Integer Partitions With Even Parts Below Odd Parts and the Mock Theta Functions

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Abstract

The paper begins with a study of a couple of classes of partitions in which each even part is smaller than each odd. In one class, a Dyson-type crank exists to explain a mod 5 congruence. The second part of the paper treats the arithmetic and combinatorial properties of the third order mock theta function $\nu(q)$ and relates the even part of $\nu(q)$ to the partitions initially considered. We also consider a surprisingly simple combinatorial relationship between the cranks and the ranks of the partition of n.

1 Introduction

Let $\mathcal{EO}(n)$ denote the number of partitions of n in which every even part is less than each odd part. As we shall see (and as others have seen), $\mathcal{EO}(n)$ has natural and elementary properties which we will catalog in section 2.

Let $\overline{\mathcal{EO}}(n)$ denote the number of partitions counted by $\mathcal{EO}(n)$ in which ONLY the largest even part appears an odd number of times (We allow partitions with no even parts where we tacitly assume 0 is the largest even and appears once). For example, $\mathcal{EO}(8) = 12$ with the relevant partitions being 8, $6+2$, $7+1$, $4+4$, $4+2+2$, $5+3$, $5+1+1+1$, $2+2+2+2$, $3+3+2$, $3+3+1+1$, $3+1+1+1+1+1$, $1+1+1+1+1+1+1+1$; and $\overline{\mathcal{EO}}(8) = 5$, with the relevant partitions being 8, $4+2+2$, $3+3+2$, $3+3+1+1$, $1+1+1+1+1+1+1+1$.

The next item to be treated is the third order mock theta function [19, p. 62]

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}, \tag{1.1}$$

where

$$(A;q)_n = (1-A)(1-Aq)\dots(1-Aq^{n-1}). \tag{1.2}$$

The first parts of this paper are tied together as follows. Let

$$\mathrm{eo}(q) = \sum_{n \geq 0} \mathcal{EO}(n) q^n, \tag{1.3}$$

and

$$\overline{\mathrm{eo}}(q) = \sum_{n \geq 0} \overline{\mathcal{EO}}(n) q^n. \tag{1.4}$$

Theorem 1.

$$\frac{1}{2}\left(\nu(q) + \nu(-q)\right) = \overline{\mathrm{eo}}(q). \tag{1.5}$$

Now the deep combinatorics of $\nu(-q)$ have been studied extensively in [6]. By means of an intricate use of mock theta function identities [6, Sec. 4] it was shown that if

$$\nu(-q) = \sum_{n \geq 0} p_\nu(n) q^n,$$

then $p_\nu(n)$ is the number of partitions of n in which all parts are distinct and all odd parts are less than twice the smallest part. In particular, it is shown [6, Thrm. 7.6] (note $p_\nu(n) = \mathrm{spt}_\nu(n)$) that

$$p_\nu(10n + 8) \equiv 0 (\mod 5),$$

which by Theorem 1 implies

$$\overline{\mathcal{EO}}(10n + 8) \equiv 0 (\mod 5). \tag{1.6}$$

For each partition π enumerated by $\mathcal{EO}(n)$, we may define the even-odd crank,

$$\mathrm{eoc}(\pi) = \text{largest even part} - \#(\text{odd parts of } \pi).$$

Theorem 2. *The even-odd crank separates the partitions enumerated by $\overline{\mathcal{EO}}(10n+8)$ into 5 equinumerous sets.*

From our example of $\overline{\mathcal{EO}}(8) = 5$, we see that

$$\begin{aligned}
\mathrm{eoc}(\text{“}8\text{”}) &= 8 - 0 \equiv 3 (\mod 5)\\
\mathrm{eoc}(\text{“}4+2+2\text{”}) &= 4 - 0 \equiv 4 (\mod 5)\\
\mathrm{eoc}(\text{“}3+3+2\text{”}) &= 2 - 2 \equiv 0 (\mod 5)\\
\mathrm{eoc}(\text{“}3+3+1+1\text{”}) &= 0 - 4 \equiv 1 (\mod 5)\\
\mathrm{eoc}(\text{“}1+1+\ldots+1\text{”}) &= 0 - 8 \equiv 2 (\mod 5).
\end{aligned}$$

