

# Truncated Theta Series and a Problem of Guo and Zeng

George E. Andrews

Department of Mathematics  
The Pennsylvania State University  
University Park, PA 16802 USA  
gea1@psu.edu

Mircea Merca

Academy of Romanian Scientists  
Splaiul Independentei 54  
Bucharest, 050094 Romania  
mircea.merca@profinfo.edu.ro

## Abstract

We provide partition-theoretic interpretation of two truncated identities of Gauss solving a problem by Guo and Zeng. We also reveal that these results, together with our previous truncation of Euler's pentagonal number theorem, are essentially corollaries of the Rogers-Fine identity. Finally we examine further positivity questions related to the partition function.

**Keywords:** partitions, Euler's pentagonal number theorem, theta series

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## 1 Introduction

In [3], we proved the following identity for the partition function,  $p(n)$ :

**Theorem 1.** For  $n > 0$ ,  $k \geq 1$ ,

$$(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j (p(n - j(3j + 1)/2) - p(n - j(3j + 5)/2 - 1)) = M_k(n), \quad (1.1)$$

where  $M_k(n)$  is the number of partitions of  $n$  in which  $k$  is the least integer that is not a part and there are more parts  $> k$  than there are  $< k$ .

Yee [13] has given a combinatorial proof of Theorem 1. This theorem was directly deduced from the following:

**Lemma 2.** For  $k \geq 1$ ,

$$\frac{1}{(q; q)_\infty} \sum_{j=0}^{k-1} (-1)^j q^{j(3j+1)/2} (1 - q^{2j+1}) = 1 + (-1)^{k-1} \sum_{n=1}^{\infty} \frac{q^{\binom{k}{2} + (k+1)n}}{(q; q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \quad (1.2)$$

where

$$\begin{aligned} (A; q)_n &= \prod_{j=0}^{\infty} \frac{(1 - Aq^j)}{(1 - Aq^{j+n})} \\ &= ((1 - A)(1 - Aq) \cdots (1 - Aq^{n-1})) \quad \text{if } n \text{ is a positive integer} \end{aligned}$$

and

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} 0, & \text{if } B < 0 \text{ or } B > A \\ \frac{(q; q)_A}{(q; q)_B (q; q)_{A-B}}, & \text{otherwise.} \end{cases}$$

Apart from Euler's pentagonal number theorem ( $k \rightarrow \infty$  in (1.2)):

$$(q; q)_\infty = \sum_{j=0}^{\infty} (-1)^j q^{j(3j+1)/2} (1 - q^{2j+1}), \quad (1.3)$$

there are two other central, classical theta identities (often attributed to Gauss and sometimes Jacobi) [2, p.23, eqs. (2.2.12) and (2.2.13)]:

$$\frac{(q; q)_\infty}{(-q; q)_\infty} = 1 + 2 \sum_{j=1}^{\infty} (-1)^j q^{j^2}, \quad (1.4)$$

and

$$\frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} = \sum_{j=0}^{\infty} (-1)^j q^{j(j+1)/2}. \quad (1.5)$$

We remark that the truncated theta series were recently studied in several papers by Guo and Zeng [7], Mao [10], Kolitsch [9], He, Ji and Zang [8], Chan, Ho and Mao [4], and Yee [10].

Guo and Zeng [7] proved analogues of the above Lemma 2 for (1.4) and (1.5). First they note that the reciprocal of the infinite product in (1.4) is the generating function for  $\bar{p}(n)$ , the number of overpartitions of  $n$  (cf. Corteel and Lovejoy [5]). Overpartitions are ordinary partitions with the added condition that the first appearance of any part may be overlined or not. Thus there are eight overpartitions of 3:

$$3, \bar{3}, 2 + 1, \bar{2} + 1, 2 + \bar{1}, \bar{2} + \bar{1}, 1 + 1 + 1, \bar{1} + 1 + 1.$$

They prove in analogy with Lemma 2:

**Theorem 3.** [7] *For  $|q| < 1$  and  $k \geq 1$ , there holds*

$$\begin{aligned} & \frac{(-q; q)_\infty}{(q; q)_\infty} \left( 1 + 2 \sum_{j=1}^k (-1)^j q^{j^2} \right) \\ &= 1 + (-1)^k \sum_{n=k+1}^{\infty} \frac{(-q; q)_k (-1; q)_{n-k} q^{(k+1)n}}{(q; q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \end{aligned} \quad (1.6)$$

