EULER'S PARTITION IDENTITY – FINITE VERSION

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Abstract. Euler proved that the number of partition of n into odd parts equals the number of partitions of n into distinct parts. There have been several refinements of Euler's Theorem which have limited the size of the parts allowed. Each is surprising and difficult to prove. This paper provides a finite version of Glaisher's exquisitely elementary proof of Euler's Theorem.

1. Introduction

Euler is truly the father of the theory of the partitions of integers. He discovered the following prototype of all subsequent partition identities.

> **Euler's Theorem [3].** The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

For example, if $n = 10$, then the ten odd partitions of n into distinct parts are

$$\begin{array}{ccccc} 10, & 9+1, & 8+2, & 7+3, & 7+2+1, \\ 6+4, & 6+3+1, & 5+4+1, & 5+3+2, & 4+3+2+1, \end{array}$$

and the ten partitions of n into odd parts are

$$\begin{array}{ccc} 9+1, & 7+3, & 7+1+1+1, \\ 5+5, & 5+3+1+1, & 5+1+1+1+1+1, \\ 3+3+3+1, & 3+3+1+1+1+1, & 3+1+1+1+1+1+1+1, \\ & 1+1+1+1+1+1+1+1+1+1. & \end{array}$$

Euler's proof was an elegant use of generating functions. If $\mathcal{D}(n)$ denotes the number of partitions of n into distinct parts and $\mathcal{O}(n)$ denotes the number of partitions of n into odd parts, then it is immediate (just by multiplying out the products and collecting terms) that

$$\sum_{n\geq 0} \mathcal{D}(n) q^n = \prod_{n=1}^{\infty} (1+q^n) \tag{1.1}$$

and

$$\begin{aligned} \sum_{n\geq 0} \mathcal{O}(n) q^n &= \prod_{n=1}^{\infty} (1+q^{2n-1}+q^{2(2n-1)}+\cdots) \\ &= \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}}. \end{aligned} \tag{1.2}$$

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Euler's proof is then an algebraic exercise:

$$\begin{aligned}\sum_{n\geq 0}\mathcal{D}(n)q^n &= \prod_{n=1}^{\infty}(1+q^n) \\ &= \prod_{n-1}^{\infty}\frac{1-q^{2n}}{1-q^n} \\ &= \prod_{n-1}^{\infty}\frac{1}{1-q^{2n-1}} \\ &= \sum_{n\geq 0}\mathcal{O}(n)q^n.\end{aligned} \tag{1.3}$$

It was J. W. L. Glaisher [5] who in 1883 found a purely bijective proof of Euler's Theorem. Glaisher's mapping goes as follows: start with a partition of n into odd parts (here f_i is the number of times $(2m_i-1)$ appears as a part):

$$f_1(2m_1-1)+f_2(2m_2-1)+\cdots+f_r(2m_r-1). \tag{1.4}$$

Now write each f_i in its unique binary representation as a sum of distinct powers of 2, i.e., now we have (with $a_1(i)<a_2(i)<\cdots$)

$$\begin{aligned}\sum_{i=1}^{r}f_i(2m_i-1) &= \sum_{i=1}^{r}(2^{a_1(i)}+2^{a_2(i)}+\cdots+2^{a_j(i)})(2m_i-1) \\ &= \sum_{i=1}^{r}\left(2^{a_1(i)}(2m_i-1)+\cdots+2^{a_j(i)}(2m_i-1)\right)\end{aligned} \tag{1.5}$$

and this last expression is the image partition into distinct parts. To make Glaisher's maps concrete, let us return to the case $n=10$.

$9+1 \quad\to\quad 1\cdot 9+1\cdot 1 \quad\to\quad 9+1$

$7+3 \quad\to\quad 1\cdot 7+1\cdot 3 \quad\to\quad 7+3$

$7+1+1+1 \quad\to\quad 1\cdot 7+3\cdot 1 \quad\to\quad 1\cdot 7+(2{+}1)\cdot 1 \quad\to\quad 7+2+1$

$5+5 \quad\to\quad 2\cdot 5 \quad\to\quad 10$

$5+3+1+1 \quad\to\quad 1\cdot 5+1\cdot 3+1\cdot 2 \quad\to\quad 5+3+2$

$5+1+1+1+1+1 \quad\to\quad 1\cdot 5+5\cdot 1$

$\quad\to\quad 1\cdot 5+(4{+}1)\cdot 1 \quad\to\quad 5+4+1$

$3+3+3+1 \quad\to\quad 3\cdot 3+1\cdot 1 \quad\to\quad (2{+}1)\cdot 3+1\cdot 1 \quad\to\quad 6+3+1$

$3+3+1+1+1+1 \quad\to\quad 2\cdot 3+4\cdot 1 \quad\to\quad 6+4$

$3+1+1+1+1+1+1+1 \quad\to\quad 1\cdot 3+7\cdot 1$

$\quad\to\quad 1\cdot 3+(1{+}2{+}4)\cdot 1 \quad\to\quad 4+3+2+1$

$1+1+1+1+1+1+1+1+1+1 \quad\to\quad 10\cdot 1 \quad\to\quad (8{+}2)\cdot 1 \quad\to\quad 8+2$

It is clear that this map is reversible; just collect together in groups those parts with common largest odd factor.