As we mentioned, section 2 is devoted to cataloging the known results for $\mathcal{EO}(n)$, many of which appear in the OEIS [12] under an alternative account. Section 3 is devoted to the rather deeper aspects of $\overline{\mathcal{EO}}(n)$. Section 4 is devoted to the "odd Ferrers graphs" which were introduced in the treatment of "odd Dufree symbols" from section 8 of [5]. Section 5 develops further combinatorial aspects of $\nu(q)$ including a proof of Theorem 1. In section 6, we note several simple combinatorial relationships between the ranks and cranks of the partitions of n. The paper concludes with a list of open problems.

We note that Ali Uncu [18] has treated a different subset of the partitions enumerated by $\mathcal{EO}(n)$. For his partitions, the generating function is $(q^2;q^4)_\infty^{-2}$.

2 $\mathcal{EO}(n)$

The standard methods for producing partition generating functions (cf. [3, Ch. 1]) reveal directly that

$$\begin{aligned}
\text{eo}(q) &= \sum_{n\geq 0} \mathcal{EO}(n)q^n \\
&= \sum_{n=0}^{\infty} \frac{q^{2n}}{(1-q^2)(1-q^4)\dots(1-q^{2n})(1-q^{2n+1})(1-q^{2n+3})\dots} \\
&= \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n(q^{2n+1};q^2)_\infty} \\
&= \frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(q^2;q^2)_n} \\
&= \frac{(q^3;q^2)_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty} \quad \text{(by [3, p. 17, eq. (2.2.1)])} \\
&= \frac{1}{(1-q)(q^2;q^2)_\infty}.
\end{aligned} \tag{2.1}$$

Thus

Proposition 3. *$\mathcal{EO}(n)$ is also the number of partitions of n into parts that are either even or 1. AND $\mathcal{EO}(n)$ is the number of palindromic partitions of n.*

Proof. The first assertion is immediate from the last line in (2.1). The second also follows from the last line of (2.1) where the $1/(q^2;q^2)_\infty$ produces two copies of a given partition and the $1/(1-q)$ produces the central part if there are an odd number of parts. Both these assertions appear in [12]. ☐

Returning to the 12 partitions for $\mathcal{EO}(8)$, we see that the twelve partitions of 8 involving only even parts and 1's are: 8, $6+2$, $6+1+1$, $4+4$, $4+2+2$, $4+2+1+1$, $4+1+1+1+1$, $2+2+2+2$, $2+2+2+1+1$, $2+2+1+1+1+1$, $2+1+1+1+1+1+1$, $1+1+1+1+1+1+1+1$. The palindromic partitions of 8 are: 8, $4+4$, $3+1+1+3$, $2+2+2+2$, $2+1+1+1+1+2$, $1+1+1+1+1+1+1+1$, $3+2+3$, $2+1+2+1+2$, $1+1+1+2+1+1+1$, $2+4+2$, $1+1+4+1+1$, $1+6+1$.

3 $\overline{\mathcal{EO}}(n)$

We define $\overline{\mathcal{EO}}(m,n)$ to be the number of partitions enumerated by $\overline{\mathcal{EO}}(n)$ whose even-odd crank equals m.