An immediate consequence owing to the positivity of the sum on the right is:

**Corollary 4.** [7] *For  $n, k \geq 1$ , there holds*

$$(-1)^k \left( \bar{p}(n) + 2 \sum_{j=1}^k (-1)^j \bar{p}(n - j^2) \right) \geq 0. \quad (1.7)$$

Next Guo and Zeng consider  $pod(n)$ , the number of partitions of  $n$  in which odd parts are not repeated. They note that the generating function for  $pod(n)$  is the reciprocal of the product in (1.5), and in analogy with (1.1), they prove

**Theorem 5.** [7] *For  $|q| < 1$  and  $k \geq 1$ , there holds*

$$\begin{aligned} & \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{j=0}^{2k-1} (-1)^j q^{j(2j+1)} (1 - q^{2j+1}) \\ &= 1 + (-1)^{k-1} \sum_{n=k}^{\infty} \frac{(-q; q^2)_k (-q; q^2)_{n-k} q^{2(k+1)n-k}}{(q^2; q^2)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q^2}. \end{aligned} \quad (1.8)$$

As before, they deduced

**Corollary 6.** [7] *For  $n, k \geq 1$ , there holds*

$$(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j (\text{pod}(n - j(2j + 1)) - \text{pod}(n - (j + 1)(2j + 1))) \geq 0. \quad (1.9)$$

Guo and Zeng [7, p.702] note that the sum in (1.1) has a partition-theoretic interpretation, namely  $M_k(n)$ . They go onto assert: “it is still an open problem to give partition-theoretic interpretations for our two sums in (1.7) and (1.9).”

This paper has two main objects. The first is to provide the interpretations of the sums in (1.7) and (1.9) requested by Guo and Zeng. Our second goal is to provide an unified treatment of (1.2), (1.6) and (1.8) by showing that all three are essentially instances of Rogers-Fine identity [11, p.15] which we write as

$$\sum_{n=0}^{\infty} \frac{(\alpha; q)_n \tau^n}{(\beta; q)_n} = \sum_{n=0}^{\infty} \frac{(\alpha; q)_n (\alpha \tau q / \beta; q)_n \beta^n \tau^n q^{n^2 - n} (1 - \alpha \tau q^{2n})}{(\beta; q)_n (\tau; q)_{n+1}}. \quad (1.10)$$

By means of (1.10), we shall give a new proof of (1.2), and shall prove the following revisions of (1.6) and (1.8). First we treat (1.6):

**Theorem 7.** *For  $n, k \geq 1$ ,*

$$\begin{aligned} & \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left( 1 + 2 \sum_{j=1}^k (-1)^j q^{j^2} \right) \\ &= 1 + 2(-1)^k \frac{(-q; q)_k}{(q; q)_k} \sum_{j=0}^{\infty} \frac{q^{(k+1)(k+j+1)} (-q^{k+j+2}; q)_{\infty}}{(1 - q^{k+j+1})(q^{k+j+2}; q)_{\infty}}. \end{aligned} \quad (1.11)$$

From Theorem 7, we may immediately deduce

**Corollary 8.** *For  $n, k \geq 1$ ,*

$$(-1)^k \left( \bar{p}(n) + 2 \sum_{j=1}^k (-1)^j \bar{p}(n - j^2) \right) = \bar{M}_k(n), \quad (1.12)$$

where  $\bar{M}_k(n)$  is the number of overpartitions of  $n$  in which the first part larger than  $k$  appears at least  $k + 1$  times.

For example,  $\overline{M}_2(12) = 16$ , and the partitions in question are  $4 + 4 + 4$ ,  $\overline{4} + 4 + 4$ ,  $3 + 3 + 3 + 3$ ,  $\overline{3} + 3 + 3 + 3$ ,  $3 + 3 + 3 + 2 + 1$ ,  $3 + 3 + 3 + \overline{2} + 1$ ,  $3 + 3 + 3 + 2 + \overline{1}$ ,  $3 + 3 + 3 + \overline{2} + \overline{1}$ ,  $\overline{3} + 3 + 3 + 2 + 1$ ,  $\overline{3} + 3 + 3 + \overline{2} + 1$ ,  $\overline{3} + 3 + 3 + 2 + \overline{1}$ ,  $\overline{3} + 3 + 3 + \overline{2} + \overline{1}$ ,  $3 + 3 + 3 + 1 + 1 + 1$ ,  $3 + 3 + 3 + \overline{1} + 1 + 1$ ,  $\overline{3} + 3 + 3 + 1 + 1 + 1$ ,  $\overline{3} + 3 + 3 + \overline{1} + 1 + 1$ .

