Binary Partitions and Binary Partition Polytopes

by

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Abstract. *This paper delves into the number of partitions of positive integers n into powers of 2 in which exactly m powers of 2 are used an odd number of times. The study of these numbers is motivated by their connections with the f-vectors of the binary partition polytopes,*

1. Introduction. The study of the partitions of nonnegative integers n into powers of 2 (here called *binary partitions*) has a lengthy history, going back as far as Euler and extending through Cayley, to many others since (cf. [1], Section 10.2). The main interest has been in determining or estimating the total number of such partitions of a given positive integer n. Here a refinement is considered: Given also a second integer m, with $m \geq 0$, what is the number $L(m, n)$ of binary partitions of n in which exactly m of the powers of 2 are used an odd number of times?

The question is motivated by a result of [3]. That paper studies a sequence $Q_1, Q_2, \ldots$ of convex polytopes, Q_d being a certain dual-antiprism of dimension d. It is found that the f-vector $(f_0, f_1, \ldots, f_{d-1})$ of Q_d is determined by the $L(m, n)$'s:

$$f_m = \begin{cases} L(m, 2^{d+1}) & \text{if } 0 \leq m \leq d-1 \text{ and } m \neq 1, \\ L(1, 2^{d+1}) - 1 & \text{if } m = 1. \end{cases} \tag{1}$$

Each polytope Q_d ($d \geq 2$) is determined up to combinatorial equivalence by the fact that its lattice of faces is isomorphic to the lattice of intevals of the face lattice of Q_{d-1}. In the 2-dimensional case the polytope is a square, having $f_0 = 4$ vertices and $f_1 = 4$ edges. The ten binary partitions of 8 are $1 \cdot 2^3$, $2 \cdot 2^2$, $1 \cdot 2^2 + 2 \cdot 2^1$,$1 \cdot 2^2 + 1 \cdot 2^1 + 2 \cdot 2^0$, $1 \cdot 2^2 + 4 \cdot 2^0$, $4 \cdot 2^1$, $3 \cdot 2^1 + 2 \cdot 2^0$, $2 \cdot 2^1 + 4 \cdot 2^0$, $1 \cdot 2^1 + 6 \cdot 2^0$, and $8 \cdot 2^0$; and it can be seen that the f-vector of Q_2 agrees with its description above.

In the following section the values $L(m, n)$ are studied. The binary partition polytopes are described and some results on congruences modulo 2 are established in Section 3. The f-vectors of the binary partition polytope, modulo 4, satisfy the recurrence relation of Pascal's triangle. This is proven in Section 4. In Section 5, binary partitions with bounded parts, a given number of which are used an odd number of times, are counted.

2. Values of L. Table 1 has values $L(m,n)$, for $0 \le m \le 4$ and $0 \le n \le 16$. Theorem 2.1, below, presents a recurrence relation that can be used to compute these numbers.

The (total) number of binary partitions of n, here denoted $b(n)$, is the sum of the numbers $L(m,n)$ as m varies. Also, $b(n) = L(0,2n)$. Clearly the sequence of numbers $b(n)$ and the values of $L(m,n)$ are closely related. Theorem 2.2 provides a summation formula involving b that gives $L(m,n)$.

THEOREM 2.1. *For $m \ge 1$ and $n \ge 1$,*

$$L(m,n) = \begin{cases} L(m,n-2) + L(m,\frac{n}{2}) & \text{if } n \text{ is even and } n \ge 2, \\ L(m-1,n-1) & \text{if } n \text{ is odd and } m \ge 1. \end{cases}$$

Proof. The generating function $F(z,q)$ for $L(m,n)$ is given by

$$\begin{aligned} F(z,q) &= \sum_{m,n \ge 0} L(m,n) z^m q^n \\ &= \prod_{j=0}^{\infty} (1 + zq^{2^j} + q^{2\cdot 2^j} + zq^{3\cdot 2^j} + \ldots) \\ &= \prod_{j=0}^{\infty} (1 + \frac{zq^{2^j}}{1-q^{2^{j+1}}} + \frac{q^{2\cdot 2^j}}{1-q^{2^{j+1}}}) \\ &= \prod_{j=0}^{\infty} \frac{1+zq^{2^j}}{1-q^{2^j+1}}. \end{aligned}$$