Proposition 4.

$$\overline{\mathrm{eo}}(z,q) := \sum_{n\geq 0}\sum_{m=-\infty}^{\infty} \overline{\mathcal{EO}}(m,n) z^m q^n = \frac{(q^4;q^4)_\infty}{(z^2q^2;q^4)_\infty (q^2/z^2;q^4)_\infty}. \tag{3.1}$$

Proof.

$$\begin{aligned}
&\overline{\mathrm{eo}}(z,q)\\
=&\sum_{n\geq 0} \frac{z^{2n}q^{2n}}{(1-q^{2+2})(1-q^{4+4})\dots(1-q^{2n+2n})(1-z^{-2}q^{(2n+1)+(2n+1)})(1-z^{-2}q^{(2n+3)+(2n+3)})\dots}\\
=&\sum_{n\geq 0} \frac{z^{2n}q^{2n}}{(q^4;q^4)_n(z^{-2}q^{4n+2};q^4)_\infty}\\
=&\frac{1}{(z^{-2}q^2;q^4)_\infty}\sum_{n\geq 0}\frac{(z^{-2}q^2;q^4)_n z^{2n}q^{-2n}}{(q^4;q^4)_n}\\
=&\frac{1}{(z^{-2}q^2;q^4)_\infty}\cdot\frac{(q^4;q^4)_\infty}{(q^2z^2;q^4)_\infty} \quad \text{(by [3, p. 17, eq. (2.2.1)])}\\
=&\frac{(q^4;q^4)_\infty}{(z^2q^2;q^4)_\infty(q^2/z^2;q^4)_\infty}.
\end{aligned}$$

□

Corollary 5.

$$\overline{\mathrm{eo}}(q) = \sum_{n\geq 0}\overline{\mathcal{EO}}(n)q^n = \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty^2} \tag{3.2}$$

Proof. Set $z=1$ in Proposition 4. □

We now define $\overline{\mathrm{EO}}_5(i,n)$ to be the number of partitions enumerated by $\overline{\mathcal{EO}}(n)$ whose even-odd crank is congruent to $i(\mod 5)$. We restate Theorem 2 as follows:

Theorem 2. *For* $0\leq i\leq 4$,

$$\overline{\mathrm{EO}}_5(i,10n+8) = \frac{1}{5}\overline{EO}(10n+8). \tag{3.3}$$

Proof. By the very definition of $\overline{\mathcal{EO}}(n)$, we see that $\overline{\mathcal{EO}}(m,n)=0$ if n is odd. Hence

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{\mathcal{EO}}(m,n)z^mq^n=\sum_{n\geq 0}\sum_{m=-\infty}^{\infty}\overline{\mathcal{EO}}(m,2n)z^mq^{2n} \tag{3.4}$$
$$=\frac{(q^4;q^4)_\infty}{(z^2q^2;q^4)_\infty(q^2/z;q^4)_\infty}$$

Thus

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{\mathcal{EO}}(m,2n)z^mq^n=\frac{(q^2;q^2)_\infty}{(z^2q;q^2)_\infty(q/z^2;q^2)_\infty}. \tag{3.5}$$

Suppose now that $\zeta=e^{2\pi i/5}$, I claim that the coefficient of every power of q of the form q^{5N+4} in the power series expansion of

$$\frac{(q^2;q^2)_\infty}{(\zeta^2q;q^2)_\infty(q/\zeta^2;q^2)_\infty} \tag{3.6}$$

is identically 0.

To see this we proceed as follows. For brevity, we write

$$(a_1,a_2,\ldots,a_r;q)_\infty=\prod_{i=1}^{r}(a_i;q)_\infty.$$

Thus

$$\frac{(q^2;q^2)_\infty}{(\zeta^2q,q/\zeta^2;q^2)_\infty}=\frac{(q^2;q^2)_\infty(q,\zeta q,q/\zeta;q^2)_\infty}{(q,\zeta q,q/\zeta,\zeta^2q,q/\zeta^2;q^2)_\infty} \tag{3.7}$$
$$=\frac{(q;q)_\infty(\zeta q,q/\zeta;q^2)_\infty}{(q^5;q^{10})_\infty}$$

