The Alladi-Schur Polynomials and Their Factorization

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Dedicated to Krishna Alladi on his 60^{th} birthday.

Abstract

K. Alladi first observed the following variant of I. Schur's 1926 partition theorem. Namely, the number of partitions of n in which all parts are odd and none appears more than twice equals the number of partitions of n in which all parts differ by at least 3 and more than 3 if one of the parts is a multiple of 3. Subsequently the theorem was refined to count also the number of parts in the relevant partitions. In this paper, a surprising factorization of the related polynomial generating functions is developed.

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1 Introduction

In 1926, I. Schur [6] proved the following result:

Theorem. *Let $A(n)$ denote the number of partitions of n into parts congruent to* $\pm 1 \pmod 6$*. Let $B(n)$ denote the number of partitions of n into distinct nonmultiples of 3. Let $D(n)$ denote the number of partitions of n of the form $b_1 + b_2 + \cdots + b_s$ where $b_i - b_{i+1} \geq 3$ with strict inequality if $3|b_i$. Then*

$$A(n) = B(n) = D(n).$$

K. Alladi [1] has pointed out (cf. [2, p. 46, eq. (1.3)]) that if we define $C(n)$ to be the number of partitions of n into odd parts with none appearing more than twice, then also

$$C(n) = D(n).$$

Recently [3] it was shown that a refinement (in the spirit of Gleissberg's refinement [4] of Schur's original theorem [6]) is valid:

Theorem. *Let $C(m,n)$ denote the number of partitions of n into m parts, all odd and none appearing more than twice. Let $D(m,n)$ denote the number of partitions of n into parts of the type enumerated by $D(n)$ with the added condition that the total number of parts plus the number of even parts is m (i.e. m is the weighted count of parts where each even is counted twice). Then*

$$C(m,n) = D(m,n).$$

The proof relied on a study of the generating function of $D_N(m,n)$ the number of partitions of the type enumerated by $D(m,n)$ with the added restriction that each part be $\leq N$. Thus

$$d_N(x) = \sum_{n,m\geq 0} D_N(m,n)x^m q^n.$$

In fact, the above theorem was directly deduced from the functional equations

$$d_{6n+2}(x) = (1+xq+x^2q^2)d_{6n-1}(xq^2), \tag{1.1}$$

$$d_{6n-1}(x) = (1+xq+x^2q^2)\{d_{6n-4}(xq^2) + xq^{6n-1}(1-qx)d_{6n-7}(xq^2)\}, \tag{1.2}$$

where $d_{-1}(x)$ is defined to be 1.

It turns out that much more than this is true.

Theorem 1. *For $n \geq 3$, with $d_{-1}(x) = 1$,*

$$d_{2n}(x) = (1+xq+x^2q^2)d_{2n-3}(xq^2), \tag{1.3}$$

$$d_{2n-1}(x) = (1+xq+x^2q^2)\{d_{2n-4}(xq^2) + xq^{2n-1}(1-qx)d_{2n-7}(xq^2)\}. \tag{1.4}$$

From Theorem 1, it is possible to provide a factorization of the $d_n(x)$. We define

$$p_n(x) = \prod_{j=1}^{n}(1+xq^{2j-1}+x^2q^{4j-2}). \tag{1.5}$$

Theorem 2. *If* $n \not\equiv 3 \pmod 6$, *then* $p_{\lfloor \frac{n+4}{6} \rfloor}(x)$ *divides* $d_n(x)$. *If* $n \equiv 3 \pmod 6$, *then* $p_{\lfloor \frac{n-2}{6} \rfloor}(x)$ *divides* $d_n(x)$.

Finally it is possible to give a full account of the quotient arising in the division given in Theorem 2.

Theorem 3.

$$d_{6n-1}(x) = p_n(x) \sum_{j=0}^{n} c(n,j) x^j, \tag{1.6}$$

where

$$c(n,j) = \sum_{r=0}^{j} \sum_{0 \le 3i \le r} \frac{(-1)^i q^{4nj-2nr+j+3i(i-1)} (q^2;q^2)_n}{(q^2;q^2)_{n-j} (q^2;q^2)_{j-r} (q^2;q^2)_{r-3i} (q^6;q^6)_i}. \tag{1.7}$$

From Theorem 3, one can deduce explicit formulas for the other $d_{6n-i}(x)$, and we will discuss this in the conclusion.