Hence

$$F(z,q^2) = \prod_{j=0}^{\infty} \frac{1+zq^{2^{j+1}}}{1-q^{2^{j+2}}} = \frac{1-q^2}{1+zq} F(z,q),$$

or

$$(1+zq)F(z,q^2) = (1-q^2)F(z,q).$$

Therefore

$$\sum_{m,n \ge 0} L(m,n) z^m q^{2n} + \sum_{m,n \ge 0} L(m,n) z^{m+1} q^{2n+1} = \sum_{m,n \ge 0} L(m,n) z^m q^n (1-q^2).$$

Comparing the coefficients of $z^m q^n$ on both sides yeilds the case n even of the theorem. When n is odd, it yields $L(m,n-2) + L(m-1,\frac{n-1}{2})$, which is easily seen to imply the odd case of the theorem statement. Alternatively, a bijective correspondence is given by, from a binary partiton of n (odd), 1 being a necessary part, removing it. $\square$

Upon setting $L(0,0) = 1$, $L(m,0) = 0$ for $m > 0$, and $L(0,n) = 0$ when n is odd, the recurrence gives the value of $L(m,n)$ for all other m and n. Of course, $L(m,n) = 0$ when $n < 2^m - 1$. The values of $b(n)$ appear in the even positions of the first row: $b(n) = L(0,2n)$.

$m\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	0	1	0	2	0	2	0	4	0	4	0	6	0	6	0	10
1	0	1	1	1	2	2	3	2	5	4	7	4	10	6	12	6	17
2	0	0	0	1	0	2	1	3	1	5	3	7	4	10	7	12	8
3	0	0	0	0	0	0	0	1	0	1	0	3	0	4	1	7	1
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0

Table 1: Some values of $L(m,n)$

For an integer $n \geq 0$, $\nu(n)$ denotes the number of 1's in the binary expansion of n.

Theorem 2.2. *We have*

$$L(m,n) = \sum_{\substack{0 \leq l \leq \frac{n}{2},\\ \nu(n-2l)=m}} b(l).$$

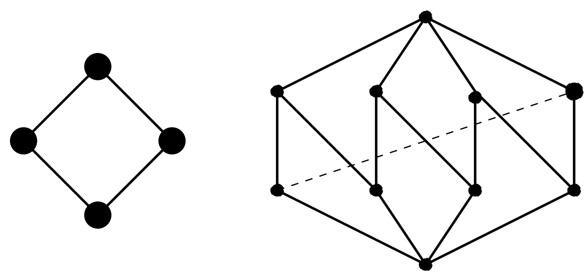
Proof.

$$\begin{aligned}
\sum_{m,n\geq 0} L(m,n) z^m q^n &= \prod_{j=0}^{\infty} \frac{1+zq^{2^j}}{1-q^{2^j}} \\
&= \sum_{k\geq 0} z^{\nu(k)} q^k \sum_{l\geq 0} b(l) q^{2l} \\
&= \sum_{k,l\geq 0} b(l) z^{\nu(k)} q^{k+2l} \\
&= \sum_{m,n\geq 0} \sum_{\substack{0\leq l\leq n/2,\\ \nu(n-2l)=m}} b(l) z^m q^n.
\end{aligned}$$

$\square$

3. The binary partition polytopes, Q_d. In this section, a recursive definition will be given for the sequence of "binary partition polytopes." Two polytopes are *combinatorially equivalent* if their face lattices are isomorphic. Here, polytopes will be considered to be "the same" if they are combinatorially equivalent. If L is a lattice, a nonempty interval in L is a set of the form $\{x \in L \mid a \leq x \leq b\}$, where $a, b \in L$ and $a \leq b$. The empty set is also considered to be an interval. It is easy to see that the intervals of a lattice also form a lattice when ordered by inclusion.

The lattice of faces of a line segment $[a, b]$ is depicted in Figure 1(a); that of a square, in Figure 1(b). It is apparent that the lattice of Figure 1(b) is isomorphic to the lattice of intervals of the lattice of Figure 1(a).



(a) (b)

Figure 1. Interval lattices.

Binary partition polytopes are defined as follows. First, a single point is the binary partition polytope of dimension 0. Take Q_0 to be such a polytope. Recursively, if $d \geq 1$ and Q_{d-1} is a binary partition polytope of dimension $d-1$, then Q_d is a binary partition polytope of dimension d if its face lattice is isomorphic to the lattice of intervals of Q_{d-1}. It is proven in [3] that such a sequence of polytopes exists. It is obvious from the definition that the binary partition polytopes of dimension d fall into one combinatorial equivalence class, so we will speak of *the* binary partition polytope Q_d.

