The Bhargava-Adiga Summation and Partitions

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Abstract

The Bhargava-Adiga summation rivals the $_1\psi_1$ -summation of Ramanujan in elegance. This paper is devoted to two applications in the theory of integer partitions leading to partition questions related to Gauss's celebrated three triangle theorem.

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1 Introduction

One very useful and elegant identity discovered by Ramanujan (cf. [1]) is $\left(\left|\frac{b}{a}\right| < |t| < 1 \text{ and } |q| < 1\right)$:

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n t^n}{(b)_n} = \frac{(b/a, at, q/(at), q; q)_{\infty}}{(t, b, q/a, b/(at); q)_{\infty}},$$
(1.1)

where

$$(a)_n = (a;q)_n := \prod_{j=0}^{\infty} \frac{(1 - aq^j)}{(1 - aq^{n+j})},$$

and

$$(a_1, a_2, \dots, a_m; q)_n = (a_1)_n (a_2)_n \dots (a_m)_n.$$

From this result one sets b = aq to deduce the famous result from the theory of elliptic functions (cf. [8, p. 21, eq. (18.61)]):

$$\sum_{n=-\infty}^{\infty} \frac{t^n}{1 - aq^n} = \frac{(q, at, q/at, q; q)_{\infty}}{(t, q/t, a, q/a; q)_{\infty}}$$

$$= \frac{(q; q)_{\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} a^n t^n}{\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} a^n \sum_{m=-\infty}^{\infty} (-1)^m q^{m(m-1)/2} t^m},$$
(1.2)

where the second line follows from Jacobi's triple product [2, p. 21, Th. 2.8].

As Bhargava and Adiga point out in [5], the $_1\psi_1$ -summation may be used to derive the classical theorems on representations of integers by two and four squares, and in [1], Adiga treats two and four triangular numbers.

The focus of this paper will be on the following result also by Bhargava and Adiga [6, eq. (1.1)]

$$\sum_{n=-\infty}^{\infty} \frac{(q/a)_n a^n}{(d)_n (1 - bq^n)} = \frac{(d/b)_{\infty} (ab)_{\infty} (q)_{\infty}^2}{(q/b)_{\infty} (d)_{\infty} (a)_{\infty} (b)_{\infty}},$$
(1.3)

where |a| < 1, |d| < 1, |q| < 1, and b not a power of q.

Note that this result has as many free parameters as Ramanujan's summation and is comparably surprising and elegant.

In the Bhargava-Adiga paper [6], they reveal the fact that now the classical theorems on two and four squares and triangles are even easier to prove than in the earlier papers where the $_1\psi_1$ was used.

In their paper [6, eq. (3.7)], they also deduce the following formula for $r_3(n)$, the number of representations of n as a sum of three squares

$$\sum_{n=0}^{\infty} (-1)^n r_3(n) q^n = 1 + 2 \sum_{m=1}^{\infty} \frac{(-1)^m q^{m(m+1)/2}}{(-q;q)_m (1+q^m)} + 4 \sum_{m=1}^{\infty} \frac{(-q)^m (-q;q)_{m-1}}{1+q^m}.$$
 (1.4)

Their Theorem 3.7 then expands the right-hand side using the classical partition interpretation of the expressions $1/(-q;q)_m$ and $(-q;q)_{m-1}$.

In [7], Bhargava, Adiga and Somashekara also consider the three square theorem, and in [10] Somashekara, Murthy and Shalini provide a beautiful generalization of (1.3) to the full $_2\psi_2$ with four free parameters.

Our first goal is to reinterpret (1.4) in terms of compact partitions. We accomplish this in Section 2.

It is also possible to deduce from (1.3) (let $q \to q^2$, then let $a \to 0$, $c = q^3$, b = q and multiply by 1/(1-q))

$$\sum_{n\geq 0} \Delta_{3}(n)q^{n} = \left(\sum_{n\geq 0} q^{\binom{n+1}{2}}\right)^{3} = \left(\frac{(q^{2}; q^{2})_{\infty}}{(q; q^{2})_{\infty}}\right)^{3}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n}q^{n^{2}+n}}{(q; q^{2})_{n+1}(1-q^{2n+1})} + \sum_{n\geq 0} \frac{q^{3n+1}(q; q^{2})_{n}}{(1-q^{2n+1})}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n}q^{n^{2}+n}}{(q; q^{2})_{n+1}(1-q^{2n+1})} + \sum_{n\geq 0} \frac{q^{n}(q; q^{2})_{n}}{1-q^{2n+1}}$$

$$-\sum_{n=0}^{\infty} (-1)^{n}q^{3n^{2}+2n}(1+q^{2n+1}), \qquad (1.5)$$

where the final expression follows from noting that $q^{3n+1} = q^n - q^n(1 - q^{2n+1})$, and recalling [4, p. 238, Entry 9.5.2]

$$\sum_{n>0} q^n (q; q^2)_n = \sum_{n=0}^{\infty} (-1)^n q^{3n^2 + 2n} (1 + q^{2n+1}).$$

In Section 3, we shall relate (1.5) to partitions with only odd parts, and, as a consequence, we consider some open problems in the concluding Section 4.