Next we revise (1.8):

**Theorem 9.** For  $n, k \geq 1$ ,

$$\begin{aligned} & \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{j=0}^{2k-1} (-q)^{j(j+1)/2} \\ &= 1 - (-1)^k \frac{(-q; q^2)_k}{(q^2; q^2)_{k-1}} \sum_{j=0}^{\infty} \frac{q^{k(2j+2k+1)} (-q^{2j+2k+3}; q^2)_\infty}{(q^{2k+2j+2}; q^2)_\infty}. \end{aligned} \quad (1.13)$$

**Corollary 10.** For  $n, k \geq 1$ ,

$$(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j (\text{pod}(n - j(2j + 1)) - \text{pod}(n - (j + 1)(2j + 1))) = MP_k(n),$$

where  $MP_k(n)$  is the number of partitions of  $n$  in which the first part larger than  $2k - 1$  is odd and appears exactly  $k$  times. All other odd parts appear at most once.

For example,  $MP_2(19) = 10$ , and the partitions in question are  $9 + 9 + 1$ ,  $9 + 5 + 5$ ,  $8 + 5 + 5 + 1$ ,  $7 + 7 + 3 + 2$ ,  $7 + 7 + 2 + 2 + 1$ ,  $7 + 5 + 5 + 2$ ,  $6 + 5 + 5 + 3$ ,  $6 + 5 + 5 + 2 + 1$ ,  $5 + 5 + 3 + 2 + 2 + 2$ ,  $5 + 5 + 2 + 2 + 2 + 2 + 1$ .

A further interesting corollary of Theorem 9 relates to  $p(n)$ .

**Corollary 11.** If at least one of  $n$  and  $k$  is odd,

$$(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j (p(n - j(2j + 1)) - p(n - (j + 1)(2j + 1))) \geq 0.$$

In section 2, we shall prove three key lemmas, each a special case of the Rogers-Fine identity (1.10). In section 3, we deduce Theorem 7 and Corollary 8 from Lemma 12. In section 4, we deduce Theorem 9 and Corollary 10 from Lemma 13. For completeness, in section 5 we deduce Lemma 2 from Lemma 14. Section 6 deduces Corollary 11 from Theorem 9. In the conclusion, we discuss related work and an open problem.

## 2 Background Lemmas

In this section, we collect three special cases of the Rogers-Fine identity (1.10). The key to this unification of proofs of Lemma 2, Theorem 3 and Theorem 5 is this use of (1.10), and this is made possible by treating what may be called the complimentary problem; namely, we find directly a formula for the tail of the relevant theta series instead of the truncation. It should be noted that each of our three result in this section is effectively in the literature. The next result is equivalent to [6, p.15, eq.(14.31)] and [1, p.571, eq.(3.6)].

**Lemma 12.**

$$\sum_{n=0}^{\infty} \frac{(\alpha; q)_n \alpha^n}{(-\alpha; q)_{n+1}} = \sum_{n=0}^{\infty} (-1)^n \alpha^{2n} q^{n^2}. \quad (2.1)$$

*Proof.* Set  $\beta = -\alpha q$ ,  $\tau = \alpha$  in (1.10) and multiply by  $1/(1 + \alpha)$ .  $\square$

**Lemma 13.** [1, p.571, eq.(3.5)]

$$\sum_{n=0}^{\infty} \frac{(\alpha; q^2)_n \alpha^n}{(\alpha q; q^2)_n} = \sum_{n=0}^{\infty} \alpha^n q^{\binom{n}{2}}. \quad (2.2)$$

*Proof.* In (1.10), replace  $q$  by  $q^2$ , then set  $\beta = \alpha q$  and  $\tau = \alpha$ . The result is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\alpha; q^2)_n \alpha^n}{(\alpha q; q^2)_n} &= \sum_{n=0}^{\infty} \alpha^{2n} q^{2n^2-n} (1 + \alpha q^{2n}) \\ &= \sum_{n=0}^{\infty} \alpha^n q^{\binom{n}{2}}. \end{aligned}$$

$\square$

The next result was first proved by L.J. Rogers [11, p.333] (cf. [1, p.570, eqs. (3.2) and (3.4)]) and M.V. Subbarao [12] combinatorially.