In [3] it is proven that the number of faces of Q_d is $b(2^{d+1})$, and that, for $0 \leq m \leq d$, the number of faces of dimension m of Q_d is $L(m, 2^{d+1})$ when $m \neq 1$ and is $L(1, 2^{d+1}) - 1$ when $m = 1$. Combining that result with Theorem 2.2 yields the following.

THEOREM 3.1. *For $0 \leq m \leq d$, The number of faces of Q_d of dimension m*

is

$$\sum_{\substack{1 \le l \le 2^d, \\ \nu(2^d - l) = m}} b(l).$$

Proof. This is the summation from Theorem 2.2 with $n = 2^{d+1}$ and with $l = 0$ excluded from the summation. $\square$

The paper [3] also describes a partial ordering relation on binary partitions of 2^{d+1} that yields a lattice isomorphic to the face lattice of Q_d. The next theorem, in which $\mathcal{B}_d$ denotes the set of binary partitions of 2^d, gives the gist of it.

We will denote a binary partition a in the form $a = \alpha_0 \cdot 2^0 + \ldots + \alpha_k \cdot 2^k$, where the parts of the partiton are the powers 2^j that occur with a positive coefficient $\alpha_j > 0$. The number of parts is the sum, $\sum \alpha_j$. Terms $\alpha_l \cdot 2^l$ with $\alpha_l = 0$ may be omitted. If $a = \sum \alpha_j \cdot 2^j$ and $b = \sum \beta_j \cdot 2^j$ then $a + b$ denotes the binary partition $\sum (\alpha_j + \beta_j) \cdot 2^j$.

Theorem 3.2. *For each integer $d \ge 0$ there is a partial ordering $\preceq_d$ of $\mathcal{B}_d$ with respect to which $\mathcal{B}_d$ is a lattice having least element the single part partition $1 \cdot 2^d$. For $n \ge 1$ there are functions $l_d, r_d : \mathcal{B}_d \setminus \{2^d\} \to \mathcal{B}_{d-1}$ such that, for each $a \in \mathcal{B}_d \setminus \{1 \cdot 2^d\}$, $l_d(a) \preceq r_d(a)$, and a is the partition whose parts are those of $l_d(a)$ and $r_d(a)$. Furthermore, if $a \in \mathcal{B}_d \setminus \{1 \cdot 2^d\}$, $b, c \in \mathcal{B}_{d-1}$, the parts of a are those of b and c, and $b \preceq_{d-1} c$, then $b = l_d(a)$ and $c = r_d(a)$. If $a, b \in \mathcal{B}_d$, then $a \preceq_d b$ if and only if $l_d(b) \preceq l_d(a)$ and $r_d(a) \preceq_{d-1} r_d(b)$. The lattice $\mathcal{B}_d$ is isomorphic to the lattice of faces of Q_{d-1}.*

From the theorem it follows that the partitions of $\mathcal{B}_d$, other than $1 \cdot 2^d$, correspond to pairs $a, b \in \mathcal{B}_{d-1}$, with $a \preceq_{d-1} b$, Associating the partition $1 \cdot 2^d$ with the empty set completes the picture: The poset $\mathcal{B}_d$ is isomorphic to the lattice of intervals of $\mathcal{B}_{d-1}$. The partial ordering relation on $\mathcal{B}_d$ is determined from that of $\mathcal{B}_{d-1}$ by the rule, $a \preceq_d b$ if and only if a is the partion $1 \cdot 2^d$ or there exist $a', a'', b', b'' \in \mathcal{B}_{d-1}$ such that $a = a' + a''$, $b = b' + b''$, $b' \preceq_{d-1} a'$, and $a'' \preceq_{d-1} b''$.

From this point on we consider the set $\mathcal{B}_d$ of binary partitions of 2^d to be partially ordered as in the theorem; and we will drop the subscript on the symbol $\preceq$. From the fact that $\mathcal{B}_d$ is isomorphic to the lattice of faces of the polytope Q_{d-1} it follows that $\mathcal{B}_d$ has a height function $h : \mathcal{B}_d \to \mathbb{Z}$ for which

$h(a)$ is the length of any maximal chain $a_0 = 1 \cdot 2^d \prec a_1 \prec \ldots \prec a_k = a$; $h(a) = k$.