Now we recall both the triple product identity [2, p. 461, Th. 3.4] and the quintiple product identity [2, p. 466, Th. 3.9]

$$\sum_{n=-\infty}^{\infty}(-x)^nq^{n(n-1)/2}=(q,x,q/x;q)_\infty, \tag{3.8}$$

and

$$\sum_{n=-\infty}^{\infty}q^{n(3n-1)/2}(x^{3n}-x^{1-3n})=(q,q/x,x;q)_\infty(qx^2,q/x^2;q^2)_\infty \tag{3.9}$$

Thus (since $\zeta^4=\zeta^{-1}$)

$$\frac{(q^2;q^2)_\infty}{(\zeta^2q,q/\zeta^2;q^2)_\infty}=\frac{(q,q/\zeta^2,\zeta^2;q)_\infty(q\zeta,q/\zeta;q^2)_\infty}{(1-\zeta^2)(q^5;q^{10})_\infty(q/\zeta^2,q\zeta^2;q)_\infty} \tag{3.10}$$

$$
= \frac{(q, q\zeta, q/\zeta; q)_\infty \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2}(\zeta^{6n} - \zeta^{2-6n})}{(1-\zeta^2)(q^5; q^{10})_\infty (q^5; q^5)_\infty}
$$

$$
= \frac{\sum_{j=-\infty}^{\infty} (-1)^j \zeta^j q^{j(j-1)/2} \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2}(\zeta^n - \zeta^{2-n})}{(1-\zeta)(1-\zeta^2)(q^5; q^{10})_\infty (q^5; q^5)_\infty}
$$

Now the only way that

$$
j(j-1)/2 + n(3n-1)/2 \equiv 4 (\mod 5)
$$

is if $n \equiv 1(\mod 5)$ and $j \equiv 3(\mod 5)$. But if $n \equiv 1(\mod 5)$, then $\zeta^n - \zeta^{2-n} = \zeta - \zeta = 0$. Hence the power series expansion of the expression in (3.6) has no powers of q of the form q^{5N+4}.

Finally

$$
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{\mathcal{EO}}(m, 2n)\zeta^m q^n = \frac{(q^2; q^2)_\infty}{(\zeta^2 q; q^2)_\infty (q/\zeta^2; q^2)_\infty}
$$

Therefore examining the coefficient of q^{5n+4}:

$$
\sum_{m=-\infty}^{\infty} \overline{\mathcal{EO}}(m, 2(5n+4))\zeta^m = 0
$$

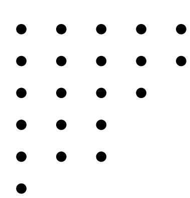
$$
= \sum_{i=0}^{4} \overline{\mathrm{EO}}_5(i, 10n+8)\zeta^i = 0.
$$

But ζ is a root of $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$, and thus the $\overline{\mathrm{EO}}_5(i, 10n+8)$ must all be identical (otherwise ζ would be the root of a different minimal polynomial). Hence (3.8) follows. □

4 Odd Ferrers Graphs

In section 8 of [5, pp. 60-62], a variation of the standard Ferrers graph was given in order to provide a combinatorial interpretation of the mock theta function $\omega(q)$.

The standard Ferrers graph for a partition is as follows: for the partition $5 + 5 + 4 + 3 + 3 + 1$, we arrange left justified rows of dots representing each part:



Equivalently we could replace the dots by 1's, and the number being partitioned would be obtained by adding up all the 1's:

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & & \\ 1 & & & & \end{array}$$

The odd Ferrers graphs introduced in [5] consist of a Ferrers graph using 2's with a surrounding border of 1's. Thus

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 & \\ 1 & 2 & 2 & 2 & 2 & 2 & \\ 1 & 2 & 2 & & & & \\ 1 & 2 & & & & & \\ 1 & & & & & & \end{array}$$