**Lemma 14.**

$$\sum_{n=0}^{\infty} \frac{(-1)^n \beta^{2n} q^{\binom{n+1}{2}}}{(\beta q; q)_n} = \sum_{n=0}^{\infty} \beta^{3n} q^{n(3n+1)/2} (1 - \beta^2 q^{2n+1}). \quad (2.3)$$

*Proof.* In (1.10), replace  $\beta$  by  $\beta q$  and  $\alpha$  by  $\beta^2 q/\tau$ . Then let  $\tau \rightarrow 0$ , and the result follows immediately.  $\square$

### 3 Overpartitions

We are now ready to prove Theorem 7 and Corollary 8.

**Proof of Theorem 7.** Dividing both sides of (1.11) by

$$\frac{(-q; q)_\infty}{(q; q)_\infty} = \frac{1}{\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}},$$

we see that we need only prove

$$(-1)^n q^{(n+1)^2} \sum_{j=0}^{\infty} \frac{(q^{n+1}; q)_j q^{j(n+1)}}{(-q^{n+1}; q)_{j+1}} = \sum_{j=n+1}^{\infty} (-1)^j q^{j^2},$$

and this is merely Lemma 12 with  $\alpha$  replaced by  $q^{n+1}$ .

**Proof of Corollary 8.** We see that in

$$\frac{(-q; q)_k}{(q; q)_k} \sum_{j=0}^{\infty} \left( \frac{2q^{(k+1)(k+j+1)}}{1 - q^{k+j+1}} \right) \left( \frac{(-q^{k+j+2}; q)_\infty}{(q^{k+j+2}; q)_\infty} \right)$$

the expression preceding the summation generates parts  $\leq k$ . The first expression inside the sum accounts for the  $\geq k+1$  appearances of the first part  $> k$ , and the second expression inside the sum accounts for all the larger parts.

### 4 Partitions with Distinct Odd Parts

We can now prove Theorem 9 and Corollary 10.

**Proof of Theorem 9.** Multiplying both sides of (1.13) by the reciprocal of

$$\frac{(q^2; q^2)_\infty}{(-q^2; q^2)_\infty} = \sum_{j=0}^{\infty} (-q)^{\binom{j+1}{2}},$$

we see that we need only prove

$$(-1)^n q^{\binom{2n+1}{2}} \sum_{j=0}^{\infty} \frac{(q^{2n}; q^2)_{j+1} q^{2jn}}{(-q^{2n+1}; q^2)_{j+1}} = \sum_{s=2n}^{\infty} (-q)^{\binom{s+1}{2}},$$

and adding  $(-1)^n q^{\binom{2n}{n}}$  to both sides while shifting  $j$  to  $j - 1$ , we see that we must show that

$$(-1)^n q^{\binom{2n}{n}} \sum_{j=0}^{\infty} \frac{(q^{2n}; q^2)_j q^{2jn}}{(-q^{2n+1}; q^2)_j} = \sum_{s=2n-1}^{\infty} (-q)^{\binom{s+1}{2}}.$$

This last identity follows immediately from Lemma 13 with  $\alpha = q^{2n}$  and multiplying by  $(-q)^{\binom{2n}{n}}$ .

**Proof of Corollary 10.** We see that in

$$\frac{(-q; q^2)_k}{(q^2; q^2)_{k-1}} \sum_{j=0}^{\infty} q^{k(2j+2k+1)} \frac{(-q^{2j+2k+3}; q^2)_{\infty}}{(q^{2j+2k+2}; q^2)_{\infty}}$$

the expression preceding the summation generates parts  $\leq 2k - 1$ . The term  $q^{k(2j+2k+1)}$  produces the  $k$  appearances of the first part,  $2j+2k+1$ , larger than  $2k - 1$ , and the final expression generates the parts larger than  $2j + 2k + 1$ .