2 Compact Partitions

N. J. Fine [8, p. 57] was perhaps the first to discuss partitions without gaps (allbeit he was discussing only partitions with odd parts). In this section we consider a mild generalization of Fine's idea.

Definition 1. A partition π of n is called **compact** if every integer between the largest and smallest parts of π also appears as a part.

For example, the compact partitions of 5 are 5, 3+2, 2+2+1, 2+1+1+1, 1+1+1+1+1.

The conjugates of compact partitions are partitions in which only the largest part may repeat. Thus the conjugate of the compact partitions of 5 are 1+1+1+1+1, 2+2+1, 3+2, 4+1, 5.

This observation allows us to observe that if c(n) denotes the number of compact partitions of n, then

$$\sum_{n\geq 0} c(n)q^n = 1 + \sum_{n\geq 1} \frac{q^n}{1-q^n} (-q;q)_{n-1}.$$
 (2.1)

We note that c(n) is sequence A034296 in the OEIS [9].

Now if we consider the subset of compact partitions that contain a 1 (i.e. partitions without gaps), then again the conjugates of these partitions are generated by

$$1 + \sum_{n \ge 1} q^n (-q; q)_{n-1} = (-q; q)_{\infty}$$

$$= \sum_{n \ge 0} \frac{q^{\binom{n+1}{2}}}{(q; q)_n}$$
(2.2)

by [2, Ch.'s 1 and 2].

We now require some partition statistics associated with an integer partition π :

 $L(\pi) :=$ the number of largest parts of π

 $\#(\pi) :=$ the number of parts of π

 $\sigma(\pi) :=$ the smallest part of π

$$\chi(1 \in \pi) = \begin{cases} 1 & \text{if 1 is a part of } \pi \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1. If $c_0(n)$ is the number of compact partitions π where π is counted with weight

$$(-1)^{\#(\pi)+\sigma(\pi)-1}$$

then

$$\sum_{n\geq 1} c_0(n)q^n = \sum_{n\geq 1} \frac{(-1)^n q^n (-q;q)_{n-1}}{1+q^n}.$$
 (2.3)

Proof. The series in (2.3) considers the conjugates of compact partitions counted with +1 if the number of largest parts has the opposite parity of the largest part and -1 otherwise. Referring to the conjugate partition (i.e. the compact partition) we see that the largest part conjugates to the number of

parts and the number of largest parts conjugates to the smallest part. Hence the series (2.3) generates partitions counted by the weight

$$(-1)^{\#(\pi)+\sigma(\pi)-1}$$

as asserted. \Box

Lemma 2.2 If $c_1(n)$ denotes the number of compact partitions of n containing a 1 and counted with weight

$$(-1)^{\#(\pi)}L(\pi),$$

then

$$\sum_{n\geq 1} c_1(n)q^n = \sum_{n\geq 1} \frac{(-1)^n q^{\binom{n+1}{2}}}{(-q)_n (1+q^n)}.$$
 (2.4)

Proof. The right hand expression in (2.4) is

$$\sum_{n>0} (-1)^n \frac{q}{1+q} \cdot \frac{q^2}{1+q^2} \cdots \frac{q^n}{(1+q^n)^2}.$$

This series generates compact partitions containing 1 and counted with weight $L(\pi) \cdot (-1)^{\#(\pi)}$ (because $x/(1-x)^2 = x + 2x^2 + 3x^3 + \dots$). I.e. the right hand side of (2.4) is the generating function for $c_1(n)$.

Theorem 1. For $n \geq 1$,

$$(-1)^{n} r_{3}(n)/2 = c_{1}(n) + 2c_{2}(n)$$

$$= \sum_{\pi \in \mathcal{C}(n)} (-1)^{\#(\pi)} (L(\pi)\chi(1 \in \pi) + 2(-1)^{\sigma(\pi)-1}),$$
(2.5)

where $\mathcal{C}(n)$ is the set of compact partitions of n.

