# 4-Shadows in $q$-Series and the Kimberling Index

By
George E. Andrews

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**Abstract**

An elementary method in $q$-series, the method of 4-shadows, is introduced and applied to several poblems in $q$-series and partitions. This among other things lead to treatment of the Kimberling index for partitions.

Key words: Partitions, Kimberling index, 4-shadows, Garden of Eden partitions

Subject Classification Code: 05A17, 05A19, 11P83, 11F11

## 1 Introduction

In my tribute to Hans Raj Gupta [4], I added a mathematical appendix titled "Congruences Mod 4." My object was to provide a mathematical conclusion to the article that would expand on some of Gupta's work.

The basis for this study is the following (quoted directly from [4]):

**Mod 4 Lemma.** *Suppose $f(z)$ is a power series in $z$ whose coefficients may be multiple power series in other variables with integral coefficients. Then*

$$f(z)^2 \equiv f(-z)^2 \pmod 4$$

Note: This is easily extended to $f(z)^{2^n} \equiv f(-z)^{2^n} \pmod{2^{n+1}}$

*Proof.*
$$f(z)^2 - f(-z)^2 = (f(z) + f(-z))\,(f(z) - f(-z))\,,$$
and the right-hand side is twice the even part of $f$ times twice the odd part of $f$. Hence 4 divides all coefficients on the right-hand side. □

In [4], this result is used to prove results connected to $q$-series and partitions. Namely,

$$\sum_{n\geq 0} p(n)q^n := \frac{1}{(q;q)_\infty} \equiv \phi(-q)\psi(-q)\sum_{n=0}^{\infty} p(n)q^{4n} \pmod 4 \tag{1.1}$$

where

$$(A;q) = \prod_{n=0}^{\infty}(1 - Aq^n)\,, \tag{1.2}$$

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{1.3}$$

and

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{1.4}$$

Also given as a corollary of the Mod 4 Lemma:

$$\sum_{n\geq 0} p_2(n)q^n := \frac{1}{(q;q)_\infty^2} \equiv \psi(-q)^2\sum_{n\geq 0} p_2(n)q^{4n} \pmod 4. \tag{1.5}$$

The genesis for the Mod 4 Lemma came from the following observation about the generating functions for $p(n)$, the number of partitions of $n$, and the third order mock theta function of Ramanujan. Namely from [1, p.21], we know that

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n^2}, \tag{1.6}$$

and [12, p.64]

$$f(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2} = \frac{1}{(q;q)_\infty}\left(1 + 4\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}\right) \tag{1.7}$$

Thus from (1.6) and (1.7), we see immediately that

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \equiv \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \sum_{n\geq 0} p(n)q^n \pmod 4. \tag{1.8}$$

The question was: is there a simple way of proving (1.8) without invoking (1.6) and (1.7)? The answer was from the Mod 4 Lemma by noting the $z = \pm 1$ cases of

$$\frac{1}{(zq;q)_n^2} \equiv \frac{1}{(-zq;q)_n^2} \pmod 4. \tag{1.9}$$

Now the appendix of [4] was intended as a brief and elementary conclusion to my essay on Gupta. However, the Mod 4 Lemma leads to some intriguing mathematical discoveries and mysteries. We proceed as follows.

We start with a classical $q$-series representation of a modular form such as the $q$-series in (1.6). We refer to the $q$-series in (1.7) as a "4-shadow." In each case we shall also consider the difference between the two functions divided by 4 which we call the "shadow difference." So the shadow difference between the generating functions for $p(n)$ and $f(q)$ is

$$\frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n} \tag{1.10}$$

Our first application of the Mod 4 Lemma is to

$$\Theta_3(q) := \frac{\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2}}{\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}}.$$

In this instance, the 4-shadow that arises is

$$\overline{\Theta}_3(q) = 1 + 2\sum_{n=1}^{\infty} \frac{(q;q)_{n-1} q^{n^2}}{(q^n;q)_n (1-q^n)}.$$