Theorem 3.3. *If $1 \le k \le d-1$ then the number of elements of $\mathcal{B}_d$ of height k in $\mathcal{B}_d$ is even.*

Proof. We need only exhibit an involution on $\mathcal{B}_d$ that preserves height and fixes only the top and bottom elements.

The 4-element lattice $\mathcal{B}_2$ has a unique involutory automorphism other than the identity. We denote this automorphism by α_2. The set of fixed points of this automorphism is a lattice that is isomorphic to $\mathcal{B}_1$.

Any automorphism α of a lattice L induces an automorphism $\widehat{\alpha}$ of its lattice of intervals, by $\widehat{\alpha}([a,b]) = [\alpha(a), \alpha(b)]$. If the fixed-point set of α is $M \subseteq L$ then the fixed-point set of $\widehat{\alpha}$ is isomorphic to the lattice of intervals of M.

Inductively, let α_d ($d = 3, 4, \ldots$) be the automorphism of $\mathcal{B}_d$ induced by α_{d-1} as above. It follows from the above that the lattice of fixed points of α_d is isomorphic to $\mathcal{B}_{d-1}$, for $d = 2, 3, \ldots$.

We can define an involution β_d on $\mathcal{B}_d$ that preserves the heights of elements and has as its fixed-point set only the top and bottom elements of $\mathcal{B}_d$. Define $\beta_2 = \alpha_2$. Inductively, for $d > 2$, define

$$\beta_d(x) = \begin{cases} \alpha_d(x) & \text{if } x \notin F_k \\ \bar{\beta}_{d-1}(x) : & \text{otherwise,} \end{cases}$$

where F_d denotes the fixed-point set of α_d and $\bar{\beta}_{d-1}$ is an involution on F_d that fixes only the top and bottom elements and preserves height. $\square$

From [3], for $k > 0$, the elements at height k in $\mathcal{B}_d$ are the binary partitions of 2^d having exactly $k-1$ powers of 2 appearing as parts an odd number of times, excepting $1 \cdot 2^d$ when $k = 2$. It follows that $L(m, 2^d)$ is congruent to 0 modulo 2 when $m \neq 1$ or d. The final theorem of this section determines congruence of $L(m,n)$ modulo 2 for all n. In the next section, the f-vectors of the polytopes Q_d will be determined explicitly, modulo 4.

Theorem 3.4. *For $n \ge 2$, $L(m,n)$ is even unless $m = \nu(n)$ or $m = \nu(n-2)$, and m is not congruent to 4 or 5 modulo 8; in these cases $L(m,n)$ is odd.*

Proof. When $n \ge 2$, $b(n)$ is even. Also $b(0) = b(1) = 1$. From these facts and Theorem 2.2 it follows that $L(m,n) \equiv \epsilon_1 + \epsilon_2 \mod 2$, where

$$\epsilon_1 = \begin{cases} 1 & \text{if } \nu(n) = m \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \epsilon_2 = \begin{cases} 1 & \text{if } \nu(n-2) = m \\ 0 & \text{otherwise.} \end{cases}$$

Consider the binary expansion $n-2 = \alpha_0 + \alpha_1 \cdot 2 + \alpha_2 \cdot 2^2 + \ldots$, where each α_i is 0 or 1. If $\alpha_1 = 0$ then $\nu(n) = \nu(n-2)+1$. Otherwise let s be the smallest index > 1 such that $\alpha_s = 0$, so that $n-2 = \alpha_0 + 2 + 2^2 + \ldots + 2^{s-1} + \alpha_{s+1}2^{s+1} + \ldots$. Then $\nu(n) = \nu(n-2) - s + 2 \leq \nu(n-2)$. Equality holds if and only if $s = 2$, in which case $n \equiv 4$ or $5 \mod 8$. It follows that $\epsilon_1 = \epsilon_2$ when $n \equiv 4$ or $5 \mod 8$, in which case $\epsilon_1 + \epsilon_2 \equiv 0 \mod 2$. In all other cases, $\nu(n) \neq \nu(n-2)$, so in those cases $\epsilon_1 + \epsilon_2$ is odd if and only if $\nu(n-2) = m$ or $\nu(n) = m$. □

4. The f-vectors of the binary partition polytopes, modulo 4. We write $f(m,d)$ for the number of m-dimensional faces of the d-dimensional binary partition polytope, Q_d. We know that

$$f(m,d) = \begin{cases} L(m, 2^{d+1}) & \text{if } m \neq 1 \\ L(1, 2^{d+1}) - 1 & \text{if } m = 1. \end{cases} \tag{2}$$

We will determine the numbers $f(m,d)$ up to congruence modulo 4 by using Theorem 2.2 and some facts about the function $b(n)$.