Proof. Bhargava and Adiga have given us the proof via (1.4). Namely, we divide (1.4) by 2, apply Lemma 2.1 to the second series on the right, apply Lemma 2.2 to the first series on the right, and then compare coefficients of q^n .

As an example of Theorem 1, we examine n = 8. $r_3(8) = 12$; so $(-1)^8 r_3(8)/2 = 6$. The following are the 10 compact partitions of 8 tabulated so that we can compute the sum on the right of (2.5).

π	$(-1)^{\#(\pi)}$	$L(\pi)$ if $\chi(1 \in \pi) = 1$	$2(-1)^{\sigma(\pi)-1}$	weight
$1+1+\cdots+1$	+1	8	2	10
$2+1+\cdots+1$	-1	1	2	-3
2+2+1+1+1+1	+1	2	2	4
2+2+2+1+1	-1	3	2	-5
2+2+2+2	+1	_	-2	-2
3+3+2	-1	_	-2	+2
3+2+1+1+1	-1	1	2	-3
3+2+2+1	+1	1	2	+3
4 + 4	+1	_	-2	-2
8	-1	_	-2	2
				01 1F C

21 - 15 = 6

3 Partitions with Odd Parts

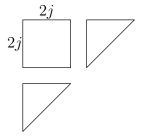
Just as Section 2 was devoted to a partition-theoretic interpretation of (1.4), we now formulate a partition-theoretic interpretation of (1.5).

Lemma 3.1. [9, A067357] If sfo(n) denotes the number of self-conjugate partitions of 4n + 1 into odd parts, then

$$\sum_{n\geq 0} \operatorname{sfo}(n)q^n = \sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q^2)_{n+1}}.$$
(3.1)

Remark. The series on the right-hand side of (3.1) is the third order mock theta function v(q) [8, p. 61, eq. (26.82)].

Proof. First we note that there cannot be a self-conjugate partition of 4n+1 with an even sided Durfee square. Schematically the Ferrers graph of the partition would be



In addition the bottom edge of the Durfee square cannot be exposed (otherwise 2j would be one of the parts and 2j is not odd). Hence there are 2j odd numbers in the upper triangle to the right of the Durfee square. Hence since the partition is self-conjugate the number being partitioned is

$$(2j)^2 + 2 \times (2j) \times \sum \text{odds} \equiv 0 \mod 4.$$

Therefore the Durfee square must be odd sided. Thus using a modified 2-modular representation for the Ferrers graph, we have

	1	1	1		1	1	1		1
1	2	2	2 2		2 2	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	2 2	• • •	 2/
1 1		2	2		2	:	2	•••	
	÷	:	÷		:	:	/		
1	2	2	2	• • •	2	$\frac{2}{2}$			
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as a Ferrers-like generic representation of the partitions in question. And from this we see that when the Durfee square has side 2n + 1, the relevant generating function is

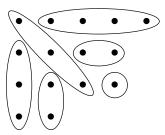
$$\frac{q^{(2n+1)^2}}{(1-q^4)(1-q^{12})\dots(1-q^{8n+4})},$$

and summing over all allowable Durfee squares, we see that

$$\sum_{n>0} \operatorname{sfo}(n) q^{4n+1} = \sum_{n>0} \frac{q^{4n^2+4n+2}}{(q^4; q^8)_{n+1}},$$
(3.2)

which is equivalent to (3.1).

We now recall the concept of the Frobenius symbol for a partition. This is best understood by an example. Suppose we consider the Ferrers graph of 5+4+4+2



We delete the diagonal (of 3 dots). The grouping to the right is 4, 2 and 1 and below is 3, 2 and 0. This produces the Frobenius symbol for this partition, namely

$$\begin{pmatrix} 4 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix}.$$

Now it is obvious that the Frobenius symbol of a self-conjugate partition has identical top and bottom rows.

We define the bottom, $b(\pi)$, of a partition π as the smallest entry in the Frobenius symbol for π .

Lemma 3.2. Let $d_1(n)$ denote the weighted sum over all the self-conjugate partitions of 4n + 1 into odd parts where the weight is

$$(-1)^{\frac{s(\pi)-1}{2}}(b(\pi)/2+1)$$

with $s(\pi)$ being the side of the Durfee square. Then

$$\sum_{n\geq 0} d_1(n)q^n = \sum_{n\geq 0} \frac{(-1)^n q^{n^2+n}}{(q;q^2)_{n+1}(1-q^{2n+1})}.$$
 (3.3)

Proof. We replace q by q^4 on the right side of (3.3) and multiply by q:

$$\sum_{n>0} \frac{(-1)^n q^{(2n+1)^2}}{(1-q^4)(1-q^{12})\dots(1-q^{8n-4})(1-q^{8n+4})^2},$$

and as before the denominator supplies the two conjugate flanges of the Ferrers graph except that now the final $1/(1-q^{8n+4})^2$ contributes $(b(\pi)-1)/2$ to the weight, and the $(-1)^n$ in the numerator is just $(-1)^{(s(\pi)-1)/2}$.