The shadow difference is

$$\begin{aligned} D_3(q) &:= \left(\Theta_3(q) - \overline{\Theta}_3(q)\right)/8 \\ &= q^5 + q^6 + 3q^7 + 4q^8 + 7q^9 + 10q^{10} + 16q^{11} + 20q^{12} \\ &\quad + 31q^{13} + 41q^{14} + 56q^{15} + 74q^{16} + 101q^{17} + 129q^{18} \\ &\quad + 172q^{19} + 219q^{20} + 284q^{21} + 363q^{22} + \cdots , \end{aligned}$$

and the coefficients up through $q^{21}$ are the OEIS sequence A237833, the number of partitions of $n$ in which the largest part minus the least part is greater than the number of parts. However, the coefficients in $D_3(q)$ never again coincide with this partition function.

Partitions of this sort were first computed by Clark Kimberling [10] and are related to five successive sequences in the OEIS, A237830–A237834.

In $q$-series and partitions, this is, perhaps, a record for the agreement of two series before a discrepancy arises. Such an anomaly as this requires an explanation. Section 3 is devoted to the Kimberling index and Kimberling's partition functions.

Section 4 considers a natural 4-shadow connected with (1.4), and Section 5 looks at the partition mysteries arising therefrom including the relationship to Garden of Eden partitions [8],[9].

The conclusion considers open problems.

## 2 A $\phi(q)$ Quotient

Our first application of 4-shadowing is to the modular form

$$\Theta_3(q) := \frac{\phi(-q^3)}{\phi(-q)}. \tag{2.1}$$

$\Theta_3(q)$ as a generating function first arose in the work of Corteel and Lovejoy [6, Th 1.5 with $k = 3$] and is the generating function for overpartitions into parts not divisible by 3.

L.J. Slater [11, p.152, eq. (6), corrected] showed that

$$\Theta_3(q) = \sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n^2}}{(q;q)_n (q;q^2)_n} = 1 + 2\sum_{n=1}^{\infty} \frac{(-q;q)_{n-1}^2 q^{n^2}}{(q;q)_{2n-1}(1-q^n)}. \tag{2.2}$$

Thus we have as a 4-shahow for $\Theta_3(q)$ :

$$\begin{aligned} \overline{\Theta}_3(q) &= 1 + 2\sum_{n=1}^{\infty} \frac{(q;q)_{n-1}^2 q^{n^2}}{(q;q)_{2n-1}(1-q^n)} \\ &= 1 + 2\sum_{n=1}^{\infty} \frac{(q;q)_{n-1} q^{n^2}}{(q^n;q)_n (1-q^n)}, \end{aligned} \tag{2.3}$$

and the Mod 4 Lemma together with the factor 2 in each term tells us that

$$\Theta_3(q) \equiv \overline{\Theta}_3(q) \pmod 8. \tag{2.4}$$

Now (1.7) is the underlying deeper result behind the 4-shadowing of the partition function. So our next step is to obtain the deeper result underlying (2.4).

**Theorem 1.**

$$\overline{\Theta}(q) = \frac{1}{(q;q)_\infty}\left(1 - \sum_{n=1}^{\infty}(-1)^n(2n^2-1)q^{n(3n-1)/2}(1+q^n)\right). \tag{2.5}$$

*Remark.* Congruence (2.4) follows from (2.5) because (2.1) may be rewritten [11, p.152 eq.(6)] as

$$\Theta_3(q) = \frac{1}{(q;q)_\infty}\left(1 + \sum_{n=1}^{\infty} q^{n(3n-1)/2}(1+q^n)\right), \tag{2.6}$$

thus

$$\begin{aligned}\Theta_3(q) - \overline{\Theta}_3(q) &= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty} q^{n(3n-1)/2}(1+q^n)\left(1+(-1)^n(2n^2-1)\right) \\ &\equiv 0 \pmod 8.\end{aligned} \tag{2.7}$$

*Proof.* We recall the weak, $a = 1$ instance of Bailey's lemma [2, p.27, eq.(3.33)].

