Symmetric Expansions of Very Well Poised Basic Hypergeometric Series

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Dedicated to a grand mathematician and a good friend, Mourad Ismail.

Abstract

The classical transformation of the very well poised ${}_{2k+4}\phi_{2k+3}$ reduces the symmetry of the original series from the full symmetric group, S_{2k}, in the $2k$ parameters to S_2^k symmetry. Thus the symmetry drops from a group of $(2k)!$ elements to a group of 2^k elements. In this paper, a more symmetric expansion is obtained where the image symmetry group is $S_k \times S_2^k$.

Key words: Symmetric expansions, q-series, Rogers-Ramanujan identities
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1 Introduction

Symmetric expansions have played a vital role in the study of basic or q-hypergeometric functions. Indeed the road to the Rogers-Ramanujan identies started with L.J. Rogers in 1893 [9]. He observed a hidden symmetry in the Heine series

$$\sum_{n\geq 0} \frac{(a)_n(b)_n t^n}{(q)_n(c)_n}, \tag{1.1}$$

where $(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. He set himself the task of finding an expansion of this series that made all the symmetries transparent. A full account of the evolution of Rogers's papers [9],[10],[11] into the elaborate expansions of today is given in [4]. It should also be noted that D. Bowman has greatly extended Rogers's original efforts [7].

The Rogers-Ramanujan identities are two elegant identities:

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1.2}$$

and

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.3}$$

They first appear on the tenth page of Rogers's paper [11] which was the natural follow-up to [9] and [10]. The quintessential q-hypergeometric proof of (1.2) and (1.3) was give by G.N. Watson in 1929 in his wonderful identity [12]:

$$\begin{aligned} &{}_8\phi_7 \left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, b_2, c_2, q^{-N}; q, \frac{a^2q^{2+N}}{b_1c_1b_2c_2} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b_1}, \frac{aq}{c_1}, \frac{aq}{b_2}, \frac{aq}{c_2}, aq^{N+1} \end{matrix} \right) \\ &\qquad = \frac{(aq)_N \left(\frac{aq}{b_2c_2}\right)_N}{\left(\frac{aq}{b_2}\right)_N \left(\frac{aq}{c_2}\right)_N} {}_4\phi_3 \left(\begin{matrix} \frac{aq}{b_1c_1}, b_2, c_2, q^{-N}; q, q \\ \frac{aq}{b_1}, \frac{aq}{c_1}, \frac{b_2c_2q^{-N}}{a} \end{matrix} \right), \end{aligned} \tag{1.4}$$

where

$${}_{R+1}\phi_R \left(\begin{matrix} \alpha_0, \alpha_1, \ldots, \alpha_R; q, t \\ \beta_1, \ldots, \beta_R \end{matrix} \right) := \sum_{j\geq 0} \frac{(\alpha_0)_j (\alpha_1)_j \cdots (\alpha_R)_j t}{(q)_j (\beta_1)_j \cdots (\beta_R)_j}. \tag{1.5}$$

The left side series in (1.4) is called "well-poised" because the product of every column is the same (in this case the product is aq) and the adverb "very" is added to describe the special second and third columns. The series on the right side of (1.4) is called "balanced" because the product of the four upper entries times q equals the product of the three lower entries.

Watson deduced (1.1) and (1.2) from (1.4) by letting b_1, c_1, b_2, c_2, and N all $\to \infty$ and then setting $a = 1$ to obtain an equivalent result to (1.1) and obtaining (1.2) by $a = q$.

At this point, we note that the left side is a symmetric function on the four parameters b_1, c_1, b_2, c_2 while on the right side the symmetry has been reduced to $b_1 \leftrightarrow c_1$ and $b_2 \leftrightarrow c_2$.