1. $b(0) = b(1) = 1$.
2. If n is odd then $b(n) = b(n-1)$.
3. $b(n) = b(n-2) + b(\lfloor \frac{n}{2} \rfloor)$.
4. For even $n \geq 2$ of the form $n = 2^k(2l+1)$, (so in particular $k \neq 0$) we have $b(n) \equiv 2k \mod 4$.
5. If $2 \leq n < 2^d$ then $b(n + 2^d) \equiv b(n) \mod 4$.

Except for the last, proofs of these statements can be found in [2]; the last follows easily from the third by induction.

Lemma 4.1. *For $0 \leq m \leq d-1$,*

$$f(m,d) \equiv 2|\{k : 2 \leq k \leq 2^d, \nu(2^d - k) = m, \\ \textit{and } b(k) \equiv 2 \mod 4\}| \mod 4.$$

Proof. By Theorem 3.1,

$$f(m,d) = \sum_{\substack{1\le k\le 2^d,\\ \nu(2^d-k)=m}} b(k).$$

Also $\nu(2^d-1) = d > d-1$, so the sum doesn't extend over 1 when m is in the stated range. The result then follows at once upon noting that when $k \ge 2$ and $b(k) \not\equiv 2 \mod 4$, $b(k) \equiv 0 \mod 4$, since $b(k)$ is even for $k \ge 2$. $\square$

LEMMA 4.2. *When* $0 \le m \le d-2$,

$$f(m+1,d+1) \equiv f(m,d) + f(m+1,d) \mod 4.$$

Proof. By the preceding lemma,

$$\begin{aligned} f(m+1,d+1) \equiv 2|\{k : 2 \le k \le 2^{d+1}, \nu(2^{d+1}-k) = m+1,\\ \text{and } b(k) \equiv 2 \mod 4\}| \mod 4. \end{aligned}$$

The number on the right side is the sum of the numbers

$$\begin{aligned} 2|\{k : 2 \le k \le 2^{d}, \nu(2^{d+1}-k) = m+1,\\ \text{and } b(k) \equiv 2 \mod 4\}| \mod 4 \end{aligned}$$

and

$$\begin{aligned} 2|\{k : 2^d < k \le 2^{d+1}, \nu(2^{d+1}-k) = m+1,\\ \text{and } b(k) \equiv 2 \mod 4\}| \mod 4. \end{aligned}$$

The first of these is

$$\begin{aligned} 2|\{k : 2 \le k \le 2^{d}, \nu(2^{d}-k) = m,\\ \text{and } b(k) \equiv 2 \mod 4\}| \mod 4, \end{aligned}$$

since $\nu(2^{d+1}-k)=\nu(2^d+(2^d-k))=1+\nu(2^d-k)$ when $2\le k\le 2^d$; so the number is $f(m,d)$. The second is

$$\begin{aligned}2|\{k:2<&k\le 2^d,\nu(2d-k)=m+1,\\ &\text{and } b(k+2^d)\equiv 2 \mod 4\}| \mod 4.\end{aligned}$$

Here we have used the fact that $\nu(2^{d+1}-(2^d+1))=\nu(2^d-1)=d>d-1\ge m+1$. For $2\le k\le 2^d$, $b(k+2^d)\equiv b(k)$, so this number is $(m+1,d)$. $\square$

Theorem 4.3. *For $0\le m<d$,*

$$f(m,d)\equiv 2\binom{d+2}{m+1} \mod 4.$$

Proof. For $m=0$ we get $f(0,d)=b(2^d)\equiv\begin{cases}2 & d \text{ even}\\ 0 & d \text{ odd.}\end{cases}$ and $f(0,d)\equiv 2(d+2)=\binom{d+2}{1}$. For $m=d-1$,

$$\begin{aligned}f(d-1,d)&=f(0,d-1)+f(0,d-2)+\ldots+f(0,1)+2\\ &=b(2^{d-1})+b(2^{d-2})+\ldots+b(2)+2.\end{aligned}$$