We may translate this into numerical terms as a sum of parts where we have an ordinary partition into odd parts except for one "odd" part which may be odd or even and is larger than half of the largest of the other parts. For the above graph the related "odd" partition is

$$7 + 11 + 11 + 5 + 3 + 1.$$

As noted in [5, p. 60], the related generating function is $q\omega(q)$, namely

$$q\omega(q) = \sum_{n\geq 0} \frac{q^{n+1}}{(q;q^2)_{n+1}} \tag{4.1}$$

$$= \sum_{n\geq 0} \frac{q^{(2n+1)^2}}{(q;q^2)^2_{n+1}}. \tag{4.2}$$

Of more immediate interest to us for the next section, is the fact that the third order mock theta function $\nu(q)$ is related to self-conjugate odd Ferrers graphs.

The conjugate of an odd Ferrers graph is obtained by interchanging rows and columns (or equivalently by reflecting along the diagonal). Thus the conjugate of the odd Ferrers graph just given is

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & \\ 1 & 2 & 2 & 2 & & \\ 1 & 2 & 2 & & & \\ 1 & 2 & 2 & & & \\ 1 & 2 & 2 & & & \\ 1 & & & & & \end{array}$$

Self-conjugate odd Ferrers graphs are ones that are unaltered under conjugation. In [5, p. 68, corrected] it was shown that the generating function for slef-conjugate odd Ferrers graphs is

$$q\nu(-q^2) = \sum_{m\geq 0} \frac{q^{2m^2+2m+1}}{(q^2;q^4)_{m+1}}, \tag{4.3}$$

where, as noted previously, $\nu(q)$ is the third order mock theta function [16, p. 62]

$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}. \tag{4.4}$$

5 The combinatorics of $\nu(q)$

We begin this section by collecting from the literature three further representations of $\nu(q)$. First,

$$\nu(q) = \sum_{n=0}^{\infty} (-q)^n (q;q^2)_n; \tag{5.1}$$

second

$$\nu(-q^2) = \sum_{n\geq 0} \frac{(-1)^n q^n}{(q;-q^2)_{n+1}}; \tag{5.2}$$

and third,

$$\nu(-q) = \sum_{n\geq 0} q^n (-q^{n+1};q)_n (-q^{2n+2};q^2)_\infty \tag{5.3}$$

Identity (5.1) was proved combinatorially by P. A. MacMahon [11, Sec. 512, pp. 260-261] in a somwhat disguised form. It is given explicitly by N. J. Fine [9, p. 61, eq. (26.85)]. Identity (5.2) (with $q \to iq^{1/2}$) is also given by Fine [9, p.61, eq. (26.87)]. Identity (5.3) is given in [6, Th. 4.1].

Recall that $\nu(-q)$ is the generating function for $p_\nu(n)$:

$$\nu(-q) := \sum_{n\geq 0} p_\nu(n) q^n. \tag{5.4}$$

There are a couple of interpretations of $p_\nu(n)$ listed for sequence A067357 (provided by D. Hickerson, M. Somos, and P. A. MacMahon)[14]

(5.5) $p_\nu(n)$ is the number of self-conjugate partitions of $4n+1$ into odd parts.

(5.6) $p_\nu(n)$ is the number of partitions of n in which even parts are distinct and every even number smaller than the largest part is also a part.

We note in passing that upon replacing q by $-q^2$ in (5.1) and multiplying by q, one produces an alternative to (4.3) revealing that

(5.7) $p_\nu(n)$ is the number of self-conjugate odd Ferrers graphs for $2n+1$.

As mentioned in the introduction [6, Th. 4.1]

(5.8) $p_\nu(n)$ is the number of partitions of n into distinct non-negative parts where every odd is less than twice the smallest part (note that 0 maybe a part).

However, the following interpretation is new.