Namely if

$$\beta_n = \sum_{j=0}^{n} \frac{\alpha_j}{(q;q)_{n-j}(q;q)_{n+j}}, \tag{2.8}$$

then

$$\sum_{n=0}^{\infty} q^{n^2}\beta_n = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{n^2}\alpha_n. \tag{2.9}$$

The pair of sequences $(\alpha_n, \beta_n)$ is called a Bailey pair with $a = 1$ (cf.[2, p.26]). Now by [3, pp. 258–259, eqs. (3.9) and (3.11)], we see that

$$\left((-1)^{n-1}n^2q^{\binom{n}{2}}(1+q^n), \frac{(q;q)_{n-1}^2}{(1-q^n)(q;q)_{2n-1}}\right) \tag{2.10}$$

is a Bailey pair at $a=1$ with stipulation that $\alpha_0=\beta_0=0$.

By [2, p.27, last paragraph]

$$\left((-1)^n q^{\binom{n}{2}}(1+q^n), \delta_{n0}\right) \tag{2.11}$$

is a Bailey pair when $a=1$. Hence multiplying (2.10) by 2 and adding (2.11), we see that

$$\left((-1)^{n-1}(2n^2-1)q^{\binom{n}{2}}(1+q^n), \frac{2(q;q)_{n-1}^2}{(1-q^n)(q;q)_{2n-1}}\right) \tag{2.12}$$

forms a Bailey pair at $a=1$ subject to the stipulation that the first entry in each sequence is 1.

Therefore applying (2.12) to (2.9), we see that

$$\begin{aligned}\overline{\Theta}_3(q) &= 1+2\sum_{n=1}^{\infty}\frac{(q;q)_{n-1}^2 q^{n^2}}{(q;q)_{2n-1}(1-q^n)} \\ &= \frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^{n-1}(2n^2-1)q^{\binom{n}{2}}(1+q^n)\right).\end{aligned}$$

□

A noted in the introduction, the shadow difference

$$D_3(q) = \left(\Theta_3(q) - \overline{\Theta}_3(q)\right)/8$$

appears to be closely related to a class of partitions considered by C. Kimberling. We explore his partitions and their relationship to $D_3(q)$ in the next section.

# 3 Kimberling Partitions

We first define the Kimberling index, $K(\pi)$, of a partition $\pi$,

$$K(\pi) = (\text{largest part of } \pi) - (\text{least part of } \pi) - (\text{number of parts of } \pi). \tag{3.1}$$

Thus the Kimberling index of $11+7+7+4+3$ is $11-3-5=3$.

We now define five partition functions: $K_>(n)$, $K_<(n)$, $K_\le(n)$, $K_=(n)$ and $K_\ge(n)$. These are the numbers of partitions of $n$ for which the Kimberling index is $>$, $<$, $\leq$, $=$ or $\geq 0$ respectively. These five sequences correspond, in order, to sequences A237803–A237834 in the OEIS [10].

There are obvious relations among these partition functions

$$K_<(n) + K_=(n) + K_>(n) = p(n) \tag{3.2}$$

$$K_<(n) + K_=(n) = K_\leq(n) \tag{3.3}$$

$$K_>(n) + K_=(n) = K_\geq(n). \tag{3.4}$$

Obviously any two of the $K$'s suffices, along with $p(n)$, to determine the other three $K$'s. It is most convenient to treat the generating functions for $K_<(n)$ and $K_\leq(n)$.

In order to do this efficiently we need some background from $q$-series and the theory of partitions. First, we need the Gaussian polynomials of $q$-binomial coefficients:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{cases} \frac{(q;q)_N}{(q;q)_M (q;q)_{N-M}}, & 0 \leq M \leq N \\ 0, & \text{otherwise} \end{cases} \tag{3.5}$$

The polynomial $\begin{bmatrix} N+M \\ M \end{bmatrix}$ is the generating function for partitions in which each part is $\leq N$ and the number of partis is $\leq M$ [1, p.33, Th. 3.1].