Forty five years later, (1.1) and (1.2) were extended to a multiple series generalization [2]:

$$\sum_{n_1,\dots,n_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+\cdots N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \prod_{\substack{n=1 \\ n\neq 0,\pm i \pmod{2x+1}}}^{\infty} \frac{1}{1-q^n}, \tag{1.6}$$

where $N_m = n_m + n_{m+1} + \cdots + n_{k-1}$.

Then in 1976, the massive generalization of (1.4) was proved [3]: For $k \geq 1$, N a nonnegative integer,

$$\begin{aligned}
&{}_{2k+4}\phi_{2k+3}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, b_2, c_2, \dots, b_k, c_k, q^{-N}; q, \frac{a^k q^{k+N}}{b_1\cdots b_k c_1\cdots c_k} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b_1}, \frac{aq}{c_1}, \frac{aq}{b_2}, \frac{aq}{c_2}, \dots, \frac{aq}{b_k}, \frac{aq}{c_k}, aq^{N+1} \end{matrix}\right] \\
&= \frac{(aq)_N \left(\frac{aq}{b_k c_k}\right)_N}{\left(\frac{aq}{b_k}\right)_N \left(\frac{aq}{c_k}\right)_N} \sum_{m_1,\dots,m_{k-1}\geq 0} \frac{\left(\frac{aq}{b_1 c_1}\right)_{m_1} \left(\frac{aq}{b_2 c_2}\right)_{m_2} \cdots \left(\frac{aq}{b_{k-1} c_{k-1}}\right)_{m_{k-1}}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{k-1}}} \\
&\times \frac{(b_2)_{m_1}(c_2)_{m_1}(b_3)_{m_1+m_2}(c_3)_{m_1+m_2}\cdots(b_k)_{m_1+\cdots+m_{k-1}}}{\left(\frac{aq}{b_1}\right)_{m_1}\left(\frac{aq}{c_1}\right)_{m_1}\left(\frac{aq}{b_2}\right)_{m_1+m_2}\left(\frac{aq}{c_2}\right)_{m_1+m_2}\cdots\left(\frac{aq}{b_{k-1}}\right)_{m_1+\cdots+m_{k-1}}} \\
&\times \frac{(c_k)_{m_1+\cdots+m_{k-1}}}{\left(\frac{aq}{c_{k-1}}\right)_{m_1+\cdots+m_{k-1}}} \times \frac{\left(q^{-N}\right)_{m_1+m_2+\cdots+m_{k-1}}}{\left(b_k c_k \frac{q^{-N}}{a}\right)_{m_1+m_2+\cdots+m_{k-1}}} \\
&\times \frac{(aq)^{m_{k-2}+2m_{k-3}+\cdots+(k-2)m_1} q^{m_1+m_2+\cdots+m_{k-1}}}{(b_2c_2)^{m_1}(b_3c_3)^{m_1+m_2}\cdots(b_{k-1}c_{k-1})^{m_1+m_2+\cdots+m_{k-2}}}.
\end{aligned} \tag{1.7}$$

Note now that the S_{2k} symmetry of the left side reduces to S_2^k symmetry on the right. Often this loss of symmetry seems quite significant. In almost all applications, the pairs (b_i, c_i) are naturally kept together; so it wolud be valuable to have a transformation of the left side of (1.7) that was symmetric in these pairs. To produce such a transformation is the object of this paper.

Theorem 1. *For $k \geq 1$, N a nonnegative integer,*

$$
{}_{2k+4}\phi_{2k+3}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, b_2, c_2, \ldots, b_k, c_k, q^{-N}; q, \frac{a^k q^{k+N}}{b_1\cdots b_k c_1 \cdots c_k} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b_1}, \frac{aq}{c_1}, \frac{aq}{b_2}, \frac{aq}{c_2}, \ldots, \frac{aq}{b_k}, \frac{aq}{c_k}, aq^{N+1} \end{matrix}\right]
$$

$$
= \sum_{m_1,\ldots,m_k \geq 0} \prod_{i=1}^{k} \frac{\left(\frac{aq}{b_i c_i}\right)_{m_i} q^{m_i}}{(q)_{m_i}\left(\frac{aq}{b_i}\right)_{m_i}\left(\frac{aq}{c_i}\right)_{m_i}} \times K_k(a, N; m_1, m_2, \ldots, m_k), \tag{1.8}
$$

where K_k is symmetric in $m_1, m_2, \ldots, m_k$ and has the following properties for $k > 1$.