Theorem 6. *Let $\overline{\mathcal{EO}}_4(n)$ denote the difference between the number of odd Ferrers graphs with an even number of rows (below the top row of 1's) summing to $1(\mod 4)$ minus those with an odd number of such rows. Then*

$$\overline{\mathcal{EO}}_4(n) = \begin{cases} 0 & \text{if } n \text{ odd} \\ p_\nu\left(\frac{n}{2}\right) & \text{if } n \text{ even.} \end{cases} \tag{5.9}$$

Proof. Let us change q to $-q$ in (5.2), multiply the result by q and then compare with the series in (4.1) which is the generating function for all odd Ferrers graphs. Thus we are comparing

$$\sum_{n\geq 0} \frac{q^n}{(1+q)(1-q^3)(1+q^5)\dots(1-(-1)^n q^{2n+1})}$$

with

$$\sum_{n\geq 0} \frac{q^n}{(1-q)(1-q^3)\dots(1-q^{2n+1})}.$$

It is now immediately apparent tht the latter series is counting the same partitions as the former with the condition that the count is $+1$ for partition with an even number of rows (below the top row) adding to $1(\mod 4)$ and -1 for those with an odd number of such rows.

Finally, the assertion of the theorem follows from the fact that by (5.2)

$$q\nu(-q^2) = \sum_{n\geq 0} \frac{q^{n+1}}{(-q;-q^2)_{n+1}}.$$

□

We are now finally fully in position to prove Theorem 1.

Proof of Theorem 1. A succinct proof follows immediately from the top line of [19, p. 72] which asserts (rewritten in our notation)

$$\nu(q) + q\omega(q^2) = \frac{(q^4;q^4)_\infty}{(q^2;q4)^2_\infty}.$$

Thus

$$\frac{1}{2}\left(\nu(q)+\nu(-q)\right)=\frac{(q^4;q^4)_\infty}{(q^2;q^4)^2_\infty}; \tag{5.10}$$

and (1.5) follows by noting that the right hand side of (5.10) is $\overline{\text{eo}}(q)$.

There is a second proof of (1.5) which relies on an instance of Ramanujan's ${}_1\psi_1$-summation. Namely

$$\begin{aligned}
&\frac{1}{2}\left(\nu(-q)+\nu(q)\right)\\
=&\frac{1}{2}\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q^2)_{n+1}}+\frac{1}{2}\sum_{n=0}^{\infty}(-q)^n(q;q^2)_n \quad \text{(by (4.4) and (5.1))}\\
=&\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{q^{n^2+n}}{(q;q^2)_{n+1}}\\
=&\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{q^{n^2-n}}{(q;q^2)_{n}}\\
=&\frac{1}{2}\frac{2(-q^2;q^2)_\infty(-q^2;q^2)_\infty(q^2;q^2)_\infty}{(-q;q^2)_\infty(q;q^2)_\infty}\\
&\qquad\qquad \text{(by [4, p.115, eq. (C3) with } q\to q^2,\ t=-1,\ b=q\text{])}\\
=&\frac{(q^4;q^4)_\infty}{(q^2;q^4)^2_\infty},
\end{aligned}$$

as desired. □

Corollary 7.

$$p_\nu(2n)=\overline{\mathcal{EO}}(2n). \tag{5.11}$$

Proof. Compare coefficients of q^{2n} in (5.10). □

6 Identities for classical ranks and cranks

The previous five sections have been devoted primarily to $\nu(q)$ (with $\omega(q)$ in an auxiliary role) and associated partition functions. There are analogous results related to the three third order mock theta functions $f(q)$, $\phi(q)$ and $\psi(q)$ which we record here. The necessary background is already in the literature. All that is necessary is to put everything together.

First,we recall the definitions of rank and crank.

Definition. The rank of a partition π is the largest part minus the number of parts [8].

Definition. In any given partition π, we let $\omega(\pi)$ be the number of 1's in π, $l(\pi)$ is the largest part of π and $\mu(\pi)$ the number of parts of π larger than $\omega(\pi)$. Then the crank of π is defined to be $l(\pi)$ if $\omega(\pi)=0$ and $\mu(\pi)-\omega(\pi)$ if $\omega(\pi)>0$. [7]

Definition. $r_e(n)$ (resp. $r_o(n)$) is the number of partitions of n with even (resp. odd) rank.

