2-Durfee rectangle is of size $(n+2) \times n$ for some $n > 1$, and π_B is an overpartition in which parts are 1 or $\overline{1}$, and π_R is an overpartition in which overlined parts are $\overline{n+1}$, and non-overlined parts are (a) n or $n+1$, or (b) $n+1$ or $n+2$ with at least one $n+2$.

Note that the TH-overpartitions in D1 are generated by

$$\frac{q^2}{(1-q^2)},$$

and the TH-overpartitions in D2 are generated by

$$\frac{-q^3}{(1+q)(1-q^2)} + \frac{-q^3 \cdot q^2}{(1+q)(1-q^2)} + \frac{-(1-q)q^6(1+q^2)}{(1+q)(1-q^2)(1-q^3)},$$

where the first term generates TH-overpartitions with π_R having 2's if any, the second term generates TH-overpartitions with π_R having one $\overline{2}$ and 2's if others exist, and the third term generates TH-overpartitions with at least one 3 in π_R . Also, we note that TH-overpartitions in D3 are generated by

$$\frac{1-q}{1+q}\left(\frac{(-1)^n q^{(n+2)n}(1+q^{n+1})}{(1-q^n)(1-q^{n+1})} + \frac{(-1)^n q^{(n+2)n+(n+2)}(1+q^{n+1})}{(1-q^{n+1})(1-q^{n+2})}\right),$$

where $(1-q)/(1+q)$ accounts for the parts $\overline{1}$ and 1 in π_B, and the first term accounts for the TH-overpartitions of type (a), and the second term accounts for the TH-overpartitions of type (b). Thus the generating function for the fixed points is

$$\begin{aligned}&\frac{q^2}{1-q^2} - \left(\frac{q^3(1+q^2)}{(1+q)(1-q^2)} + \frac{(1-q)q^6(1+q^2)}{(1+q)(1-q^2)(1-q^3)}\right)\\ &+ \frac{1-q}{1+q}\left(\sum_{n=2}^{\infty}\frac{(-1)^n q^{(n+2)n}(1+q^{n+1})}{(1-q^n)(1-q^{n+1})} + \sum_{n=2}^{\infty}\frac{(-1)^n q^{(n+2)n+(n+2)}(1+q^{n+1})}{(1-q^{n+1})(1-q^{n+2})}\right). \qquad (22)\end{aligned}$$

For further cancellation, we do the following steps:

- Step 3: Among the TH-overpartitions generated by the second sum in (22), we consider TH-overpartitions with at least one $(n+2)$ and one $\overline{n+1}$ in π_R. We delete $\overline{n+1}$ and create one $(n+1)$ in π_B. Since there is at least one $(n+2)$ in π_R, this operation is well defined resulting in a TH-overpartition with its 2-Durfee rectangle of size $(n+3)\times(n+1)$ and π_R with parts $(n+1)$ or $(n+2)$. Since the number of parts is increased by 1, it cancels the TH-overpartitions generated by $(-1)^n q^{(n+2)n}/(1-q^n)(1-q^{n+1})$ in the first sum for $n>2$. Thus (22) reduces to

$$\frac{q^2}{1-q^2}-\left(\frac{q^3(1+q^2)}{(1+q)(1-q^2)}+\frac{(1-q)q^6(1+q^2)}{(1+q)(1-q^2)(1-q^3)}\right)+\frac{(1-q)q^{(2+2)2}}{(1+q)(1-q^2)(1-q^3)} + \frac{1-q}{1+q}\left(\sum_{n=2}^{\infty}\frac{(-1)^n q^{(n+2)n+(n+1)}}{(1-q^n)(1-q^{n+1})}+\sum_{n=2}^{\infty}\frac{(-1)^n q^{(n+2)n+(n+2)}}{(1-q^{n+1})(1-q^{n+2})}\right), \tag{23}$$

where the fourth term accounts for the term for $n=2$ in the first sum in (22).