$$
K_k(a, N; m_1, \ldots, m_k) = 0 \qquad \text{if } N > m_1 + m_2 + \cdots + m_k. \tag{1.9}
$$

$$
K_k(a, N; m_1, \ldots, m_k) = \tag{1.10}
$$

$$
q^{-\sigma_2(m_1,\ldots,m_k) - \sigma_1(m_1,\ldots,m_k)}(aq)_N (q)_N, \ \text{ if } N = m_1 + \cdots + m_k.
$$

$$
K_k(a, N; m_1, \ldots, m_k) = \tag{1.11}
$$

$$
\sum_{j=0}^{N} \frac{(a)_j (1 - aq^{2j})(q^{-N})_j q^{N_j}}{(q)_j (1-a)(aq^{N+1})_j} \prod_{r=1}^{k} (q^{-j})_{m_r} (aq^j)_{m_r},
$$

where $\sigma_s(m_1, \ldots, m_k)$ is the s^{th} elementary symmetric function in $m_1, \ldots, m_k$.

I would note that neither (1.9) nor (1.10) is at all an immediate consequence of (1.11). Indeed one would hope that there might be representations of K_k that would make (1.9) and (1.10) as well as the symmetry clear. To that end, we have

Theorem 2.

$$
K_1(a, N; m_1) = \delta_{N, m_1}, \tag{1.12}
$$

$$
K_2(a, N; m_1, m_2) = \tag{1.13}
$$

$$
\begin{bmatrix} m_1 + m_2 \\ N \end{bmatrix} \frac{(-1)^N q^{\binom{N}{2}} (1 - aq^N)(q^{-N})_{m_1} (q^{-N})_{m_2} (a)_{m_1+m_2}}{(1-a)(q)_{m_1+m_2}}
$$

$$K_3(a,N;m_1,m_2,m_3) = \tag{1.14}$$

$$\begin{bmatrix} m_1+m_2+m_3 \\ N \end{bmatrix} \frac{(-1)^N q^{\binom{N}{2}}(1-aq^N)(q^{-N})_{m_1}(q^{-N})_{m_2}(q^{-N})_{m_3}(a)_{m_1+m_2+m_3}}{(1-a)(q)_{m_1+m_2+m_3}}$$

$$\times\ {}_4\phi_3\left(\begin{matrix} q^{-m_1}, q^{-m_2}, q^{-m_3}, \frac{q^{1-N}}{a}; q, q \\ q, q^{-N}, \frac{q^{1-m_1-m_2-m_3}}{a} \end{matrix}\right).$$

Section 2 will be devoted to a proof of Theorem 1 as well as (1.13). Section 3 will be devoted to the remaining two assertions in Theorem 2. Section 4 concludes with possible applications and open problems.

2 Proof of Theorem 1

We start with the easiest assertion, namely (1.11). To prove this result we require the following formulation of the q-Pfaff-Saalschiitz identity [8, p.32, eq.(2.2.1)]:

$$\sum_{r=0}^{m} \frac{(q^{-m})_r(aq^m)_r \left(\frac{aq}{bc}\right)_r q^r}{(q)_r \left(\frac{aq}{b}\right)_r \left(\frac{aq}{c}\right)_r} = \frac{a^m q^m (b)_m (c)_m}{b^m c^m \left(\frac{aq}{b}\right)_m \left(\frac{aq}{c}\right)_m}. \tag{2.1}$$