The paper concludes with a discussion of other possible factorization theorems in the theory of partitions.

It should be emphasized that, in some real sense, the intrinsic theorem is the Alladi-Schur theorem. Not only do we see that

$$d_\infty(x) = p_\infty(x),$$

but also the partial products of $p_\infty(x)$ as revealed in Theorems 2 and 3 are naturally arising as n increases. None of the other variants of Schur's theorem reveals the successive appearance of the relevant partial products.

2 Proof of Theorem 1.

Theorem 1 is actually an extension of Lemma 3 in [3] to its full generality, and the proof builds upon what was proved there.

Proof of Theorem 1. Let

$$\chi(n) = \begin{cases} 1 & \text{if } 3|n \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

then the recurrences (2.2)–(2.4) of [3] can be rewritten as

$$d_{2n}(x) = d_{2n-1}(x) + x^2 q^{2n} d_{2n-3-\chi(2n)}(x), \tag{2.2}$$

$$d_{2n-1}(x) = d_{2n-2}(x) + xq^{2n-1} d_{2n-4-\chi(2n-1)}(x). \tag{2.3}$$

Next we define

$$\mathcal{F}(n) = d_{2n+2}(x) - (1 + xq + x^2q^2) d_{2n-1}(xq^2), \tag{2.4}$$

and

$$\mathcal{G}(n) = d_{2n-1}(x) - (1 + xq + x^2q^2)(d_{2n-4}(xq^2) + xq^{2n-1}(1 - xq) d_{2n-7}(xq^2)). \tag{2.5}$$

To prove (1.3) and (1.4), we only need to show that for $n \geq 3$,

$$\mathcal{F}(n) = \mathcal{G}(n) = 0.$$

Now

$$\begin{aligned}
\mathcal{F}(3) =& d_8(x) - (1 + xq + xq^2) d_5(xq^2) \\
=& (1 + xq^5 + xq^7)(1 + xq + x^2q^2)(1 + xq^3 + x^2q^6) \\
& - (1 + xq + x^2q^2)\{(1 + xq^3 + x^2q^6)(1 + xq^5 + xq^7)\} \\
=& 0,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(3) =& d_5(x) - (1 + xq + xq^2)\{d_2(xq^2) + xq^5(1 - xq) d_{-1}(xq^2)\} \\
=& 1 + xq + x^2q^2 + xq^3 + x^2q^4 + xq^5 + x^3q^5 \\
& + x^2q^6 + x^3q^7 - (1 + xq + xq^2)(1 + x^2q^3 + x^2q^6) \\
& + xq^5(1 - xq)(1 + xq + xq^2) \\
=& 0.
\end{aligned}$$

In the following, we write for simplicity

$$\lambda(x) = 1 + xq + x^2q^2.$$

Now Lemma 3 of [3] asserts that for $n \geq 1$,

$$\mathcal{F}(3n) = \mathcal{G}(3n) = 0.$$

Hence by (2.4) and (2.5)

$$\mathcal{F}(3n - 1) = \mathcal{F}(3n - 1) - \mathcal{G}(3n) - x^2q^{6n}\mathcal{F}(3n - 3)$$

$$
\begin{aligned}
=&(d_{6n}(x) - \lambda(x)d_{6n-3}(xq^2)) \\
&- (d_{6n-1}(x) - \lambda(x)d_{6n-4}(xq^2) - \lambda(x)xq^{6n-1}(1-xq)d_{6n-7}(xq^2)) \\
&- x^2q^{6n}(d_{6n-4}(x) - \lambda(x)d_{6n-7}(xq^2)) \\
=&(d_{6n}(x) - d_{6n-1}(x) - x^2q^{6n}d_{6n-4}(x)) \\
&- \lambda(x)(d_{6n-3}(xq^2) - d_{6n-4}(xq^2) - xq^{6n-1}d_{6n-7}(xq^2)) \\
=&0,
\end{aligned}
$$

by (2.2) and (2.3). So $\mathcal{F}(3n-1)$ is identically 0 for $n \geq 2$.