Since $b(2^k)\equiv 2 \mod 4$ when k is odd and $\equiv 0 \mod$ when k is even, this sum is congruent to $0 \mod 4$ when d is congruent to 2 or $3 \mod 4$ and to $2 \mod 4$ when d is congruent to 1 or $4 \mod 4$; that is, it is congruent to $2\binom{d+1}{d}$, as given by the statement of the lemma. Considering the recurrence relation of Lemma 4.2 and the equality $\binom{d+3}{m+2}=\binom{d+2}{m+1}+\binom{d+2}{m+2}$ when $0\le m\le d-2$, it is clear that the congruence is valid for all m,d with $0\le m<d$. $\square$

5. Binary partitions with bounded parts. Let $L_p(m,n)$ denote the number of partitions of n into powers of 2, m of which are used an odd number of times, all parts at most 2^p. Let $F_p(z,q)=\sum_{m,n\ge 0}L_p(m,n)z^mq^n$

be the generating function for L.

$$F_p(z,q) = \prod_{j=0}^{p} (1 + zq^{2^j} + q^{2\cdot 2^j} + zq^{3\cdot 2^j} + \ldots)$$
$$= \prod_{j=0}^{p} \frac{1 + zq^{2^j}}{1 - q^{2^{j+1}}} = \frac{F(z,q)}{F(z, q^{2^{p+1}})}.$$

Then the following equation holds:

$$F_p(z,q^2) = \frac{1 + zq^{2^{p+1}}}{1 - q^{2^{p+2}}} \frac{1 - q^2}{1 + zq} F_p(z,q).$$

Equivalently,

$$(1 + zq - q^{2^{p+2}} - zq^{2^{p+2}+1}) F_p(z,q^2) = (1 - q^2 + zq^{2^{p+1}} - zq^{2^{p+1}+2}) F_p(z,q),$$

and

$$(1 + zq - q^{2^{p+2}} - zq^{2^{p+2}+1}) \sum_{m,n\geq 0} L_p(m,n) z^m q^{2n}$$
$$= (1 - q^2 + zq^{2^{p+1}} - zq^{2^{p+1}+2}) \sum_{m,n\geq 0} L_p(m,n) z^m q^n.$$

From the coefficient of $z^m q^{2n}$ we get

$$\begin{aligned} L_p(m,n) - L_p(m, n - 2^{p+1}) = \\ L_p(m,2n) - L_p(m, 2n-2) + L_p(m-1, 2n - 2^{p+1}) \\ - L_p(m-1, 2n - 2^{p+1} - 2) \end{aligned}$$

and from the coefficient of $z^{m+1} q^{2n+1}$ we get

$$\begin{aligned} L_p(m,n) - L_p(m, n - 2^{p+1}) = \\ L_p(m+1, 2n+1) - L_p(m+1, 2n-1) + L_p(m, 2n + 1 - 2^{p+1}) \\ - L_p(m, 2n - 2^{p+1} - 1). \end{aligned}$$

From this the following analogue of Theorem 2.1 is a consequence.

THEOREM 5.1. *For even n,*

$$L_p(m,n) = L_p(m,\frac{n}{2}) - L_p(m,\frac{n}{2} - 2^{p+1}) + L_p(m,n-2) \\ - L_p(m-1,n-2^{p+1}) + L_p(m-1,n-2^{p+1}-2);$$

for odd n,

$$L_p(m,n) = L_p(m-1,\frac{n-1}{2}) - L_p(m-1,\frac{n-1}{2} - 2^{p}) + L_p(m,n-2) \\ - L_p(m-1,n-2^{p+1}) + L_p(m-1,n-2^{p+1}-2).$$

The coefficients $L(m,n)$ and $L_p(m,n)$ of the power series expansions of $F(z,q)$ and $F_p(z,q)$ agree for those terms having degree less than 2^{p+1} in q. For those having degree 2^{p+1} in q, the single difference is in the coefficient of $zq^{2^{p+1}}$, which in $F(z,q)$ exceeds by 1 its value in $F_p(z,q)$.

We can write $F_p(z,q) = \sum_{n=0}^{\infty} \rho_n(z)q^n$, where each $\rho_n(z)$ is a polynomial in z of degree at most p. The coefficient of $q^{2^{p+1}}$ is of interest in connection with the binary partition polytope Q_p. It is $\rho_{2^{p+1}}(z) = f_0 + f_1 z + \ldots + f_{p-1}x^{p-1} + z^p$, where $(f_0, f_1, \ldots, f_{p-1})$ is the f-vector of Q_p, as is clear from Theorem 3.2.