- Step 4: We now take a TH-overpartition generated by the second sum in (23) with π_R having at least two $(n+2)$'s and one $(n+1)$ in π_R. By moving the $(n+1)$ to π_B, we obtain a TH-overpartition in which the 2-Durfee rectangle is of size $(n+3)\times(n+1)$, π_B is an overpartition with no parts >1, and π_R is an overpartition with parts $(n+1)$ or $(n+2)$, at least one $(n+2)$. Thus this cancels the TH-overpartitions generated by the first sum for $n>2$, and (23) reduces to

$$\frac{q^2}{1-q^2}-\left(\frac{q^3(1+q^2)}{(1+q)(1-q^2)}+\frac{(1-q)q^6(1+q^2)}{(1+q)(1-q^2)(1-q^3)}\right)+\frac{(1-q)q^{(2+2)2}}{(1+q)(1-q^2)(1-q^3)} + \frac{(1-q)q^{(2+2)2+(2+1)}}{(1+q)(1-q^2)(1-q^3)}+\frac{1-q}{1+q}\left(\sum_{n=2}^{\infty}\frac{(-1)^n q^{(n+2)n+(n+2)}}{1-q^{n+1}}+\sum_{n=2}^{\infty}\frac{(-1)^n q^{(n+2)n+2(n+2)}}{1-q^{n+2}}\right), \tag{24}$$

where the fifth term accounts for the term for $n=2$ in the first sum of (23), the first sum accounts for the TH-overpartitions generated by the second sum in (23) with π_R having exactly one $(n+2)$ and $(n+1)$'s only if others exist, and the second sum accounts for the TH-overpartitions in which π_R has at least two $(n+2)$'s and no $(n+1)$'s.

- Step 5: We now take a TH-overpartition generated by the second sum in (24) with at least three $(n+2)$'s. We now move one $(n+2)$ from π_R to π_B to create a TH-overpatition with an $(n+3)\times(n+1)$ rectangle with π_R having at least one $(n+3)$ and $(n+2)$'s only if others exist, which is generated by the first sum in (24). Thus (24) reduces to

$$\frac{q^2}{1-q^2}-\left(\frac{q^3(1+q^2)}{(1+q)(1-q^2)}+\frac{(1-q)q^6(1+q^2)}{(1+q)(1-q^2)(1-q^3)}\right)+\frac{(1-q)q^{(2+2)2}}{(1+q)(1-q^2)(1-q^3)} + \frac{(1-q)q^{(2+2)2+(2+1)}}{(1+q)(1-q^2)(1-q^3)}+\frac{(1-q)q^{(2+2)2+(2+2)}}{(1+q)(1-q^3)}+\frac{1-q}{1+q}\sum_{n=2}^{\infty}(-1)^n q^{(n+2)n+2(n+2)}, \tag{25}$$

where the sixth term is the term for $n=2$ in the first sum of (24) and the sum generates the fixed TH-overpartitions, namely the 2-Durfee rectangle is of size $(n+2)\times n$ and π_R has only two $(n+2)$'s.

- Step 6: We can define further cancellations combinatorially for

$$\frac{q^2}{1-q^2}-\frac{q^3(1+q^2)}{(1+q)(1-q^2)}-\frac{(1-q)q^6(1+q^2)}{(1+q)(1-q^2)(1-q^3)}+\frac{(1-q)q^{(2+2)2}}{(1+q)(1-q^2)(1-q^3)} + \frac{(1-q)q^{(2+2)2+(2+1)}}{(1+q)(1-q^2)(1-q^3)}+\frac{(1-q)q^{(2+2)2+(2+2)}}{(1+q)(1-q^3)},$$

which reduces to

$$\frac{q^2+(1-q)(q^4+q^9)}{1+q}.$$

But we skip the details.

Also, note that the sum in (25) can be written as

$$\sum_{n=4}^{\infty}(-1)^n q^{n^2}.$$

Thus we have

$$\sum_{n=1}^{\infty} THE(n)q^n = \frac{q^2}{1+q} + \frac{1-q}{1+q}\sum_{n=2}^{\infty}(-1)^n q^{n^2},$$

from which we see that the remaining TH-overpartitions are the ones with the Durfee square of size $n \times n$ and parts 1 or $\overline{1}$ below it. We can easily see that this proves (21). If $n = k^2$ for some k, then possible sides d of the Durfee square are $\leq k$ and for d, $1 < d < k$, there are $k^2 - d^2$ copies of 1's and one of them can be overlined, so there are two such TH-overpartitions. If $d = 1$ or k, there is only one such TH-overpartition each. Thus $(2k-2)$ TH-overparitions will remain. We can similarly show the $n \neq k^2$ case, and we skip the details.

