# Euler's Partition Identity and Two Problems of George Beck 

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#### Abstract

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Euler's famous partition identity asserts that the number of partitions of an integer $n$ into odd parts equals the number of partitions of $n$ into distinct parts. This paper examines what happens if one even part might be allowed among the odd parts or one part might be repeated thrice among distinct parts. This study yields proofs of two conjectures of George Beck.


## 1 Introduction

Our starting point is sequence A090867 in the On-Line Encyclopedia of Integer Sequence [3]. The sequence in question, $a(n)$, counts the number of partitions of $n$ such that the set of even parts has only one element. Thus $a(5)=4$ where the relvant partitions are $4+1,3+2,2+2+1$ and $2+1+1+1$.

The sequence $a(n)$ is a natural one to investigate in light of Euler's theorem [1, p. 5, Cor. 1.2]:

The number of partitions of $n$ into odd parts equals the number of partitions of $n$ into distinct parts.

Thus the partitions counted by $a(n)$ are much like Euler's partitions with off parts except now a single even number occurs as a part (possibly repeated).

Also on the page for A090867, we find the following conjecture by George Beck:
Conjecture. $a(n)$ is also the difference between the number of parts in the odd partitions of $n$ and the number of parts in the distinct partitions of $n$ (offset $0)$. For example, if $n=5$, there are 9 parts in the odd partitions of $5(5,311$, 11111) and 5 parts in the distinct partitions of $5(5,41,32)$, with difference 4.

- George Beck, Apr 222017

Let us define $b(n)$ to be the difference between the number of parts in the odd partitions of $n$ and the number of parts in the distinct partitions of $n$.

While we are at it, let us define $c(n)$ to be the number of partitions of $n$ in which exactly one part is repeated. Thus $c(5)=4$ with the relevant partitions being $3+1+1,2+2+1,2+1+1+1$, and $1+1+1+1+1$.

Theorem 1. For all $n \geq 1, a(n)=b(n)=c(n)$.
So far we have seen that the theorem is true for $n=5$. Our proof relies on generating functions and differentiation. There are many combinatorial proofs of Euler's Theorem (cf. [2]). Can one prove this new theorem combinatorially?

In section 3, we treat a further conjecture of George Beck also related to Euler's theorem.

## 2 Proof of Theorem 1

As mentioned in the introduction, we require the generating functions for our sequences:

$$
\begin{align*}
& A(q)=\sum_{n \geq 0} a(n) q^{n}  \tag{2.1}\\
& B(q)=\sum_{n \geq 0} b(n) q^{n} \tag{2.2}
\end{align*}
$$

and

$$
\begin{equation*}
C(q)=\sum_{n \geq 0} c(n) q^{n} \tag{2.3}
\end{equation*}
$$

To prove our theorem, we shall show that each of $A(q), B(q)$ and $C(q)$ is equal to

$$
\begin{equation*}
\prod_{n=1}^{\infty} \frac{1}{1-q^{2 n-1}} \sum_{m=1}^{\infty} \frac{q^{2 m}}{1-q^{2 m}} \tag{2.4}
\end{equation*}
$$

The equality of the generating functions then proves the theorem.
Throughout our proof we shall require the elegant, elementary identity of Euler used by him to prove his theorem [1, p.5, eq. (1.2.5)] for $|q|<1$,

$$
\begin{equation*}
\prod_{n=1}^{\infty} \frac{1}{1-q^{2 n-1}}=\prod_{n=1}^{\infty} \frac{\left(1-q^{2 n}\right)}{\left(1-q^{n}\right)}=\prod_{n=1}^{\infty}\left(1+q^{n}\right) \tag{2.5}
\end{equation*}
$$

Let us do the easiest part first.

$$
\begin{aligned}
C(q) & =\sum_{n=1}^{\infty}\left(q^{2 n}+q^{2 n+2 n}+q^{2 n+2 n+2 n}+\cdots\right) \prod_{m=1}^{\infty} \frac{1}{1-q^{2 m-1}} \\
& =\left(\sum_{n=1}^{\infty} \frac{q^{2 n}}{1-q^{2 n}}\right) \prod_{m=1}^{\infty} \frac{1}{1-q^{2 m-1}},
\end{aligned}
$$

and we have establish that $C(q)$ is the expression in (2.4).