Hence

$${}_{2k+4}\phi_{2k+3}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, b_2, c_2, \ldots, b_k, c_k, q^{-N}; q, \frac{a^k q^{k+N}}{b_1\cdots b_k c_1 \cdots c_k} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b_1}, \frac{aq}{c_1}, \frac{aq}{b_2}, \frac{aq}{c_2}, \ldots, \frac{aq}{b_k}, \frac{aq}{c_k}, aq^{N+1} \end{matrix}\right]$$

$$= \sum_{j=0}^{N} \frac{(a)_j(1-aq^{2j})(q^{-N})_j q^{Nj}}{(q)_j(1-a)(aq^{N+1})_j} \prod_{i=1}^{k} \frac{a^j q^j (b_i)_j (c_i)_j}{b_i^j c_i^j \left(\frac{aq}{b_i}\right)_j \left(\frac{aq}{c_i}\right)_j}$$

$$= \sum_{j=1}^{N} \frac{(a)_j(1-aq^2)(q^{-N})_j q^{Nj}}{(q)_j(1-a)(aq^{N+1})_j} \times \sum_{m_1,\ldots,m_k\geq 0} \frac{(q^{-j})_{m_i}(aq^j)_{m_i}\left(\frac{aq}{b_ic_i}\right)_{m_i} q^{m_i}}{(q)_{m_i}\left(\frac{aq}{b_i}\right)_{m_i}\left(\frac{aq}{c_i}\right)_{m_i}}$$

$$= \sum_{m_1,\ldots,m_k\geq 0} \prod_{i=1}^{k} \frac{\left(\frac{aq}{b_i c_i}\right)_{m_i} q^{m_i}}{(q)_{m_i}\left(\frac{aq}{b_i}\right)_{m_i}\left(\frac{aq}{c_i}\right)_{m_i}} \tag{2.2}$$

$$\times \sum_{j=0}^{N} \frac{(a)_j(1-aq^{2j})(q^{-N})_j q^{Nj}}{(q)_j(1-a)(aq^{N+1})_j} \prod_{i=1}^{k} (q^{-j})_{m_i}(aq^j)_{m_i},$$

as asserted in (1.11).

In order to treat the other two assertions we need to rewrite K_k as a very well-poised series. Given the symmetry of K_k in the m's, we shall assume that m_k is at least as large as all the other m_i. Also we note that

$$(aq^j)_{m_i} = \frac{(a)_{m_i+j}}{(a)_j}$$

and

$$(q^{-j})_{m_i} = (1-q^j)\cdots(1-q^{j-m_i+1}) \cdot q^{-jm_i+\binom{m_i}{2}}(-1)^{m_i}$$

$$= \frac{(q)_j}{(q)_{j-m_i}}(-1)^{m_i} q^{-jm_i+\binom{m_i}{2}}$$

Hence

$$K_k(a,N;m_1,\ldots,m_k)$$
$$= \sum_{j=0}^{N} \frac{(a)_j(1-aq^{2j})(q^{-N})_j q^{Nj}}{(q)_j(1-a)(aq^{N+1})_j} \times \prod_{i=1}^{k} \frac{(a)_{m_i+j}(q)_j(-1)^{m_i} q^{\binom{m_i}{2}-jm_i}}{(a)_j(q)_{j-m_i}}.$$

Now $\frac{1}{(q)_M} = 0$ for $M < 0$, thus if j is less than any m_i the term is zero. So we may replace j by $j + m_k$ and no non-zero terms will be deleted, and to make clear the role of m_k we replace m_k by t. Thus

$$K_k(a,N;m_1,\ldots,m_{k-1},t)$$

$$= \sum_{j\geq 0} \frac{(a)_{j+t}(1-aq^{2j+2t})(q^{-N})_{j+t} q^{N(j+t)}}{(q)_{j+t}(1-a)(aq^{N+1})_{j+t}}$$