Next we require an identity from Ramanujan's Lost Notebook [5, p.230, Entry 9.3.5] which was discovered independently by N.J. Fine [7, p.53, eq.(25.94)] in greater generality. We use Fine's notation. Let

$$Q(a;q) := \sum_{n\geq 0} \frac{(aq^{m+1};q)_m q^m}{(q;q)_m}, \tag{3.6}$$

then

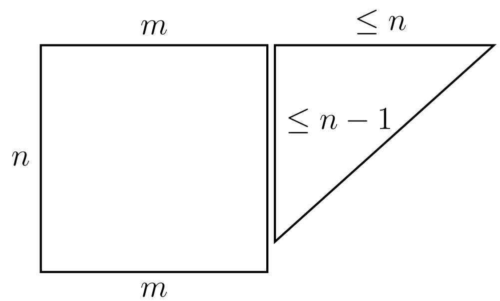
$$Q(a,q) = \frac{1}{(q;q)_\infty} \sum_{m\geq 0} (-a)^m q^{3m(m+1)/2} \tag{3.7}$$

**Theorem 2.**

$$\sum_{n\geq 1} K_\leq(n) q^n = \sum_{m\geq 1} \frac{q^m (q^{m+1};q)_{m-1}}{(q;q)_m} \tag{3.8}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} n q^{n(3n-1)/2} (1+q^n). \tag{3.9}$$

*Proof.* Let us consider those partitions with Kimberling index $\leq 0$ that have $n$ parts and smallest part $m$. Then schematically, the Ferrers graph of such partitions is of the form




The generating function for the nodes in the rectangle is $q^{mn}$ and in the triangle $\begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}$. Hence

$$\begin{aligned}\sum_{n\geq 1} K_{\leq}(n) q^n &= \sum_{n,m\geq 1} q^{nm} \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix} \\ &= \sum_{n\geq 1} \frac{q^n}{1-q^n} \frac{(q;q)_{2n-1}}{(q;q)_{n-1}(q;q)_n} \\ &= \sum_{n\geq 1} \frac{q^n (q^{n+1};q)_{n-1}}{(q;q)_n},\end{aligned}$$

which establishes (3.8).

As for (3.9), we note that by (3.7)

$$\begin{aligned}&Q(aq^{-1},q) - aqQ(aq,q) \\ &= \frac{1}{(q;q)_\infty}\left(\sum_{k=0}^{\infty}(-a)^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty}(-a)^k q^{k(3k-1)/2}\right) \\ &= \frac{1}{(q;q)_\infty}\left(1 + \sum_{k=1}^{\infty}(-1)^k a^k q^{k(3k-1)/2}(1+q^k)\right)\end{aligned} \tag{3.10}$$

On the other hand, by (3.6)

$$
\begin{aligned}
&Q(aq^{-1},q) - aqQ(aq,q) \\
&= 1 + \sum_{m\geq 1} \frac{q^m(aq^m;q)_m}{(q;q)_m} - \sum_{m\geq 1} \frac{aq^m(aq^{m+1};q)_{m-1}}{(q;q)_{m-1}} \\
&= 1 + \sum_{m\geq 1} \frac{q^m(aq^{m+1};q)_{m-1}}{(q;q)_m}\left((1-aq^m) - a(1-q^m)\right) \qquad (3.11) \\
&= 1 + (1-a)\sum_{m} \geq 1 \frac{q^m(aq^{m+1};q)_{m-1}}{(q;q)_m}.
\end{aligned}
$$

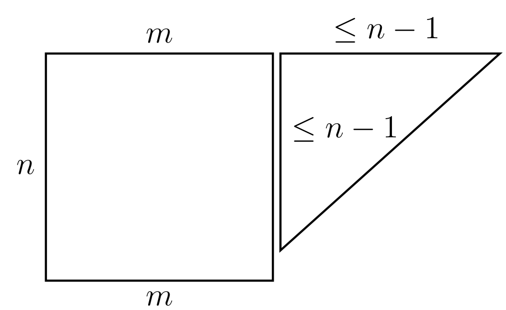
Thus we may identify the right-hand sides of (3.10) and (3.11). Finally we differentiate both expressions with respect to $a$ and set $a = 1$. This yields (3.9) and concludes this proof. □

**Theorem 3.**

$$\sum_{n\geq 1} K_<(n) = \sum_{n\geq 1} \frac{q^n(q^n;q)_{n-1}}{(q;q)_n} \tag{3.12}$$

$$= \frac{1}{(q;q)_\infty(1-q)} \sum_{n=0}^{\infty} (-1)^{n-1} q^{3n(n-1)/2+1}(1-q^{2n}). \tag{3.13}$$