The value $L_p(m,n)$ is the number of solutions in integers $x_0, \ldots, x_p$, with m of the x_i's odd, of the system

$$x_0 + 2x_1 + 4x_2 + \ldots + 2^p x_p = n, \\ x_0, x_1, \ldots, x_p \geq 0.$$

This system of inequalities defines a simplex of dimension p in $\mathbb{R}^{p+1}$. Let T_p denote this simplex in the case $n = 1$. In the general case the simplex is then nT_p. The vertices of T_p are $(1,0,\ldots,0)$, $(0,\frac{1}{2},\ldots,0)$, $\ldots$, $(0,0,\ldots,\frac{1}{2^p})$, and the simplex 2^pT_p has all vertices in $\mathbb{Z}^{p+1}$. The simplex $2^{p+1}T_p$ has all vertices in $2\mathbb{Z}^{p+1}$ and its relative interior contains a unique point of $\mathbb{Z}^{p+1}$, namely, $(2,1,1,\ldots,1)$. This is the unique element of $(2^{p+1}T_p) \cap \mathbb{Z}^{p+1}$ that has exactly p odd components.

Some facts from the theory of valuations on convex polytopes have a bearing. A valuation on convex polytopes is a function ν on convex polytopes in, say, $\mathbb{R}^{p+1}$ such that, for any two polytopes P and Q such that $P \cup Q$ is convex, the equation $\nu(P \cup Q) + \nu(P \cap Q) = \nu(P) + \nu(Q)$ holds. The valuations of interest are the functions $\nu_m(P)$ that take the convex polytope P to the integer that is the number of points of $Z^{p+1} \cap P$ that have exactly m odd components. Each of these valuations is invariant under translation by elements of $2\mathbb{Z}^{p+1}$; that is, for each $x \in \mathbb{Z}^{p+1}$ and each polytope P, $\nu_m(P + 2x) = \nu_m(P)$. It follows by a theorem of McMullen [4] that for any polytope having vertices in $2\mathbb{Z}^{p+1}$ and for each m, the function $n \mapsto \nu_m(nP)$ is given by a polynomial in n. Notice that the valuation ν_0 simply counts the elements of $\mathbb{Z}^{p+1} \cap P$ that have no odd components, or equivalently, the elements in $2\mathbb{Z}^{p+1} \cap P$. When all vertices of P lie in $2\mathbb{Z}^{p+1}$, the resulting polynomial is just the Ehrhart polynomial of the polytope $\frac{1}{2}P$.

From this it follows that for each m the coefficient of z^m in $\rho_n(z)$, where n is restricted to multiples of 2^{p+1}, is given by a polynomial in n.

This can also be seen as follows. We have $F_p(z, q) = \prod_{j=0}^{p} \frac{1+zq^{2^j}}{1-q^{2^{j+1}}}$. This can be written as

$$\frac{1}{(1 - q^{2^{j+1}})^{p+1}} \prod_{j=0}^{p} (1 + zq^{2^j})(1 + q^{2^j})^j.$$

We can extract the terms that are the powers of q^n for which $2^{p+1}|n$ by averaging over the 2^{p+1}-th roots of unity:

$$\sum_{n=0}^{\infty} \rho_{n2^{p+1}}(z) q^{n2^{p+1}} = \frac{1}{2^{p+1}} \sum_{k=0}^{2^{p+1}-1} F_p(z, qe^{ki\pi/2^p}).$$

We see that this can be written as $\frac{\eta(z, q^{2^{p+1}})}{(1-q^{2^p+1})^{p+1}}$, where $\eta(z, t)$ is a polynomial in z and t. Substituting t for $q^{2^{p+1}}$ we have

$$\sum_{n=0}^{\infty} \rho_{n2^{p+1}}(z) t^n = \frac{\eta(z, t)}{(1 - t)^{p+1}}.$$

A basic fact concerning generating functions implies that the coefficients as functions of n are given by polynomials of degree at most p.

The function

$$\frac{\eta(0,t)}{(1-t)^{p+1}}$$

is the generating function for the Ehrhart series of $2^{p+1}T_p$.

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