$$\prod_{i=1}^{k-1} \frac{(a)_{m_i+j+t}(q)_{j+t}(-1)^{m_i} q^{\binom{m_i}{2}-(j+t)m_i}}{(a)_{j+t}(q)_{j+t-m_i}} \frac{(a)_{j+2t}(q)_{j+t}(-1)^t q^{\binom{t}{2}-(j+t)t}}{(a)_{j+t}(q)_j}$$

$$= \frac{(a)_{2t}(1-aq^{2t})(q^{-N})_t q^{Nt-\binom{t+1}{2}}(-1)^t}{(aq^{N+1})_t(1-a)} \tag{2.3}$$

$$\times \prod_{i=1}^{k-1} \frac{(a)_{m_i+t}(q)_t(-1)^{m_i} q^{\binom{m_i}{2}-tm_i}}{(a)_t(q)_{t-m_i}}$$

$$\times {}_{2k+2}\phi_{2k+1}\left(\begin{matrix} aq^{2t}, q\sqrt{aq^{2t}}, -q\sqrt{aq^{2t}}, aq^{m_1+t}, q^{t+1}, \ldots, aq^{m_{k-1}+t}, q^{t+1}, q^{-N+t}; q, q^{N-t-m_1-\cdots-m_{k-1}} \\ \sqrt{aq^{2t}}, -\sqrt{aq^{2t}}, q^{t-m_1+1}, aq^t, \ldots, q^{t-m_{k-1}+1}, aq^t, aq^{N+t+1} \end{matrix}\right)$$

Now (2.3) allows us to obtain a recurrence for K_k by applying (1.8) to the inner series appearing in (2.3). Hence

$$\begin{aligned} &K_k(a, N; m_1, m_2, \ldots, m_{k-1}, t) \\ &= \frac{(a)_{2t}(1-aq^{2t})(q)_N q^{-t}}{(aq^{N+1})_t(1-a)(q)_{N-t}} \prod_{i=1}^{k-1}(aq^{t+1})_{m_i}(q^{-t})_{m_i} \\ &\sum_{\mu_1,\ldots,\mu_{k-1}\geq 0} \prod_{i=1}^{k-1} \frac{(q^{-m_i})_{\mu_i} q^{\mu_i}}{(q)_{\mu_i}(q^{t-m_i+1})_{\mu_i}(aq^{t+1})_{\mu_i}} \\ &K_{k-1}(aq^{2t}, N-t; \mu_1, \mu_2, \ldots, \mu_{k-1}). \end{aligned} \tag{2.4}$$

We now proceed to prove (1.9) and (1.10) by mathematical induction on k. The initial case is $k = 2$. By (2.3) with $m_1 = m, m_2 = t$,

$$K_2(a, N; m, t)$$

$$= \frac{(a)_{2t}(1-aq^{2t})(q)_N q^{-t}(aq^t)_m(q^{-t})_m}{(aq^{N+1})_t(1-a)(q)_{N-t}}$$

$${}_6\phi_5\left(\begin{matrix} aq^{2t}, q\sqrt{aq^{2t}}, -q\sqrt{aq^{2t}}, aq^{m+t}, q^{t+1}, q^{-N+t}; q, q^{N-t-m} \\ \sqrt{aq^{2t}}, -\sqrt{aq^{2t}}, q^{t-m+1}, aq^t, aq^{N+t+1} \end{matrix}\right)$$

$$= \frac{(aq)_{2t}(q)_N(a)_{m+t}(q^{-t})_m(aq^{2t+1})_{N-t}(q^{-m})_{N-t}q^{-t}}{(aq^{N+1})_t(q)_{N-t}(a)_t(q^{t+1-m})_{N-t}(aq^t)_{N-t}}$$

by [8, p.238, eq.(II.21)]

$$= \frac{(a)_{m+t}(q)_N(q^{-t})_m(q^{-m})_{N-t}(1-aq^N)q^{-t}}{(1-a)(q)_{N-t}(q^{t+1-m})_{N-t}(aq^{2n+1})_{N-t}}$$

$$= \begin{bmatrix} m+t \\ N \end{bmatrix} \frac{(-1)^N q^{\binom{N}{2}}(1-aq^N)(a)_{m+t}(q^{-N})_m(q^{-N})_t(q)_m(q)_t}{(1-a)(q)_{m+t}}. \qquad (2.5)$$