*Proof.* The proof is exactly like that of (3.8) except that the schematic diagram of a generic Ferrers graph is: Hence



$$\begin{aligned}\sum_{n\geq 1} K_<(n)q^n &= \sum_{n,m\geq 1} q^{nm}\begin{bmatrix}2n-2\\ n-1\end{bmatrix}\\ &= \sum_{n\geq 1}\frac{q^n}{1-q^n}\frac{(q;q)_{2n-2}}{(q;q)_{n-1}^2}\\ &= \sum_{n\geq 1}\frac{q^n(q^n;q)_{n-1}}{(q;q)_n},\end{aligned}$$

which establishes (3.13).

For the proof of (3.14) we additionally require Euler's pentagonal number theorem [1, p.11, eq. (1.3.1)]:

$$(q;q)_\infty = 1 + \sum_{k=1}^{\infty}(-1)^k q^{k(3k-1)/2}(1+q^k). \tag{3.14}$$

Now equating the two right-hand sides of (3.10) and (3.11) and setting $a = q^{-1}$, we find

$$\begin{aligned}&1 + (1-q^{-1})\sum_{m\geq 1}\frac{q^m(q^m;q)_{m-1}}{(q;q)_m}\\ &\quad = \frac{1}{(q;q)_\infty}\sum_{k\geq 0}(-1)^k q^{3\binom{k}{2}}(1+q^k).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{m\geq 1}\frac{q^m(q^m;q)_{m-1}}{(q;q)_m} &= \frac{1}{(1-q^{-1})(q;q)_\infty}\Bigg(1 + \sum_{k\geq 1}(-1)^k q^{3\binom{k}{2}}(1+q^k) - 1\\ &\qquad - \sum_{k\geq 1}(-1)^k q^{k(3k-1)/2}(1+q^k)\Bigg)\\ &= \frac{1}{(1-q^{-1})(q;q)_\infty}\left(\sum_{k\geq 1}(-1)^k q^{3\binom{k}{2}} - \sum_{k\geq 1}(-1)^k q^{k(3k+1)/2}\right)\\ &= \frac{q}{(1-q)(q;q)_\infty}\sum_{k\geq 1}(-1)^{k-1}q^{3k(k-1)/2}(1-q^{2k}),\end{aligned}$$

which proves (3.14). □

**Corollary 4.**

$$\sum_{n\geq 1} K_{>}(n)q^n = \sum_{n\geq 1} \frac{q^n\left(1-(q^{n+1};q)_{n-1}\right)}{(q;q)_n} \tag{3.15}$$

$$= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^n(n-1)q^{n(3n-1)/2}(1+q^n). \tag{3.16}$$

*Proof.*

$$\sum_{n\geq 1} K_{>}(n)q^n = \sum_{n\geq 1}\left(p(n)-K_{\leq}(n)\right)q^n$$
$$= \frac{1}{(q;q)_\infty} - \sum_{n\geq 1} K_{\leq}(n)q^n.$$

Identities (3.15) and (3.16) now follow from (3.8), (3.9) and (3.14). □

It is now transparent why $D_3(q)$ and the generating function for $K_{>}(n)$ agree to so many terms. By (2.5) and (2.6)

$$\begin{aligned} D_3(q) &= \left(\Theta_3(q) - \overline{\Theta}_3(q)\right)/8 \\ &= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty} q^{n(3n-1)/2}(1+q^n)\left(1+(-1)^n(2n^2-1)\right)/8 \\ &= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\left(q^5+q^7-2q^{15}-2q^{17}+4q^{22}+4q^{26}-\cdots\right) \end{aligned} \tag{3.17}$$

On the other hand, by Corollary 4, equation (3.16)

$$\sum_{n\geq 1} K_{>}(n)q^n = \frac{1}{(q;q)_\infty}\left(q^5+q^7-2q^{15}-2q^{17}+3q^{22}+3q^{26}-\cdots\right). \tag{3.18}$$

Thus the discrepancy beginning at $q^{22}$ is now clear.