Next

$$
\begin{aligned}
\mathcal{G}(3n+2) =&\mathcal{G}(3n+2) - \mathcal{F}(3n) - xq^{6n+3}\mathcal{G}(3n) \\
=&(d_{6n+3}(x) - \lambda(x)d_{6n}(xq^2) - xq^{6n+3}(1-xq)\lambda(x)d_{6n-3}(xq^2)) \\
&- (d_{6n+2}(x) - \lambda(x)d_{6n-1}(xq^2)) \\
&- xq^{6n+3}(d_{6n-1}(x) - \lambda(x)d_{6n-4}(xq^2) - \lambda(x)xq^{6n-1}(1-xq)d_{6n-7}(xq^2)) \\
=&(d_{6n+3}(x) - d_{6n+2}(x) - xq^{6n+3}d_{6n-1}(x)) \\
&- \lambda(x)(d_{6n}(xq^2) - d_{6n-1}(xq^2) - x^2q^{6n+4}d_{6n-4}(xq^2)) \\
&- xq^{6n+3}(1-xq)\lambda(x)(d_{6n-3}(xq^2) - d_{6n-4}(xq^2) \\
&- xq^{6n-1}d_{6n-7}(xq^2)) \\
=&0,
\end{aligned}
$$

by (2.2) and (2.3). So $\mathcal{G}(3n+2)$ is identically 0, for $n \geq 1$.

Next,

$$
\begin{aligned}
\mathcal{F}(3n-2) =& - (\mathcal{G}(3n) - \mathcal{F}(3n-2) - xq^{6n-1}\mathcal{F}(3n-3)) \\
=& - (d_{6n-1}(x) - \lambda(x)d_{6n-4}(xq^2) - xq^{6n-1}(1-xq)\lambda(x)d_{6n-7}(xq^2)) \\
&+ (d_{6n-2}(x) - \lambda(x)d_{6n-5}(xq^2)) \\
&+ xq^{6n-1}(d_{6n-4}(x) - \lambda(x)d_{6n-7}(xq^2)) \\
=& - (d_{6n-1}(x) - d_{6n-2}(x) - xq^{6n-1}d_{6n-4}(x)) \\
&+ \lambda(x)(d_{6n-4}(xq^2) - d_{6n-5}(xq^2) - x^2q^{6n}d_{6n-7}(xq^2)) \\
=&0,
\end{aligned}
$$

by (2.2) and (2.3). Thus $\mathcal{F}(3n-2)$ is identically 0, for $n \geq 2$.

Finally

$$
\mathcal{G}(3n+1) = - (\mathcal{F}(3n) - \mathcal{G}(3n+1) - x^2q^{6n+2}\mathcal{G}(3n))
$$

$$
\begin{aligned}
=& -(d_{6n+2}(x) - \lambda(x)d_{6n-1}(xq^2)) \\
&+ (d_{6n+1}(x) - \lambda(x)d_{6n-2}(xq^2) - xq^{6n+1}(1-xq)\lambda(x)d_{6n-5}(xq^2)) \\
&+ x^2q^{6n+2}(d_{6n-1}(x) - \lambda(x)d_{6n-4}(xq^2) - xq^{6n-1}(1-xq)\lambda(x)d_{6n-7}(xq^2)) \\
=& -(d_{6n+2}(x) - d_{6n+1}(x) - x^2q^{6n+2}d_{6n-1}(x)) \\
&+ \lambda(x)(d_{6n-1}(xq^2) - d_{6n-2}(xq^2) - xq^{6n+1}d_{6n-4}(xq^2)) \\
&+ xq^{6n+1}\lambda(x)(1-xq)(d_{6n-4}(xq^2) - d_{6n-5}(xq^2) - x^2q^{6n}d_{6n-7}(xq^2)) \\
=&0,
\end{aligned}
$$

by (2.2) and (2.3). Thus $\mathcal{G}(3n+1)$ is identically 0 for $n \geq 1$, and Theorem 1 is proved. □

3 Proof of Theorem 2.

This result is essentially a corollary of Theorem 1, but is of major significance in Theorem 3 and is the factorization referred to in the title.