Now the factor $(q)_{m+t-N}$ in the denominator reveals that K_2 is 0 if $N > m+t$, and if $N = m+t$, then

$$\begin{aligned} &K_2(a,N;m,t) \\ &\quad = \frac{(-1)^N q^{\binom{N}{2}}(1-aq^N)(a)_N}{(1-a)(q)_N} \times \frac{(q)_N^2(-1)^{m+t}q^{-N(m+t)+\binom{m}{2}+\binom{t}{2}}}{(q)_t(q)_m} \\ &\quad = q^{-\binom{N+1}{2}+\binom{m}{2}+\binom{t}{2}}(aq)_N(q)_N \\ &\quad = q^{-mt-m-t}(aq)_N(q)_N. \end{aligned}$$

Thus we have established (1.9) and (1.10) in the case $k = 2$.

Now we must utilize the recurrance to complete the induction proof of (1.9) and (1.10). We assume that (1.9) and (1.10) are valid for each k less than a given k. Suppose $N > m_1 + m_2 + \cdots + m_{k-1} + t$ in (2.4). We see that the terms in the sum on the right-hand side of (2.4) must vanish if any $\mu_i > m_i$ because of the factor $(q^{-m_i})_{\mu_i}$. Given that

$$N > m_1 + m_2 + \cdots + m_{k-1} + t,$$

we see that

$$N - t > m_1 + \cdots + m_{k-1} \geq \mu_1 + \cdots \mu_{k-1}.$$

Hence every term of the sum in (2.4) is 0; therefore (1.9) is valid for K_k.

Next suppose that

$$N = m_1 + \cdots m_{k-1} + t.$$

The previous argument shows that now the only non-vanishing term in the inner sum occurs for $\mu_i = m_i$, $1 \leq i \leq k-1$. Hence, in this case, by (2.4) and the induction hypothesis

$$
\begin{aligned}
&K_k(a, N; m_1, \ldots, m_{k-1}, t) \\
&\quad = \frac{(a)_{2t}(1-aq^{2t})(q)_N q^{-t}}{(aq^{N+1})_t(1-a)(q)_{N-t}} \prod_{i=1}^{k-1} (aq^{t+1})_{m_i}(q^{-t})_{m_i} \\
&\quad\quad \times \prod_{i=1}^{k-1} \frac{(q^{-m_i})_{m_i} q^{m_i}}{(q)_{m_i}(q^{t-m_i+1})_{m_i}(at^{t+1})_{m_i}} \\
&\quad\quad \times q^{-\sigma_2(m_1,\ldots,m_{k-1})-\sigma_1(m_1,\ldots,m_{k-1})}(aq^{2t+1})_{N-t}(q)_{N-t} \\
&\quad = \frac{(aq)_{2t}(q)_N q^{-t}}{(aq^{N+1})_t(q)_{N-t}} \prod_{i=1}^{k-1} \frac{(aq^{t+1})_{m_i}(-1)^{m_i} q^{-tm_i+\binom{m_i}{2}}(q)_t}{(q)_{t-m_i}} \\
&\quad\quad \times \prod_{i=1}^{k-1} \frac{q^{-\binom{m_i}{2}}(q)_{m_i}(q)_{t-m_i}}{(q)_{m_i}(q)_t(aq^{t+1})_{m_i}} \\
&\quad\quad \times q^{-\sigma_2(m_1,\ldots,m_{k-1})-\sigma_1(m_1,\ldots,m_{k-1})}(aq^{2t+1})_{N-t}(q)_{N-t} \\
&\quad = q^{-\sigma_2(m_1,\ldots,m_{k-1},t)-\sigma_1(m_1,\ldots,m_{k-1},t)}(aq)_N(q)_N.
\end{aligned}
$$