Now there have been a number of refinements of Euler's Theorem which have, in one way or another, placed restrictions on the size of the parts used. Bousquet-Mélou and Eriksson [1] have a version in which their "lecture hall partitions" occur. Nathan Fine [4] has a version involving the Dyson rank. I. Pak [6] devotes Section 3 of his exhaustive study of partition identities to a variety of refinements of Euler's Theorem.

The point of this short note is to provide a simple Glaisher style proof of the following finite version of Euler's Theorem due to Bradford, Harris, Jones, Komarinski, Matson, and O'Shea that was first stated in [2].

> **Theorem [2; Sec. 3].** The number of partitions of n into odd parts each $\leq 2N$ equals the number of partitions of n into parts each $\leq 2N$ in which the parts $\leq N$ are distinct.

It should be noted that the bijective proofs in [2] as well as those by Bousquet-Melou and Erickson in [1] and Yee in [7] prove much more than the above theorem and are thus much more complicated than our Glaisher-like bijection.

2. First proof of the theorem

This result has an Eulerian proof that has exactly the simplicity of Euler's original proof.

Let $\mathcal{O}_N(n)$ denotes the number of partitions of n in which each part is odd and $\leq 2N$, and $\mathcal{D}_N(n)$ denotes the number of partitions of n in which each part is $\leq 2N$ and all parts $\leq N$ are distinct. Thus

$$\sum_{n\geq 0} \mathcal{O}_N(n)q^n = \prod_{n=1}^{N} \frac{1}{1-q^{2n-1}}$$

and

$$\sum_{n\geq 0} \mathcal{D}_N(n)q^n = \prod_{n=1}^{N} \frac{1+q^n}{1-q^{N+n}}.$$

Finally

$$\begin{aligned}\sum_{n\geq 0} \mathcal{D}_N(n)q^n &= \prod_{n=1}^{N} \frac{1-q^{2n}}{(1-q^n)(1-q^{N+n})}\\ &= \frac{\prod_{n=1}^{N}(1-q^{2n})}{\prod_{n=1}^{2N}(1-q^n)}\\ &= \prod_{n=1}^{N} \frac{1}{1-q^{2n-1}}\\ &= \sum_{n\geq 0} \mathcal{O}_N(n)q^n.\end{aligned}$$

3. A Glaisher-type proof

Now we return to Glaisher's proof, with the following alteration. Namely, for each odd part $(2m_i - 1)$ (all being $\leq 2N$) there is a unique $j_i \geq 0$ such that

$$N < (2m_i - 1)2^{j_i} \leq 2N.$$

Now instead of rewriting each f_i completely in binary, we instead write f_i (with $a_1(i) < a_2(i) < \cdots < a_m(i) < f_i$) as

$$2^{a_1(i)} + 2^{a_2(i)} + \cdots + 2^{a_m(i)} + g_i 2^{j_i},$$

where, of course, g_i might be 0. Thus, instead of (1.5) we now have

$$\begin{aligned}\sum_{i=1}^{r} &= f_i(2m_i - 1)\\ &= \sum_{i=1}^{r}(2^{a_1(i)} + 2^{a_2(i)} + \cdots + 2^{a_m(i)} + g_i 2^{j_i})(2m_i - 1)\\ &= \sum_{i=1}^{r}\left(2^{a_1(i)}(2m_i - 1) + 2^{a_2(i)}(2m_i - 1) + \cdots\right) + \sum_{i=1}^{r} g_i 2^{j_i}(2m_i - 1)\end{aligned}$$

and the latter expression is a partition wherein the parts $\leq N$ are distinct and each is $\leq 2N$.

As an example, let us consider a partition of 78 into odd parts each $\leq 2N = 2\cdot 6$:

$$11 + 11 + 7 + 7 + 5 + 5 + 5 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3.$$

Now 11 and 7 lie in $(6, 12]$, $2 \cdot 5 \in (6, 12]$, and $4 \cdot 3 \in (6, 12]$. Hence this partition is

$$\begin{aligned}2 \cdot 11 &+ 2 \cdot 7 + 3 \cdot 5 + 9 \cdot 3\\ &= 2 \cdot 11 + 2 \cdot 7 + (1 + 2) \cdot 5 + (1 + 2 \cdot 4) \cdot 3\\ &= 11 + 11 + 7 + 7 + 5 + 10 + 3 + 2 \cdot (4 \cdot 3)\\ &= 11 + 11 + 7 + 7 + 5 + 10 + 3 + 12 + 12\end{aligned}$$

and this last expression has all parts ≤ 12 and no repeated parts ≤ 6.

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References

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