Proof of Theorem 2. The succinct assertion of Theorem 3 may be stated more comprehensibly as follows. We are to prove that there exist polynomials

$$\triangle(i,n) = \triangle(i,n,x,q),$$

such that

$$
\begin{aligned}
d_{6n+1}(x) &= p(n)\triangle(-1,n) && (3.1)\\
d_{6n}(x) &= p(n)\triangle(0,n) && (3.2)\\
d_{6n-1}(x) &= p(n)\triangle(1,n) && (3.3)\\
d_{6n-2}(x) &= p(n)\triangle(2,n) && (3.4)\\
d_{6n-3}(x) &= p(n-1)\triangle(3,n) && (3.5)\\
d_{6n-4}(x) &= p(n)\triangle(4,n) && (3.6)
\end{aligned}
$$

Now by (1.3), note

$$
\begin{aligned}
d_7(x) &= p(1)(1 + x(q^3 + q^5 + q^7) + x^2(q^6 + q^{10})) && (3.7)\\
d_6(x) &= p(1)(1 + x(q^3 + xq^5) + x^2q^6) && (3.8)\\
d_5(x) &= p(1)(1 + xq^3 + xq^5) && (3.9)\\
d_4(x) &= p(1)(1 + xq^3) && (3.10)\\
d_3(x) &= p(0)(1 + x(q + q^3) + x^2q^2) && (3.11)
\end{aligned}
$$

$$d_2(x) = p(1) \tag{3.12}$$

so the case $n = 1$ is established.

Now assume (3.1)–(3.6) are proved up to but not including a given n. Then by (2.2) and (2.3),

$$\begin{aligned} d_{6n+1}(x) &= d_{6n}(x) + xq^{6n+1}d_{6n-2}(x) \\ &= p(n)(\triangle(0,n) + xq^{6n+1}\triangle(2,n), \end{aligned}$$

$$\begin{aligned} d_{6n}(x) &= d_{6n-1}(x) + x^2q^{6n}d_{6n-4}(x) \\ &= p(n)(\triangle(1,n) + x^2q^{6n}\triangle(4,n)) \end{aligned}$$

$$\begin{aligned} d_{6n-1}(x) &= d_{6n-2}(x) + xq^{6n-1}d_{6n-4}(x) \\ &= p(n)(\triangle(2,n) + xq^{6n-1}\triangle(4,n)) \end{aligned}$$

$$\begin{aligned} d_{6n-2}(x) &= (1 + xq + x^2q^2)d_{6n-5}(xq^2) \\ &= p(n)\triangle(-1, n-1, xq^2, q), \end{aligned}$$

$$\begin{aligned} d_{6n-3}(x) &= d_{6n-2}(x) + xq^{6n-3}d_{6n-7}(x) \\ &= p(n)\triangle(2,n) + xq^{6n-3}p(n-1)\triangle(1,n-1) \\ &= p(n-1)((1 + xq^{2n-1} + x^2q^{4n-2})\triangle(2,n) + xq^{6n-3}\triangle(1,n-1)), \end{aligned}$$

and finally by (1.3),

$$\begin{aligned} d_{6n-4}(x) &= (1 + xq + xq^2)d_{6n-5}(xq^2) \\ &= p(n)\triangle(-1, n-1, xq^2, q), \end{aligned}$$

and our theorem is proved. □

4 Proof of Theorem 3.

This result seems to require a rather elaborate proof. In order to make Theorem 3 comprehensible, we shall prove a number of preliminary lemmas.

We begin by defining

$$\overline{c}(n,j) := \sum_{r=0}^{j} \sum_{0 \le 3i \le r} \frac{(-1)^i q^{4nj-2nr+j+3i(i-1)}(q^2;q^2)_n}{(q^2;q^2)_{n-j}(q^2;q^2)_{j-r}(q^2;q^2)_{r-3j}(q^6;q^6)_i}. \tag{4.1}$$

Clearly Theorem 3 reduces to proving that, in fact, $c(n,j) = \overline{c}(n,j)$.