Thus (1.9) and (1.10) have been established by mathematical induction on k. □

3 Proof of Theorem 2

As noted previously (1.13) was proved in section 2. Equation (1.12) follows immediately from the classical summation [8, p.238, eq.(II.21)]:

$$
{}_6\phi_5 \left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, q^{-N}; q, \frac{aq^{1+N}}{b_1c_1} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b_1}, \frac{aq}{c_1}, aq^{N+1} \end{matrix} \right) = \frac{(aq)_N \left(\frac{aq}{b_1c_1}\right)_N}{\left(\frac{aq}{b_1}\right)_N \left(\frac{aq}{c_1}\right)_N} \tag{3.1}
$$

To treat K_3, we must utilize (2.3). So we shall assume m_3 is not exceeded by m_1 or m_2. Symmetry allows these assumptions without loss of generality.

Hence

$$K_3(a,N;m_1,m_2,m_3)$$

$$= \frac{(a)_{2m_3}(1-aq^{2m_3})(q^{-N})_{m_3}q^{Nm_3-\binom{m_3+1}{2}}(-1)^{m_3}}{(aq^{N+1})_{m_3}(1-a)}$$

$$\times \frac{(a)_{m_1+m_3}(a)_{m_2+m_3}(q)^2_{m_3}(-1)^{m_1+m_2}q^{\binom{m_1}{2}+\binom{m_2}{2}-m_3(m_1+m_2)}}{(a)^2_{m_3}(q)_{m_3-m_1}(q)_{m_3-m_2}}$$

$$\times {}_8\phi_7\left(\begin{matrix}aq^{2m_3}, q^{m_3+1}\sqrt{a}, -q^{m_3+1}\sqrt{a}, aq^{m_1+m_3}, q^{m_3+1}, aq^{m_2+m_3}, q^{m_3+1}, q^{-N+m_3}; q, q^{N-m_1-m_2-m_3} \\ q^{m_3}\sqrt{a}, -q^{m_3}\sqrt{a}, q^{m_3-m_1+1}, aq^{m_3}, q^{m_3-m_2+1}, aq^{m_3}, aq^{N+m_3+1}\end{matrix}\right)$$

$$= C(a,N;m_1,m_2,m_3) \tag{3.2}$$

$$\times {}_8\phi_7\left(\begin{matrix}aq^{2m_3}, q^{m_3+1}\sqrt{a}, -q^{m_3+1}\sqrt{a}, aq^{m_1+m_3}, q^{m_3+1}, aq^{m_2+m_3}, q^{m_3+1}, q^{-N+m_3}; q, q^{N-m_1-m_2-m_3} \\ q^{m_3}\sqrt{a}, -q^{m_3}\sqrt{a}, q^{m_3-m_1+1}, aq^{m_3}, q^{m_3-m_2+1}, aq^{m_3}, aq^{N+m_3+1}\end{matrix}\right) \tag{3.3}$$

where we have written $C(a,N;m_1,m_2,m_3)$ for the multiplying product.