# 4 A $\psi(q)$ Quotient

It is natural to expect that there would be a $\psi(q)$ analog to the $\Theta_3(q)$ discussed in section 2. Indeed this is the case. We define

$$\Psi_3(q) := \frac{\psi(q^3)}{(q;q)_\infty}. \tag{4.1}$$

In the next section, we shall discuss the partition-theoretic aspects of $\Psi_3(q)$. Here we want to examine a natural 4-shadow. We rewrite [11, p.154, eq.(22)] as

$$\sum_{n\geq 0}\frac{(-q;q)_n^2 q^{n^2+n}}{(q;q)_{2n+1}}=\Psi_3(q). \tag{4.2}$$

The obvious 4-shadow is

$$\begin{aligned}\overline{\Psi}_3(q)&=\sum_{n\geq 0}\frac{(q;q)_n^2 q^{n^2+n}}{(q;q)_{2n+1}}\\&=\sum_{n\geq 0}\frac{(q;q)_n q^{n^2+n}}{(q^{n+1};q)_{n+1}}.\end{aligned} \tag{4.3}$$

**Theorem 5.**

$$\overline{\Psi}_3(q)=\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}. \tag{4.4}$$

*Remark.* The right-hand side of (4.4) is the generating function for 3-cores.

*Proof.* We refer now to the weak form of Bailey's lemma [2, p.27, eq.(3.33)] with $a=q$.

Namely if

$$\beta_n=\sum_{j=0}^{n}\frac{\alpha_j}{(q;q)_{n-j}(q^2;q)_{n+j}}, \tag{4.5}$$

then

$$\sum_{n\geq 0}\beta_n q^{n^2+n}=\frac{1}{(q^2;q)_\infty}\sum_{n\geq 0}q^{n^2+n}\alpha_n. \tag{4.6}$$

Here we again refer to [3]; namely by [3, p.260, eqs.(3.16) and (3.17) corrected with $q^2$ replaced by $q$], we see that

$$\left((-1)^{n-1}(2n+1)q^{n(n+1)/2}/(1-q),-\frac{(q;q)_n}{(q^{n+1};q)_{n+1}}\right) \tag{4.7}$$

forms a Bailey pair satisfying (4.5).

If we now substitute (4.7) into (4.6) and multiply by $-1$, we obtain

$$\begin{aligned}\overline{\Psi}_3(q)&=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}(-1)^n(2n+1)q^{3n(n+1)/2}\\&=\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty},\end{aligned} \tag{4.8}$$

by [1, p.176, Ex.7]. ∎

To conclude this section, we consider the shadow difference

$$\begin{aligned}\Delta_3(q) &= \left(\Psi_3(q) - \overline{\Psi}_3(q)\right)/4 \\ &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{3n(n+1)/2}\left(1-(-1)^n(2n+1)\right)/4 \end{aligned} \tag{4.9}$$

by (1.4) and (4.4). Hence

$$\begin{aligned}\Delta_3(q) = q^3 + q^4 + 2q^5 + 3q^6 + 5q^7 + 7q^8 + 10q^9 + 14q^{10} + 20q^{11} \\ + 27q^{12} + 37q^{13} + 49q^{14} + 66q^{15} + 86q^{16} + 113q^{17} + \cdots\end{aligned}$$

As we shall see in the next section, the series for $\Delta_3(q)$ agrees up through $q^{17}$ with the Garden of Eden partition generating function.

# 5 Partition anchors and Garden of Eden partitions

We begin with partition anchors.

**Definition.** A part $j$ of a partition $\pi$ is called an **anchor** of $\pi$ if no part greater than $j$ is repeated and no part is larger than $2j$. Additionally if a part $j$ is repeated, then $j$ is counted only once for being an anchor. We denote by $A(n)$ the total number of anchors in the partitions of $n$.