We note that, of the partitions enumerated by $d_{2n-1}(x)$, the one that provides the largest x-exponents is

$$4 + 7 + 10 + \cdots + 6n - 2,$$

yielding x^{3n}. Furthermore by (3.3), and noting that $p(n)$ is of degree $2n$ in x, we must have $\triangle(1,n)$ of degree n. So $c(n,j) = 0$ if $j < 0$ or $j > n$.

Lemma 4.

$$c(n,j) = \begin{cases} 1 & \text{if } n = j = 0 \\ 0 & \text{if } n < 0, j \leq 0; n \leq 0, j < 0, j > n \text{ and for } n > 0 \end{cases} \tag{4.2}$$

$$\begin{aligned} c(n,j) = q^{4j} c(n-1,j) &+ (q^{2n+4j-3} + q^{6n+2j-3}) c(n-1,j-1) \\ &+ (q^{4j+4n-6} - q^{6n+2j-4}) c(n-1,j-2). \end{aligned} \tag{4.3}$$

Proof. By (1.3) with n replaced by $3n-2$

$$\begin{aligned} d_{6n-4}(x) &= p(n)\triangle(4,n,x,q) \\ &= p(n)\triangle(1,n-1,xq^2,q). \end{aligned} \tag{4.4}$$

Therefore by (1.4) with n replaced by $3n$

$$d_{6n-1}(x) = \lambda(x)\{d_{6n-4}(xq^2) xq^{6n-1}(1-qx) d_{6n-7}(xq^2)\}. \tag{4.5}$$

So

$$\begin{aligned} \sum_{j=0}^{n} c(n,j)x^j = &\sum_{j=0}^{n} c(n-1,j)(xq^4)^j (1 + xq^{2n+1} + x^2 q^{4n+2}) \\ &+ xq^{6n-1}(1-xq) \sum_{j=0}^{n} c(n-1,j) x^j q^{2j}. \end{aligned}$$

Hence

$$\begin{aligned} c(n,j) = &c(n-1,j)q^{4j} + (q^{2n-3+4j} + q^{6n+2j-3}) c(n-1,j-1) \\ &+ (q^{4j+4n-6} - q^{6n+2j-4}) c(n-1,j-2) \end{aligned} \tag{4.6}$$

as desired. □

Lemma 5. For $n \geq 0$,

$$\begin{aligned} c(n,j)(1-q^{2j}) =& c(n,j-1)q^{2n+2j-1}(1-q^{4n-2j+2}) \\ &+ c(n,j-2)q^{4n+2j-2}(1-q^{2n-2j+4}) \\ &- c(n-1,j-2)q^{6n+2j-4}(1-q^{6n}). \end{aligned} \tag{4.7}$$

Proof. By (2.7) of [3] rewritten twice, first with n replaced by $n+1$, we see that

$$\begin{aligned} p_{n+1}(x)\sum_{j=0}^{n} c(n,j)(xq^2)^j =& (1+xq^{6n+1}+x^2q^{6n+2})p_n(x)\sum_{j=0}^{n} c(n,j)x^j \\ &+ x^2q^{6n}(1-q^{6n})p_n(x)\sum_{j=0}^{n-1} c(n-1,j)(xq^2)^j. \end{aligned}$$

Now noting that $p_{n+1}(x)/p_n(x) = 1+xq^{2n+1}+x^2q^{4n+2}$, and dividing this last equation by $p_n(x)$, we obtain

$$\begin{aligned} &(1+xq^{2n+1}+x^2q^{4n+2})\sum_{j=0}^{n} c(n,j)(xq^2)^j \\ &=(1+xq^{6n+1}+x^2q^{6n+2})\sum_{j=0}^{n} c(n,j)x^j \\ &\quad + x^2q^{6n}(1-q^{6n})\sum_{j=0}^{n-1} c(n-1,j)(xq^2)^j. \end{aligned}$$

Finally comparing coefficients of x^j on both sides, we deduce

$$\begin{aligned} &q^{2j}c(n,j)+q^{2n+2j-1}c(n,j-1)+q^{4n+2j-2}c(n,j-2) \\ &=c(n,j)+q^{6n+1}c(n,j-1)+q^{6n+2}c(n,j-2) \\ &\quad + q^{6n+2j-4}(1-q^{6n})c(n-1,j-2), \end{aligned}$$

which is equivalent to (4.5). □

From Lemma 5, we shall deduce a recurrence for $c(n,j)$ where only j varies.