We may directly observe that

$$\sum_{n\geq 0} \frac{q^{2n+1}(-q^2; q^4)_n}{1 - q^{4n+2}} = q + 2q^3 + 3q^5 + 3q^7 + \dots$$
 (3.4)

is the generating function for partitions into odd parts in which the largest part appears an odd number of times and all other parts appear twice. For example, the relevant partitions of 7 are 7, 5+1+1, and 1+1+1+1+1+1+1.

Lemma 3.3. Let $d_2(n)$ denote the weighted count of the partitions π of 2n+1 just described for (3.4) with the weight $(-1)^{\lambda(\pi)-1}$ where $\lambda(\pi)$ is the number of different parts of π . Then

$$\sum_{n>0} d_2(n)q^{2n+1} = \sum_{n>0} \frac{q^{2n+1}(q^2; q^4)_n}{1 - q^{4n+2}}.$$
 (3.5)

Proof. This is readily deduced from the observations following (3.4) with the added remark that replacing $(-q^2; q^4)_n$ by $(q^2; q^4)_n$ now weights the count by $(-1)^{\lambda(\pi)-1}$.

Theorem 2.

$$\triangle_3(n) = d_1(n) + d_2(n) + \epsilon(n),$$
 (3.6)

where

$$\epsilon(n) = \begin{cases} (-1)^{j-1} & \text{if } n = 3j^2 + 2j \text{ or } 3j^2 + 4j + 1\\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Lemmas 3.1 and 3.2, we may rewrite (1.5) as

$$\sum_{n>0} \Delta_3(n)q^n = \sum_{n>0} d_1(n)q^n + \sum_{n>0} d_2(n)q^n + \sum_{n>0} \epsilon(n)q^n,$$

and the result follows by coefficient comparison.

As an example of Theorem 2, let us consider n = 5. There are three representations of 5 by three triangles 3+1+1, 1+3+1 and 1+1+3. So $\Delta_3(5) = 3$.

For $d_1(5)$, we see that there are three self-conjugate partitions of $21 = (4 \cdot 5 + 1)$ into odd parts:

partition Frobenius symbol
$$11+1+...+1 \qquad \begin{pmatrix} 10\\10 \end{pmatrix} \\
9+3+3+1+1+1+1+1+1 \qquad \begin{pmatrix} 8 & 1 & 0\\8 & 1 & 0 \end{pmatrix} \\
5+5+5+3+3 \qquad \begin{pmatrix} 4 & 3 & 2\\4 & 3 & 2 \end{pmatrix}.$$

Hence

$$d_1(5) = +\left(\frac{10}{2}+1\right) - \left(\frac{0}{2}+1\right) - \left(\frac{2}{2}+1\right)$$

= 6-1-2=3.

For $d_2(5)$, we see that the relevant partitions of $11 = 2 \cdot 5 + 1$ are 11, 9+1+1, 5+3+3, 3+3+3+1+1, $1+1+\ldots+1$ whose corresponding weights are +1, -1, -1, -1, +1. Hence $d_2(5) = 1 - 1 - 1 - 1 + 1 = -1$.

Finally since $5 = 3 \cdot 1^2 + 2 \cdot 1$, we see that $\epsilon(5) = (-1)^{1-1} = 1$. Thus

$$3 = \triangle_3(5),$$

and

$$d_1(5) + d_2(5) + \epsilon(5) = 3 - 1 + 1 = 3.$$

4 Conjectures and Gauss's Eureka Theorem.

It is well known (cf. [3]) that Gauss noted with "Eureka!" his proof of the fact that every positive integer is a sum of three triangular numbers. I.e., $\Delta_3(n) > 0$ for all $n \geq 0$. This means that the expression on the right of (2.5) is always positive. This observation leads to the following.

Conjecture 1. $d_1(n) > 0$ for $n \ge 0$.

This assertion is true for $n \leq 1000$.

Conjecture 2. $d_1(n) > |d_2(n) + \epsilon(n)|$ for $n \ge 2$.

Obviously by Theorem 2, if this last conjecture was true it would imply $\triangle_3(n) > 0$.

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