From this definition, it follows immediately that

$$\sum_{n\geq 0} A(n)q^n = \sum_{n=0}^{\infty} \frac{(-q^{n+1};q)_n q^n}{(q)_n}. \tag{5.1}$$

**Theorem 6.**

$$\sum_{n\geq 0} A(n)q^n = \Psi_3(q). \tag{5.2}$$

*Proof.* This follows immediately from Ramanujan's identity, equation (3.7), with $a = -1$. ∎

**Corollary 7.** *$A(n)$ equals the number of overpartitions of $n$ into parts not divisible by 12 and only parts $\equiv 3 \pmod 6$ may be overlined.*

*Proof.*

$$\begin{aligned}\Psi_3(q) &= \frac{\psi(q^3)}{(q;q)_\infty} \\ &= \frac{(q^6;q^6)_\infty}{(q;q)_\infty(q^3;q^6)_\infty} \quad (by (1.4)) \\ &= \frac{(q^6;q^{12})_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty(q^3;q^6)_\infty} \\ &= \frac{(-q^3;q^6)_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty},\end{aligned} \tag{5.3}$$

and this last product in the generating function for overpartitions with no part divisible by 12 and only parts $\equiv 3 \pmod 6$ possibly being overlined. □

For example, $A(5) = 9$; the anchors in question are underlined, $\underline{5}$, $\underline{4}+1$, $\underline{3}+\underline{2}$, $\underline{3}+1+1$, $\underline{2}+2+1$, $\underline{2}+\underline{1}+1+1$, $\underline{1}+1+1+1+1$. The relevant overpartitions for 5 are $5$, $4+1$, $\overline{3}+2$, $3+2$, $\overline{3}+1+1$, $3+1+1$, $2+2+1$, $2+1+1+1$, $1+1+1+1+1$.

As with the Kimberling partitions and $D_3(q)$, we now discover a similar relationship between $\Delta_3(q)$ and the Garden of Eden partitions. The name "Garden of Eden partitions" has its origins in the study of Bulgarian Solitaire (cf. [8]). In [9], Hopkins and Sellers identify Garden of Eden partitions with those partitions having rank (largest part minus number of parts) less than $-1$. Indeed, if $ge(n)$ denotes the number of Garden of Eden partitions of $n$, then [9] Hopkins and Sellers show that

$$\begin{aligned}\sum_{n=0}^{\infty} ge(n)q^n &= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{3n(n+1)/2} \\ &= \frac{1}{(q;q)_\infty}\left(q^3 - q^9 + q^{18} - q^{30} + \cdots\right).\end{aligned} \tag{5.4}$$

On the other hand, by (4.9)

$$\Delta_3(q) = \frac{1}{(q;q)_\infty}\left(q^3 - q^9 + 2q^{18} - 2q^{30} + \cdots\right), \tag{5.5}$$

and we see that the discrepancy between $\Delta_3(q)$ and the generating function for $ge(n)$ first occurs for $n = 18$.

We note one final partition identity connected with Garden of Eden partitions.

**Theorem 8.** *$ge(n)$ equals the number of partitions of $n$ in which the first missing multiple of 3 is even.*

*Proof.* We rewrite (5.4) as follows

$$\sum_{n\geq 0} ge(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty} q^{3\binom{2n}{2}}(1-q^{3\cdot 2n})$$
$$= \sum_{n=1}^{\infty} \frac{q^{3+6+\cdots+3(2n-1)}}{\prod_{\substack{j=1 \\ j\neq 3\cdot 2n}}^{\infty}(1-q^j)},$$

and the latter expression is the generating function for partitions which the first missing multiple of 3 is even. □

# 6 Conclusion

Obviously, for any given $q$-series, there may well be any number of 4-shadows. In the examples we have chosen, we used the Mod 4 Lemma in the way that seemed most clear.

It should be remarked from this work that the implications of the 4-shadow and the shadow difference are surprising and different in each case. In addition, in the cases we have considered, we have found unexpected ties to various partition functions. Indeed, the generating functions of the Kimberling partitions were completely unexplored, and the results in Theorems 2 and 3 would never have been discovered without the impetus provided by the uncanny series agreement explained by (3.17) and (3.18).

There are obvious further candidates for exploration such as [11, p.157, eq.(56)] as well as many others in [11].

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The Pennsylvania State University, University Park, PA 16802

gea1@psu.edu