Lemma 6.

$$
\begin{aligned}
0 =&(q^{6j}-q^{4j})c(n-1,j) \\
&+(q^{2n+6j-3}-q^{6n+2j-3}+q^{2n+6j-5}+q^{2n+4j-3})c(n-1,j-1) \\
&+(q^{4n+6j-6}-q^{6n+4j-6}-q^{6n+4j-4} \\
&+q^{4n+6j-8}+q^{4n+6j-10}-q^{4n+4j-6})c(n-1,j-2) \\
&+q^{8n+2j-16}(q^{2n+4}-q^{2j})(q^{2n+6}-q^{2j})c(n-1,j-4).
\end{aligned}
\tag{4.8}
$$

Proof. By (4.3), we see that $c(n,j)$ is equal to a combination of $c(n-1,j-i), i=0,1,2$. Thus substitute the expressions for $c(n,j), c(n,j-1)$ and $c(n,j-2)$ arising from (4.5) into the recurrence (4.7). Collect terms and simplify to obtain (4.8). □

We now require a more succinct recurrence for $c(n,j)$.

Lemma 7.

$$
\begin{aligned}
0 =&(q^{2j}-q^{2n})c(n,j)-q^{4j}(1-q^{2n})c(n-1,j) \\
&+q^{4n+2j-3}(q^{2n}-q^{2j})(1-q^{2n})c(n-1,j-1).
\end{aligned}
\tag{4.9}
$$

Proof. Let us substitute for $c(n,j)$ in the right hand side of (4.9), the expression for $c(n,j)$ given in (4.3). Hence our assertion is equivalent to proving that

$$
\begin{aligned}
0 =&(q^{6j}-q^{4j})c(n-1,j)+(q^{2n+6j-3}-q^{6n+2j-3})c(n-1,j-1) \\
&+q^{4n+2j-6}(q^{2n}-q^{2j})(q^{2n+2}-q^{2j})c(n-1,j-2).
\end{aligned}
\tag{4.10}
$$

Let us denote the right hand side of (4.10) by $T(n,j)$. Now direct substitution reveals that (4.8) may be rewritten as

$$0 = T(n,j) - q^{2n+1}T(n,j-1) - q^{4n+2}T(n,j-2). \tag{4.11}$$

Furthermore, we see from (4.10) that

$$T(n,0) = 0.$$

Now from (4.7)

$$c(n,1) = \frac{q^{2n+1}(1-q^{4n})}{1-q^2}.$$

Hence

$$\begin{aligned}T(n,1) &= (q^6 - q^4)\frac{q^{2n+1}(1-q^{4n-4})}{1-q^2} + q^{2n+3} - q^{6n-1}\\ &= -q^{2n+3}(1-q^{4n-4}) + q^{2n+3} - q^{6n-1}\\ &= 0.\end{aligned}$$

Therefore by (4.11),

$$T(n,j) = 0$$

for all n and j. Thus (4.10) is proved, and (4.10) is equivalent to (4.9). □

Finally we need a "diagonal" recurrence for the $c(n,n)$.

Lemma 8. For $n \geq -1$,

$$\begin{aligned}0 =& c(n+2,n+2) - (q^{8n+11} + q^{8n+13})c(n+1,n+1)\\ &- (q^{10n+10} - q^{16n+16})c(n,n).\end{aligned} \tag{4.12}$$

Proof. Setting $j = n$ in (4.3), we find

$$c(n-1,n-2) = \frac{c(n,n) - (q^{6n-3} + q^{8n-3})c(n-1,n-1)}{q^{8n-6}(1-q^2)}. \tag{4.13}$$