We now apply Watson's q-analog of Whipple's theorem [8, p.242, eq.(III.18)] to the ${}_8\phi_7$. Hence

$$K_3(a,N;m_1,m_2,m_3) = C(a,N;m_1,m_2,m_3)$$

$$\times \frac{(aq^{2m_3+1})_{N-m_3}(q^{-m_2})_{N-m_3}}{(q^{m_3-m_2+1})_{N-m_3}(aq^{m_3})_{N-m_3}}$$

$$\times {}_4\phi_3\left(\begin{matrix}q^{-m_1}, aq^{m_2+m_3}, q^{m_3+1}, q^{-N+m_3}; q, q \\ q^{m_3-m_1+1}, aq^{m_3}, q^{m_2,+m_3-N+1}\end{matrix}\right)$$

$$= \frac{C(a,N;m_1,m_2,m_3)(aq^{2m_3+1})_{N-m_3}(q^{-m_2})_{N-m_3}}{(q^{m_3-m_2+1})_{N-m_3}(aq^{m_3})_{N-m_3}}$$

$$\times \frac{(q^{-m_1})_{m_1}(q^{m_2-N})_{m_1}}{(q^{m_3-m_1+1})_{m_1}(q^{m_2+m_3-N+1})_{m_1}}$$

$$\times\ {}_4\phi_3\left(\begin{matrix} q^{-m_1}, q^{m_3+1}, q^{-m_2}, aq^N; q, q \\ aq^{m_3}, q, q^{1-m_1-m_2+N} \end{matrix}\right)$$

(by [8, p.242, eq.(III.15), $n = m_1, a = q^{m_3+1}, b = aq^{m_2+m_3}, c = q^{-N+m_3}$, $d = aq^{m_3}, e = q^{m_3-m_1+1}, f = q^{m_2+m_3-N+1}$])

$$= \frac{C(a, N; m_1, m_2, m_3)(aq^{2m_3+1})_{N-m_3}(q^{-m_2})_{N-m_3}(q^{-m_1})_{m_1}(q^{m_2-N})_{m_1}}{(q^{m_3-m_2+1})_{N-m_3}(aq^{m_3})_{N-m_3}(q^{m_3-m_1+1})_{m_1}(q^{m_2+m_3-N+1})_{m_1}}$$

$$\times \frac{(aq^{m_3+m_2})_{m_1}(q^{1-m_1+N})_{m_1}}{(aq^{m_3})_{m_1}(q^{1-m_1-m_2+N})_{m_1}}$$

$$\times\ {}_4\phi_3\left(\begin{matrix} q^{-m_1}, q^{-m_2}, q^{-m_3}, \frac{q^{1-N}}{a}; q; q \\ q, \frac{q^{1-m_1-m_2-m_3}}{a}, q^{-N} \end{matrix}\right)$$

(by [8, p.242, eq.(III.15), $n = m_1$, $a = q^{-m_2}$, $b = q^{m_3+1}$, $c = aq^N$, $d = q$, $e = aq^{m_3}$, $f = q^{1-m_1-m_2-N}$]).

Simplification of the multiplying products yields (1.14). □

4 Conclusion

There are many unanswered questions about the pair-symmetric kernel K_k. Here are some of the most important.

1. Are there simplified expansions like (1.12), (1.13) and (1.14) for $k > 3$ that both exibit symmetry explicitly and that yield (1.9) and (1.10) reasonably directly. We should note that (1.14) reduces to (1.10) when $N = m_1 + m_2 + m_3$ because the ${}_4\phi_3$ in (1.14) reduces to a balanced ${}_3\phi_2$ which is summable [8, p.237, eq.(II.12)].

2. What is the relationship of (1.7) to (1.8)? In the case $k = 2$, one can pass easily from (1.8) to (1.7) by noting that both the m_1 and m_2 sums are each balanced (and thus summable ${}_3\phi_2$'s).

3. The most notable instance of using the paired symmetry of the k pairs of parameters in (1.7) occurs in the study of Durfee symbols (cf. [1],[5]). Indeed obvious symmetry of partition statistics was very difficult to establish [6]. It would be of interest to pursue this question using an expansion like (1.8) where the symmetry is clearly in evidence.

4. The proof of Theorem 1 is some sort of "multiple Bailey Lemma". It is clear that multiple simultaneous applications of (2.1) can be used instead of the sequential applications in the standard Bailey chain productions [4]. The possibilities here are endless.

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