The next easiest part is $A(q)$. Clearly $A(q)$ is the coefficient of $z$ in

$$
\begin{aligned}
& \prod_{n=1}^{\infty}\left(1+q^{n}+z q^{n+n}+z q^{n+n+n}+\cdots\right) \\
= & \prod_{n=1}^{\infty}\left(1+q^{n}+\frac{z q^{2 n}}{1-q^{n}}\right)
\end{aligned}
$$

and the coefficient of $z$ is:

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{q^{2 n}}{1-q^{n}} \prod_{\substack{m=1 \\
m \neq n}}^{\infty}\left(1+q^{m}\right) \\
= & \prod_{m=1}^{\infty}\left(1+q^{m}\right) \sum_{n=1}^{\infty} \frac{q^{2 n}}{1-q^{n}} \cdot \frac{1}{1+q^{n}} \\
= & \prod_{m=1}^{\infty}\left(1+q^{m}\right) \sum_{n=1}^{\infty} \frac{q^{2 n}}{1-q^{2 n}} \\
= & \prod_{m=1}^{\infty} \frac{1}{1-q^{2 m-1}} \sum_{n=1}^{\infty} \frac{q^{2 n}}{1-q^{2 n}} \quad(\text { by }(2.5)) .
\end{aligned}
$$

Hence $A(q)$ is also equal to the expression in (2.4).
Finally, we have the trickier problem of $B(q)$. In the infinite product

$$
\prod_{n=1}^{\infty} \frac{1}{1-z q^{2 n-1}}
$$

the coefficient of $z^{M} q^{N}$ is the number of partitions of $N$ into $M$ odd parts, and in the infinite product

$$
\prod_{n=1}^{\infty}\left(1+z q^{n}\right)
$$

the coefficient of $z^{M} q^{N}$ is the number of partitions of $N$ into $M$ distinct parts [1, Ch. 2, p. 16].

So if we differentiate each of these functions with respect to $z$ we will then be counting each partition with $M$ parts with weight $M$.

Consequently

$$
\begin{aligned}
B(q) & =\left.\frac{\partial}{\partial z}\right|_{z=1}\left(\prod_{m=1}^{\infty} \frac{1}{\left(1-z q^{2 m-1}\right)}-\prod_{m=1}^{\infty}\left(1+z q^{m}\right)\right) \\
& =\sum_{n=1}^{\infty} \frac{q^{2 n-1}}{\left(1-q^{2 n-1}\right)^{2}} \prod_{\substack{m=1 \\
m \neq n}}^{\infty} \frac{1}{1-q^{2 m-1}}
\end{aligned}
$$

$$
\begin{align*}
& -\sum_{n=1}^{\infty} q^{n} \prod_{\substack{m=1 \\
m \neq n}}^{\infty}\left(1+q^{m}\right) \\
= & \prod_{m=1}^{\infty} \frac{1}{1-q^{2 m-1}} \sum_{n=1}^{\infty} \frac{q^{2 n-1}}{1-q^{2 n-1}} \\
& -\prod_{m=1}^{\infty}\left(1+q^{m}\right) \sum_{n=1}^{\infty} \frac{q^{n}}{1+q^{n}} \\
= & \prod_{m=1}^{\infty} \frac{1}{1-q^{2 m-1}}\left(\sum_{n=1}^{\infty} \frac{q^{2 n-1}}{1-q^{2 n-1}}-\sum_{n=1}^{\infty} \frac{q^{n}}{1+q^{n}}\right) \quad(\text { by }(2.5)) \\
= & \prod_{m=1}^{\infty} \frac{1}{1-q^{2 m-1}}\left(\sum_{n=1}^{\infty} \frac{q^{2 n-1}}{1-q^{2 n-1}}-\sum_{n=1}^{\infty} \frac{q^{n}\left(1-q^{n}\right)}{1-q^{2 n}}\right) \\
= & \prod_{m=1}^{\infty} \frac{1}{1-q^{2 m-1}} \sum_{n=1}^{\infty} \frac{q^{2 n}}{1-q^{2 n}} \\
& +\prod_{m=1}^{\infty} \frac{1}{1-q^{2 m-1}}\left(\sum_{n=1}^{\infty} \frac{q^{2 n-1}}{1-q^{2 n-1}}-\sum_{n=1}^{\infty} \frac{q^{n}}{1-q^{2 n}}\right) \\
= & \prod_{m=1}^{\infty} \frac{1}{1-q^{2 m-1}} \sum_{n=1}^{\infty} \frac{q^{2 n}}{1-q^{2 n}}, \tag{2.6}
\end{align*}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{q^{2 n-1}}{1-q^{2 n-1}} & =\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q^{m(2 n-1)} \\
& =\sum_{m=1}^{\infty} \frac{q^{m}}{1-q^{2 m}}
\end{aligned}
$$

and (2.6) establishes that $B(q)$ is also equal to the expression in (2.4).