Next set $j = n-1$ in (4.9)

$$\begin{aligned}0 =& (q^{2n-2} - q^{2n})c(n,n-1) - q^{4n-4}(1-q^{2n})c(n-1,n-1)\\ &- q^{6n-3}(q^{2n} - q^{2n-2})(1-q^{2n})c(n-1,n-2).\end{aligned} \tag{4.14}$$

Now use (4.13) twice (first with n replaced by $n+1$), to reduce (4.14) to an expression that only involves instances of $c(n-i,n-i)$. Thus, after simplification, we find

$$\begin{aligned}0 =& c(n+2,n+2) - (q^{8n+11} + q^{8n+13})c(n+1,n+1)\\ &- (q^{10n+10} - q^{16n+16})c(n,n),\end{aligned} \tag{4.15}$$

as desired. □

Lemma 9. For $n \geq -2$,

$$\begin{aligned}0 =& -c(n+3,n+3)\\ &+ q^{4n+11}(1 - q^{2n+2} - q^{2n+4} + q^{4n+6} + q^{4n+8} + q^{4n+10})c(n+2,n+2)\\ &+ q^{10n+20}(1-q^{2n+4})(1 - q^{2n+2} + q^{4n+4} + q^{4n+6} + q^{4n+8})c(n+1,n+1)\\ &- q^{14n+21}(1-q^{2n+2})(1-q^{2n+4})(1-q^{6n+6})c(n,n).\end{aligned} \tag{4.16}$$

Proof. Let us denote the right hand side of (4.12) by $U(n)$. Then it is easily verified by algebraic simplification that the expression on the right side of (4.16) is

$$-U(n+1)+q^{4n+11}(1-q^{2n+2})(1-q^{2n+4})U(n),$$

we see by Lemma 8, that (4.16) is established for $n \geq -1$, and inspection reveals the truth for $n=-2$. □

We now move to recurrences for $\overline{c}(n,j)$ as defined by (4.1).

Lemma 10.

$$\begin{aligned} 0 =&(q^{2j}-q^{2n})\overline{c}(n,j)-q^{4j}(1-q^{2n})\overline{c}(n-1,j) \\ &-q^{4n+2j-3}(q^{2n}-q^{2j})(1-q^{2n})\overline{c}(n-1,?). \end{aligned} \tag{4.17}$$

Proof. Here we require the assistance of q-MultiSum [5]:

```
In[20]= qFindRecurrence[
    qPochhammer[q^2,q^2,n] *(-1)^i*q^(2*n*(2*j-r)+j+3*i*(i-1))/
      (qPochhammer[q^2,q^2,n-j] *qPochhammer[q^2,q^2,j-r]*
        qPochhammer[q^2,q^2,r-3*i] *qPochhammer[q^6,q^6,i]),
     {n,j},{i,r},{1,1},{0,0},{0,0}]//qSR[#,2] &//Timing
```

$$\begin{aligned} \text{Out}[20] =&\{0.093601, \{q^{3+2j+4n}(-q^j+q^n) \\ &(q^j+q^n)(-1+q^{1+n})(1+q^{1+n})\text{SUM}[n,j]+ \\ &q^{2+4j}(-1+q^{1+n})(1+q^{1+n})\text{SUM}[n,1+j]-(-q^j+q^n) \\ &(q^j+q^n)\text{SUM}[1+n,1+j]=0,... \end{aligned}$$

This is precisely the recurrence (4.17) with n replaced by $n+1$. □

Lemma 11.

$$\begin{aligned} 0 =&-\overline{c}(n+3,n+3) \\ &+q^{4n+11}(1-q^{2n+2}-q^{2n+4}+q^{4n+6}+q^{4n+8}+q^{4n+10})\overline{c}(n+2,n+2) \\ &+q^{10n+20}(1-q^{2n+4})(1-q^{2n+2}+q^{4n+4}+q^{4n+6}+q^{4n+8})\overline{c}(n+1,n+1) \\ &-q^{14n+21}(1-q^{2n+2})(1-q^{2n+4})(1-q^{6n+6})\overline{c}(n,n). \end{aligned} \tag{4.18}$$

