Binary Representations and Theta Functions

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January 19, 2017

Abstract

The question to be considered is whether there is a power series, $f(g)$, whose coefficients are ± 1 and for which $\prod_{n\geq 1} f(q^{2n-1}) = \sum_{-\infty}^{\infty} q^{n(3n-1)/2}$. This question will be answered affirmatively, following a study of the binary representation of integers. In addition, related theorems will be developed.

Keywords: Binary representation, theta function, partition
AMS Classification numbers: 11A63, 11P81, 05A19

1 Introduction

In [2] and [3], the second author raised questions along the following lines. Given the generating functions for $p(n)$, [1, p.3, Th. 1.1]

$$\begin{aligned} \sum_{n\geq 0} p(n)q^n &= \prod_{n=1}^{\infty} \frac{1}{1-q^n} \\ &= \prod_{n=1}^{\infty} \left(1+q^n+q^{2n}+q^{3n}+\cdots\right), \end{aligned} \tag{1.1}$$

could one merely change some of the signs in this last product so that the resulting power series has coefficients of 0, ± 1. The answer is still unknown.

A very natural step in this direction is to consider

$$\begin{aligned} B(q) &= \prod_{n=0}^{\infty} \left(1-q^{2^n}\right) \\ &= \sum_{n\geq 0} (-1)^{\#_1(n)} q^n \\ &= 1-q-q^2+q^3-q^4+q^5+q^6-q^7-\cdots, \end{aligned} \tag{1.2}$$

where $\#_1(n)$ is the number of 1's in the binary representation of n.

Now since every integer uniquely factors into a power of 2 times an odd number, we see that

$$\begin{aligned}\prod_{n=1}^{\infty} B(q^{2n-1}) &= \prod_{n=1}^{\infty}\prod_{m=0}^{\infty}\left(1-q^{(2n-1)2^m}\right)\\ &= \prod_{N=1}^{\infty}\left(1-q^N\right) \qquad (1.3)\\ &= \sum_{-\infty}^{\infty}(-1)^n q^{n(3n-1)/2},\end{aligned}$$

by Euler's pentagonal number theorem [1, p.11, Cor. 1.7].

In light of the success of (1.3), it is natural to ask whether there exists a power series $f(x)$ having ± 1 as coefficients such that

$$\prod_{n=1}^{\infty} f(q^{2n-1}) = \sum_{-\infty}^{\infty} q^{n(3n-1)/2}. \qquad (1.4)$$

Empirically, one finds that

$$\begin{aligned}f(x) &= 1+x+x^2-x^3-x^4-x^5-x^6-x^7 \qquad (1.5)\\ &\quad -x^8+x^9+x^{10}+x^{11}-x^{12}-x^{13}-x^{14}+\cdots\end{aligned}$$

In section 3, we show that $f(x)$ exists and determine the sign pattern for the coefficients. In order to achieve this goal we must consider

$$\begin{aligned}B_e(q) &= 1+2\sum_{n=1}^{\infty}(-1)^n q^{b_n}\\ &= 1-2q+2q^3-2q^4+2q^5-2q^7+2q^9\cdots,\end{aligned}$$

where b_n is the nth integer whose binary representation ends in an even number of zeros (cf. [4, sequence A003159]). The explanation of (1.4) and the coefficients of $f(x)$ relies on

Theorem 1.

$$\prod_{n=1}^{\infty} B_e(q^{2n-1}) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}. \qquad (1.6)$$

Section 2 will be devoted to background on $B_e(q)$. Section 3 will be devoted to a proof of Theorem 1, and Section 4 will provide a full explanation of (1.4) and (1.5). In Section 5, we provide an analogous representation of Gauss's triangular number series. We conclude with open questions.

2 Background on Binary representations

The object of this section is to find a nice closed form representation of $B_e(q)$.

Lemma 2.

$$\sum_{n=1}^{\infty}(-1)^n q^{b_n} = \sum_{n=1}^{\infty}(-1)^{\#_1(b_n)} q^{b_n}.$$

Proof. In light of the fact that $b_1 = 1$ so that $\#_1(b_1) = 1$, we see that all that is required to establish this result is a proof that $\#_1(b_n)$ changes parity as we pass from b_n to b_{n+1}.

There are 3 cases to consider:

1. b_n in binary ends in $2j$ zeros, with $j > 0$.
2. b_n in binary ends in $2k$ ones
3. b_n in binary ends in $2k+1$ ones

In case 1., $b_{n+1} = b_n + 1$ and b_{n+1} has one more 1 than b_n. Hence $\#_1(b_{n+1})$ has opposite parity from $\#_1(b_n)$.

In case 2., we see that in binary

$$b_n = \beta_1 \cdots \beta_s 0 \underbrace{11\ldots1}_{2k \text{ times}},$$

so $b_{n+1} = b_n + 1$. Thus

$$b_{n+1} = \beta_1 \cdots \beta_s 1 \underbrace{00\ldots0}_{2k \text{ times}},$$

and b_{n+1} has $2k-1$ fewer 1's than b_n. Hence $\#_1(b_{n+1})$ has opposite parity from $\#_1(b_n)$.