## 5 Ordinary Partitions

As mentioned in the introduction, we shall provide a new proof of Lemma 2 to make clear how all these results are related to the Rogers-Fine identity.

**Proof of Lemma 2.** Multiplying both sides of (1.2) by  $(q; q)_{\infty}$  and recalling (1.3), we see that we need only prove

$$(-1)^k (q; q)_{\infty} \sum_{n=1}^{\infty} \frac{q^{\binom{k}{2} + (k+1)n}}{(q; q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \sum_{j=k}^{\infty} (-1)^j q^{j(3j+1)/2} (1 - q^{2j+1}).$$

Now

$$\begin{aligned} & (-1)^k (q; q)_{\infty} \sum_{n=1}^{\infty} \frac{q^{\binom{k}{2} + (k+1)n}}{(q; q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ &= (-1)^k q^{k(3k+1)/2} (q^{k+1}; q)_{\infty} \sum_{n=0}^{\infty} \frac{(q^k; q)_n q^{(k+1)n}}{(q; q)_n (q^{k+1}; q)_n} \end{aligned}$$

(where we have shifted  $n$  to  $n + k$  nothing that when  $n < k$  the terms vanished)

$$\begin{aligned}
&= (-1)^k q^{k(3k+1)/2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2} + 2kn}}{(q^{k+1}; q)_n} \\
&\quad \text{(by [2, p.38, eq.(3.3.13), } a = 0, t = c = q^{k+1}, b \rightarrow 0\text{)]} \\
&= (-1)^k q^{k(3k+1)/2} \sum_{j=0}^{\infty} (-1)^j q^{3kj + j(3j+1)/2} (1 - q^{2j+2k+1}) \\
&\quad \text{(by Lemma 14 with } \beta = q^k\text{)} \\
&= \sum_{j=k}^{\infty} (-1)^j q^{j(3j+1)/2} (1 - q^{2j+1}).
\end{aligned}$$

## 6 Corollary 11

Corollary 11 may be easily deduced from either Theorem 5 (due to Guo and Zeng) or Theorem 9.

**Proof of Corollary 11.** Multiply both sides of (1.8) (or (1.13)) by  $(-q^2; q^2)_{\infty}$ . After simplification, we find

$$\frac{(-1)^{k-1}}{(q; q)_{\infty}} \sum_{j=0}^{k-1} (-1)^j q^{j(2j+1)} (1 - q^{2j+1}) = (-1)^{k-1} (-q^2; q^2)_{\infty} + G_k(q), \quad (6.1)$$

where

$$G_k(q) = (-q^2; q^2)_{\infty} \sum_{n=k}^{\infty} \frac{(-q; q^2)_k (-q; q^2)_{n-k} q^{2(k+1)n-k}}{(q^2; q^2)_n} \left[ \begin{matrix} n-1 \\ k-1 \end{matrix} \right]_{q^2}$$

has nonnegative coefficients. Note that  $(-q^2; q^2)_{\infty}$  is an even function of  $q$  with positive coefficients. Now if  $k$  is odd then every terms in (6.1) is nonnegative. If  $k$  is even, then because  $(-q^2; q^2)_{\infty}$  is an even function of  $q$ , all coefficients of odd powers are nonnegative.

## 7 Conclusions

We have successfully provided partition theoretic interpretations for the three classical theta series (1.3), (1.4) and (1.5). There were further conjectures on the full Jacobi triple product that appear in both [3] and [7]. A. J. Yee [13] provided a combinatorial proof of (1.1) and also answered most all of the questions posed in [3] and [7].

Relevant to Conjecture 6.4 of [7], it would be very appealing to have a combinatorial interpretation of

$$J_k(n) = (-1)^k \sum_{j=0}^k (-1)^j (2j+1) t(n - j(j+1)/2),$$

where

$$\frac{1}{(q; q)_\infty^3} = \sum_{n=0}^{\infty} t(n) q^n.$$

Also for  $n$  odd or  $k$  even, there is a substantial amount of numerical evidence to conjecture that the sum in Corollary 11 is less than or equal to  $M_k(n)$ .

**Conjecture.** *For  $n$  odd or  $k$  even,*

$$(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j (p(n - j(2j+1)) - p(n - (j+1)(2j+1))) \leq M_k(n).$$

Finally combinatorial proofs of Corollaries 8 and 10 would be very interesting.

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