Proof. Again we employ q-MultiSum:

```
In[40]= qFindRecurrence[qPochhammer[q^2,q^2,n]*(-1)^i*
        q^(2*n*(2*n-r)+n+3*i*(i-1))    /(qPochhammer[q^2,q^2,n-r]*
          qPochhammer[q^2,q^2,r-3*i]*qPochhammer[q^6,q^6,i]),
        {n},{r,i},{2},{1,1}]//qSR[#,1] &// Timing
```

Out[40] ={1.060807,

$$\{q^{21+14n}(-1+q^{1+n})^2(1+q^{1+n})^2(-1+q^{2+n})$$
$$(1+q^{2+n})(1-q^{1+n}+q^{2+2n})(1+q^{1+n}+q^{2+2n})\mathrm{SUM}[n]-$$
$$q^{20+10n}(-1+q^{2+n})$$
$$(1+q^{2+n})(1-q^{2+2n}+q^{4+4n}+q^{6+4n}+q^{8+4n})\mathrm{SUM}[1+n]+$$
$$q^{11+4n}(1-q^{2+2n}-q^{4+2n}+q^{6+4n}$$
$$+q^{8+4n}+q^{10+4n})\mathrm{SUM}[2+n]-\mathrm{SUM}[3+n]=0,\ldots$$

This is precisely the recurrence (4.18). □

Finally we are ready to deduce Theorem 3.

Proof of Theorem 3. First it is easy to check by hand (tedious) or by computer algebra system (rapid) that Theorem 3 is valid for each $n \le 3$. The fact that (4.18) and (4.16) are identical fourth order linear recurrences then allows us to establish by mathematical induction that for all n,

$$c(n,n) = \bar{c}(n,n). \tag{4.19}$$

Finally the identity of the recurrences (4.9) and (4.17) allows us to establish by mathematical induction on n that

$$c(n,j) = \bar{c}(n,j). \tag{4.20}$$

We should note that (4.19) is necessarily established independently because (4.9) and (4.17) reduce to $0 = 0$ when $j = n$.

5 Conclusion

It should be noted that while Theorem 3 only provides an exact formula for $\triangle(1,n)$, formulas for the other $\triangle(i,n)$ can easily be obtained from the recurrences for the $d_n(x)$. Indeed, (2.3) implies immediately with n replaced by $3n-2$

$$\triangle(4,n,x,q) = \triangle(1,n,xq^2,q), \tag{5.1}$$

and the remaining $\triangle$'s are produced from the original recurrences for the $d_n(x)$ given in (2.2) and (2.3).

It is not obvious from Theorem 3 that $c(n,j)$ has non-negative coefficients, but there is adequate numerical evidence to suggest the following:

Conjecture. For all n and j, $c(n,j)$ has non-negative coefficients.

If the conjecture is true, it is natural to ask for a partition-theoretic interpretation of them. Also if that could be accomplished, it would be truly interesting to have a bijective proof of Theorem 3. Yee's bijective, related work [7] suggests this may well be possible.

Finally, we note that the Alladi-Schur version of Schur's theorem seems most fundamental in that the generating polynomials factor into increasing partial products of the product side of the limiting identity. It is natural to ask whether this phenomenon holds for other either classical or new partition identities.

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References

[1] K. Alladi, *personal communication*

[2] G. E. Andrews, *Schur's theorem, partitions with odd parts and the Al-Salam-Carlitz polynomials*, Amer. Math. Soc. Contemporary Math., **254** (2000), 45–53.

[3] G. E. Andrews, *A refinement of the Alladi-Schur theorem*, (to appear).

[4] W. Gleissberg, *Über einen Sats von Herrn I. Schur*, Math. Zeit., **28** (1928), 372–382.

[5] A. Riese, *qMultiSum – a package for providing q-hypergeometric multiple summation identities*, J. Symbolic Comp., **35** (2003), 349–376.

[6] I. Schur, *Zur additiven Zahlentheorie*, S.–B. Preuss Akad. Wiss. Phys.–Math. Kl., 1926, pp. 488–495.

[7] A. J. Yee, *A combinatorial proof of Andrews' partition functions related to Schur's partition theorem*, Proc. Amer. Math. Soc., **130** (2002), 2229–2235.

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