In case 3., we see that in binary

$$b_n = \beta_1 \cdots \beta_s 0 \underbrace{11\ldots1}_{2k+1 \text{ times}},$$

so $b_{n+1} = b_n + 2$. Thus

$$b_{n+1} = \beta_1 \cdots \beta_s 1 \underbrace{00\ldots0}_{2k \text{ times}} 1,$$

and b_{n+1} again has $2k-1$ fewer 1's than b_n. Hence $\#_1(b_{n+1})$ has opposite parity from $\#_1(b_n)$. $\square$

Lemma 3.

$$1 + \sum_{n \geq 0} \frac{q^{2^n}}{\prod_{j=0}^{n}\left(1 - q^{2^j}\right)} = \prod_{n=0}^{\infty} \frac{1}{1 - q^{2^n}}, \tag{2.1}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{2^n}}{\prod_{j=0}^{n} \left(1-q^{2^j}\right)} \quad = \quad q. \tag{2.2}$$

Proof. Equation (2.1) is the generating function for partitions into powers of 2 expressed in two different ways. The nth term of the series generates those partitions whose largest part is 2^n. The infinite product is the standard form for the generating function [1, p.3, Th, 1,1].

As for (2.2), we shall prove

$$\sum_{n=0}^{N} \frac{(-1)^n q^{2^n}}{\prod_{j=0}^{n} \left(1-q^{2^j}\right)} \quad = \quad q + \frac{q^{2^{N+1}}(-1)^N}{\prod_{j=0}^{n} \left(1-q^{2^j}\right)}. \tag{2.3}$$

When $N=0$, (2.3) reduces to

$$\frac{q}{1-q} = q + \frac{q^2}{1-q},$$

which is immediate.

Assuming true up through a given N,

$$\begin{aligned}
\sum_{n=0}^{N+1} \frac{(-1)^n q^{2^n}}{\prod_{j=0}^{n} \left(1-q^{2^j}\right)} \quad &= \quad q + \frac{q^{2^{N+1}}(-1)^N}{\prod_{j=0}^{N} \left(1-q^{2^j}\right)} + \frac{q^{2^{N+1}}(-1)^{N+1}}{\prod_{j=0}^{N+1} \left(1-q^{2^j}\right)} \\
&= \quad q + \frac{q^{2^{N+1}}(-1)^N \left(\left(1-q^{2^{N+1}}\right)-1\right)}{\prod_{j=0}^{N+1} \left(1-q^{2^j}\right)} \\
&= \quad q + \frac{(-1)^{N+1} q^{2^{N+2}}}{\prod_{j=0}^{N+1} \left(1-q^{2^j}\right)},
\end{aligned}$$

which proves (2.3).

Equation (2.2) now follows by letting $N \to \infty$ in (2.3). $\square$

Theorem 4. $B_e(q) = (1-q)\prod_{j=0}^{\infty}\left(1-q^{2^j}\right)$

Proof. In light of Lemma 2, it follows that

$$B_e(q) \quad = \quad 1 - \sum_{n\geq 0} \left(1+(-1)^n\right) q^{2^n} \prod_{j=n+1}^{\infty} \left(1-q^{2^j}\right)$$

$$
\begin{aligned}
&= 1 - \prod_{j=0}^{\infty}\left(1-q^{2^j}\right)\sum_{n\geq 0}\frac{\left(1+(-1)^n\right)q^{2^n}}{\prod_{j=0}^{n}\left(1-q^{2^j}\right)} \\
&= 1 - \prod_{j=0}^{\infty}\left(1-q^{2^j}\right)\left(\frac{1}{\prod_{j=0}^{\infty}\left(1-q^{2^j}\right)} - 1 + q\right) \quad \text{(by Lemma 3)} \\
&= (1-q)\prod_{j=0}^{\infty}\left(1-q^{2^j}\right).
\end{aligned}
$$

□

3 Proof of Theorem 1

$$
\begin{aligned}
\prod_{n=1}^{\infty} B_e(q^{2n-1}) &= \prod_{n=1}^{\infty}\left\{\left(1-q^{2n-1}\right)\prod_{j=0}^{\infty}\left(1-q^{(2n-1)2^j}\right)\right\} \\
&= \prod_{n=1}^{\infty}\left(1-q^{2n-1}\right)\prod_{n=1}^{\infty}\left(1-q^n\right) \quad \text{(by (1.3))} \\
&= \prod_{n=1}^{\infty}\frac{(1-q^n)}{(1+q^n)} \quad \text{(by [1, p.5, eq (1.2.5)])} \\
&= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} \quad \text{(by [1, p.23, eq. (2.2.12)]),}
\end{aligned}
$$

and Theorem 1 is proved.