4.4. **Further results.** In this section, we discuss some combinatorial results that involve T-overpartitions and TH-overpartitions.

4.4.1. *Result 1.* By (7), we have

$$\begin{aligned}\frac{1}{2}\frac{(-q;q)_\infty}{(q;q)_\infty} - \frac{1}{2} &= \frac{1}{2(q;q)_\infty}\big((-q;q)_\infty - (q;q)_\infty\big)\\ &= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{(2n-1)n}}{(q;q)_{2n-1}},\end{aligned}$$

from which we can deduce that $T(n)$ counts the number of overpartitions of n in which the number of overlined parts is odd. This can also be proved combinatorially as follows. We set up a bijection between T-overpartitions and and overpartitions with an odd number of overlined parts. Let λ be a T-overpartition. If there are an odd number of overlined parts, we do nothing. If there are an even number of overlined parts, we remove the overline of the largest overlined part to change it to an non-overlined part.

4.4.2. *Result 2.* We have the following analogous result for TH-overpartitions. By (8),

$$\begin{aligned}\frac{1}{2}\frac{(-q^2;q)_\infty}{(q^2;q)_\infty} - \frac{1}{2} &= \frac{1}{2(q^2;q)_\infty}\big((-q^2;q)_\infty - (q^2;q)_\infty\big)\\ &= \frac{1}{(q^2;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{(2n-1)(n+1)}}{(q;q)_{2n-1}},\end{aligned}$$

which yields that $TH(n)$ counts the number of overpartitions of n in which the parts are greater than 1 and the number of overlined parts is odd. The same combinatorial reasoning works for this result as well.

4.4.3. *Result 3.* It is easy to show that the number of overpartitions of n also equals the number of partitions into red and blue parts with no even blue parts allowed. Namely,

$$\frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{1}{(q;q)_\infty (q;q^2)_\infty}.$$

By (7), we have

$$\begin{aligned}\frac{1}{2}\frac{(-q;q)_\infty}{(q;q)_\infty} - \frac{1}{2} &= \frac{1}{2}\left(\frac{1}{(q;q)_\infty (q;q^2)_\infty} - 1\right)\\ &= \frac{\sum_{n=1}^{\infty}(-1)^{n-1}q^{n^2}}{(q;q)_\infty (q;q^2)_\infty}\\ &= \frac{1}{(q;q)_\infty (q;q^2)_\infty}\sum_{n=0}^{\infty} q^{(2n+1)^2}(1-q^{4n+3})\\ &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{(2n+1)^2}}{(q;q^2)_{2n+1}(q^{4n+5};q^2)_\infty}.\end{aligned} \tag{26}$$

From (26) it is not too hard to show that $T(n)$ is equal to the number of partitions into red and blue parts with no even blue parts allowed and in addition, the first MISSING odd blue part is congruent to 3 mod 4. E.g. $T(5) = 12$, the topped overpartitions are $\bar{5}, \bar{4}+1, \bar{4}+\bar{1}, \bar{3}+2, \bar{3}+\bar{2}, \bar{3}+1+1, \bar{3}+\bar{1}+1, \bar{2}+2+1, \bar{2}+2+\bar{1}, \bar{2}+1+1+1, \bar{2}+\bar{1}+1+1, \bar{1}+1+1+1+1$. The twelve of the second type just mentioned are $4r+1b, 3r+1r+1b, 3r+1b+1b, 2r+2r+1b, 2r+1b+1b+1b, 2r+1r+1b+1b, 2r+1r+1r+1b, 1b+1b+1b+1b+1b, 1r+1b+1b+1b+1b, 1r+1r+1b+1b+1b, 1r+1r+1r+1b+1b, 1r+1r+1r+1r+1b$.

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