## 3 George Beck's Second Problem

In the On-Line Encyclopedia of Integer Sequences, we find also sequence A265251. The sequence in question, $a_{1}(n)$, is the number of partitions of $n$ such that there is exactly one part ocurring three times while all other parts occur only once. George Beck made the following:

Conjecture. $a_{1}(n)$ is also the difference between the number of parts in the distinct partitions of $n$ and the number of distinct parts in the odd partitions of $n$ (offset 0 ). For example, if $n=5$, there are 5 parts in the distinct partitions of $5(5,41,32)$ and 4 distinct parts in the odd partitions of 5 (namely, $5,(3,1)$, 1 in $5,311,11111$ ) with difference 1.

Here we define $b_{1}(n)$ to be the difference between the total number of parts in the partitions of $n$ into distinct parts and the total number of different parts in the partitions of $n$ into odd parts.

Theorem 2. $a_{1}(n)=b_{1}(n)$.
Proof. We let

$$
A_{1}(q)=\sum_{n \geq 0} a_{1}(n) q^{n}
$$

and

$$
B_{1}(q)=\sum_{n \geq 0} b_{1}(n) q^{n}
$$

As in the proof of Theorem 1, we see that $A_{1}(q)$ is the coefficient of $z$ in

$$
\prod_{n=1}^{\infty}\left(1+q^{n}+z q^{3 n}\right)
$$

Hence

$$
\begin{equation*}
A_{1}(q)=\prod_{n=1}^{\infty}\left(1+q^{n}\right) \sum_{n=1}^{\infty} \frac{q^{3 n}}{1+q^{n}} \tag{3.1}
\end{equation*}
$$

Next, as in our treatment of $B(q)$, we see that $B_{1}(q)$ must be

$$
\begin{aligned}
B_{1}(q) & =\left.\frac{\partial}{\partial z}\right|_{z=1}\left(\prod_{n=1}^{\infty}\left(1+z q^{n}\right)-\prod_{n=1}^{\infty}\left(1+\frac{z q^{2 n-1}}{1-q^{2 n-1}}\right)\right) \\
& =\prod_{n=1}^{\infty}\left(1+q^{n}\right) \sum_{m=1}^{\infty} \frac{q^{m}}{1+q^{m}}-\prod_{n=1}^{\infty} \frac{1}{1-q^{2 n-1}} \sum_{m=1}^{\infty} q^{2 m-1}
\end{aligned}
$$

Hence by (2.5)

$$
\begin{aligned}
& A_{1}(q)-B_{1}(q) \\
& =\prod_{n=1}^{\infty}\left(1+q^{n}\right)\left(\sum_{n=1}^{\infty} \frac{q^{3 n}}{1+q^{n}}-\sum_{n=1}^{\infty} \frac{q^{n}}{1+q^{n}}+\frac{q}{1-q^{2}}\right) \\
& =\prod_{n=1}^{\infty}\left(1+q^{n}\right)\left(-\sum_{n=1}^{\infty} \frac{q^{n}\left(1-q^{2 n}\right)}{1+q^{n}}+\frac{q}{1-q^{2}}\right) \\
& =\prod_{n=1}^{\infty}\left(1+q^{n}\right)\left(-\sum_{n=1}^{\infty} q^{n}\left(1-q^{n}\right)+\frac{q}{1-q^{2}}\right) \\
& =\prod_{n=1}^{\infty}\left(1+q^{n}\right)\left(-\frac{q}{1-q}+\frac{q^{2}}{1-q^{2}}+\frac{q}{1-q^{2}}\right) \\
& =0
\end{aligned}
$$

and Theorem 2 is proved.

## 4 Conclusion

It would be very interesting to provide bijective proofs of any of the assertions in our theorems. As we noted, there are many bijective proofs of Euler's theorem.

It might also be interesting to examine what would happen if we were to allow repetitions of two different parts or appearances of two even parts, but the differentiation technique suggests that the resulting theorems would be messy and somewhat unattractive.

## References

[1] G. E. Andrews, The Theory of Partitions, Addison-Wesley, Reading, 1976 (Reprinted: Cambridge University Press, Cambridge, 1985).
[2] G. E. Andrews, Euler's partition identity, The Mathematics Student, (to appear).
[3] The On-Line Encyclopedia of Integer Sequences, Sequence A090867, https://oeis.org.
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