Definition. $c_e(n)$ (resp. $c_o(n)$) is the number of partitions of n with even (resp. odd) crank.

From [5, p.70, eq. (13.1)], we know that

$$\begin{aligned}\sum_{n\geq 0}\left(r_e(n)-r_o(n)\right)q^n &= \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2} \\ &=: f(q),\end{aligned} \tag{6.1}$$

one of the third order mock theta functions [19, p.62].

From [7, p. 168, eq. (1.11)] with $z=-1$, it follows that

$$\sum_{n\geq 0}\left(c_e(n)-c_o(n)\right)q^n = \frac{(q;q)_\infty}{(-q;q)_\infty^2}. \tag{6.2}$$

It is remarkable that this function appears in the OEIS [14], [15] with no mention of cranks.

Two other mock theta functions play a role here [19, p. 62]

$$\begin{aligned}\phi(q) &= \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n} \\ &= \sum_{n=0}^{\infty} a_\phi(n)q^n,\end{aligned} \tag{6.3}$$

where [17] $a_\phi(n)$ counts the number of the partitions π of n into distinct odd parts with weight $+1$ if the largest part of π minus twice the number of parts is $\equiv 3(\mod 4)$ and weight -1 if $\equiv 1(\mod 4)$.

$$\begin{aligned}\psi(q) &= \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q^2)_n}, \\ &= \sum_{n=0}^{\infty} a_\psi(n)q^n,\end{aligned} \tag{6.4}$$

where [9, p.57] $a_\psi(n)$ is the number of partitions of n into odd parts such that each odd number smaller than the largest part is also a part.

There are two identities linking $f(q)$, $\phi(q)$ and $\psi(q)$. Namely [19, p.63], [9, p. 60, eq. (26.66)]

$$f(q)+\frac{(q;q)_\infty}{(-q;q)_\infty^2} = 2\phi(-q), \tag{6.5}$$

and

$$f(q) - \frac{(q;q)_\infty}{(-q;q)_\infty^2} = 4\psi(-q). \tag{6.6}$$

Hence by (6.1), (6.2) and (6.3)

Theorem 8.

$$r_e(n) - r_o(n) + c_e(n) - c_o(n) = 2(-1)^n a_\phi(n). \tag{6.7}$$

By (6.1), (6.2) and (6.4),

Theorem 9.

$$r_e(n) - r_o(n) - c_e(n) + c_o(n) = 4(-1)^n a_\psi(n). \tag{6.8}$$

Of course, it is immediate that

$$r_e(n) + r_o(n) = p(n) \tag{6.9}$$

and

$$c_e(n) + c_o(n) = p(n). \tag{6.10}$$

Thus (6.7)-(6.10) can be used to express each of $r_e(n)$, $r_o(n)$, $c_e(n)$ and $c_o(n)$ in terms of $p(n)$, $a_\phi(n)$ and $a_\psi(n)$.

7 Conclusion

F. Garvan [10], [7] also established that the crank produces equinumerous subclasses of $p(7n+5)(\mod 7)$ and the same was shown for $p(11n+6)(\mod 11)$. However, there appears to be no analog of (1.6) for 7 or 11.

The theorems given here provide a list of combinatorial challenges.

Problem 1. Prove (5.4) combinatorially. (Surely this is not too hard).

Problem 2. Prove Proposition 4 combinatorially. (hopefully more directly than invoking [1]).

We could make this list much longer asking for combinatorial proofs of Theorems 1, 2, 8 or 9. However, these are surely very difficult.

Problem 3 Find bijections between the various classes of partitions enumerated by $p_\nu(n)$.

On a more general possibility:

Problem 4 Undertake a more extensive investigation of the properties of $\overline{\mathcal{EO}}(n)$.

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