4 The Determination of $f(x)$

We define

$$
\begin{aligned}
f(x) &= \frac{B_e(x^3)}{1-x} \qquad (4.1) \\
&= \frac{1 + 2\sum_{n=1}^{\infty}(-1)^n x^{3b_n}}{1-x} \\
&= \frac{1 + \sum_{n=1}^{\infty}(-1)^n x^{3b_n} - \sum_{n=0}^{\infty}(-1)^n x^{3b_{n+1}}}{1-x}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1-x^3}{1-x} + \sum_{n=1}^{\infty}(-1)^n \frac{\left(x^{3b_n} - x^{3b_{n+1}}\right)}{1-x} \\
&= 1 + x + x^2 + \sum_{n=1}^{\infty}(-1)^n \left(x^{3b_n} + x^{3b_n+1} + \cdots + x^{3b_{n+1}-1}\right),
\end{aligned}
$$

and we see that $f(x)$ is a power series with ± 1 as the coefficients.

Theorem 5. *Equation* (1.4) *holds for the* $f(x)$ *given in* (4.1).

Proof.

$$
\begin{aligned}
\prod_{n=1}^{\infty} f(q^{2n-1}) &= \prod_{n=1}^{\infty} \frac{B_e(q^{3(2n-1)})}{1-q^{2n-1}} \\
&= \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} \cdot \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2} \quad \text{(by theorem 1)} \\
&= \prod_{n=1}^{\infty} (1+q^n) \prod_{n=1}^{\infty} \frac{(1-q^{3n})}{(1+q^{3n})} \\
&\quad \text{(by [1, p.5, eq. (1.2.5)] and [1, p.23, eq. (2.2.12)])} \\
&= \prod_{n=1}^{\infty} \left(1-q^{3n}\right)\left(1+q^{3n-1}\right)\left(1+q^{3n-2}\right) \\
&= \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} \\
&\quad \text{(by [1, p.21, eq. (2.2.10), } q \to q^{\frac{3}{2}},\ z = q^{-\frac{1}{2}}\text{])}
\end{aligned}
$$

and (1.4) is established for the $f(x)$ given by (4.1). ☐

5 Gauss's Triangular Number Series

This paper would be incomplete without a result relating binary partitions to Gauss's famous series:

$$
\psi(q) = \sum_{n \geq 0} q^{n(n+1)/2}. \tag{5.1}
$$

Here the relevant binary series is

$$
\begin{aligned}
B_g(q) &= \prod_{j \geq 0} \left(1 + (-1)^j q^{2^j}\right) \qquad (5.2) \\
&= 1 + q - q^2 - q^3 + q^4 + q^5 - q^6 - q^7 - q^8 - q^9 + q^{10} + \ldots
\end{aligned}
$$

Again we see that we have a series where all the coefficients are ± 1.

Theorem 6.

$$\prod_{n\geq 1} B_g(q^{2n-1})B_g(q^{4n-2}) \quad = \quad \psi(q). \tag{5.3}$$

Proof. We begin by noting that

$$\begin{aligned}
B_g(q)B_g(q^2) &= \prod_{j\geq 0}\left(1+(-1)^j q^{2^j}\right)\left(1+(-1)^j q^{2^{j+1}}\right) &(5.4)\\
&= (1+q)\prod_{j\geq 1}\left(1+(-1)^j q^{2^j}\right)\left(1-(-1)^j q^{2^j}\right)\\
&= (1+q)\prod_{j\geq 1}\left(1-q^{2^{j+1}}\right)\\
&= \frac{(1+q)}{(1-q)(1-q^2)}\prod_{j\geq 0}\left(1-q^{2^j}\right)\\
&= \frac{1}{(1-q)^2}\prod_{j\geq 0}\left(1-q^{2^j}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\prod_{n\geq 1} B_g(q^{2n-1})B_g(q^{4n-2})\\
&= \left(\prod_{n\geq 1}\frac{1}{(1-q^{2n-1})^2}\right)\left(\prod_{j\geq 0}\prod_{n\geq 1}\left(1-q^{(2n-1)2^j}\right)\right)\\
&= \prod_{n\geq 1}\frac{(1-q^n)(1+q^n)}{(1-q^{2n-1})} \quad \text{by [1, p.5, eq. (1.2.5)]}\\
&= \prod_{n\geq 1}\frac{(1-q^{2n})}{(1-q^{2n-1})}\\
&= \sum_{n\geq 0} q^{n(n+1)/2} \quad \text{by [1, p. 23, eq. (2.2.13)]}
\end{aligned}$$

□

6 Conclusion

In light of the important role played by the sequence b_n, one might as well ask what happens if we consider the sequences, a_n, the nth integer whose binary representation ends in an odd number of zeros [4, seq. A036554]. However, nothing new arises because $a_n = 2b_n$.

Apart from the classical theta series given by the right hand sides of (1.3), (1.4), (1.6), and (5.1) , it would be interesting to examine other classical theta series. In light of the fact that binary representation played a crucial role in all our results, it would be interesting to see if there are similar theorems related to other bases apart from 2.

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