Partitions With Parts Separated By Parity

by
George E. Andrews

Key Words: partitions, parity of parts, Ramanujan
AMS Classification Numbers: 11P84, 11P83, 11P81

Abstract

There have been a number of papers on partitions in which the parity of parts play a central role. In this paper, the parts of partitions are separated by parity, either all odd parts are smaller than all even parts or vice versa. This concept first arose in a study related to the third order mock theta function $\nu(q)$. The current study also leads back to one of Ramanujan's more mysterious functions.

1 Introduction

Probably the first appearance of parity in partitions arose in Legende's interpretation of Euler's pentagonal number theorem

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}. \tag{1.1}$$

Legendre [14, pp.128-133] interpreted (1.1) as follows:

Theorem (Legendre's Theorem)**.** *Let $D_e(n)$ (resp. $D_o(n)$) denote the number of partitions of n into an even (resp. odd) number of distinct parts. Then*

$$D_e(n) - D_o(n) = \begin{cases} (-1)^i & \text{if } n - j(3j \pm 1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

F. Franklin [12] provided the famous combinatorial proof of Legendre's theorem.

Subsequently parity has played a substantial role in assertion from Ramanujan's Lost Notebook [7, sec. 9.5] and in extension of the Rogers-Ramanujan identities [4], [13] as well as many other instances.

In this paper, we consider partitions in which parts of a given parity are all smaller than those of the other parity. We designate both cases where the even (resp. odd) parts are distinct with the couplet "*ed*" (resp. "*od*"), or when the even (resp. odd) parts may appear an unlimited number of times with the couplet "eu" (resp. "ou"). Our eight partition functions, $p_{xy}^{zw}(n)$ will designate the partition functions in question where the xy will constrain the smaller parts and the zw the larger parts, with "xy" and "zw" will be among the aforementioned couplets.

For example $p_{eu}^{ou}(n)$ denotes the number of partitions of n in which each even part is less than each odd part. It is a simple exercise (see [6, Sec. 2]) to show that

$$F_{eu}^{ou}(q) := \sum_{n\geq 0} p_{eu}^{ou}(n)q^n = \frac{1}{(1-q)(q^2;q^2)_\infty}, \tag{1.2}$$

where

$$(A;q)_n = (1-A)(1-Aq)\dots(1-Aq^{n-1}). \tag{1.3}$$

The study in [6] was primarily centered on a subset of the partitions associated with $p_{eu}^{ou}(n)$, which were closely related to the third order mock theta function $\nu(q)$. In this paper, we extend these considerations of partitions to eight cases where there is no interlacing of parity in the parts.

Among the surprises resulting from this study are the following partition identities.

Theorem 1. *The number of partitions of n into parts each >1 with each odd part less than each even part equals*

$$p_o(n) - p_e(n) - p_e(n-1),$$

where $p_o(n)$ (resp. $p_e(n)$) is the number of partitions of n into odd (resp. even) parts.

For example, there are three partitions of 10 described in the theorem, namely $3+3+4$, $5+5$, $7+3$. On the other hand $p_o(10) = 10$, $p_e(10) = 7$ and $p_e(9) = 0$.

Theorem 2. *Let $\mathcal{O}_d(n)$ denote the number of partitions of n in which the odd parts are distinct and each odd integer smaller than the largest odd part must appear as a part. Then*

$$p_{eu}^{od}(n) = \mathcal{O}_d(n).$$

The 6 partitions enumerated by $\mathcal{O}_d(9)$ are 8+1, 6+2+1, 4+4+1, 4+2+2+1, $5+3+1$ and those enumerated by $p_{eu}^{od}(9)$ are 9, $7+2$, $5+4$, $5+3+1$, $5+2+2$, $3+2+2+2$.

A further surprise occur for $p_{od}^{eu}(n)$, namely:

Theorem 3.

$$F_{od}^{eu}(-q) = \frac{1}{(q^2;q^2)_\infty}\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}(-1)^{n+j}q^{n(3n+1)/2-j^2}(1-q^{2n+1}). \tag{1.4}$$

Thus $F^{eu}_{od}(q)$ is closely related to the function

$$R(q)=\sum_{n=0}^{\infty}\frac{q^{\binom{n+1}{2}}}{(-q;q)_n}$$

which arose in two enigmatic identities from Ramanujan's Lost Notebook [3] (cf [2], [8]), and has played such an important role in the study of weak Mauss forms (cf. [9], [10], [15], [16]).

2 Proof of Theorem 1.

We begin by recalling the Rogers-Fins identity [6, p. 223] (cf. [11, p. 15]).

$$\sum_{n\geq 0}\frac{(\alpha;q)_n\tau^n}{(\beta;q)_n}=\sum_{n\geq 0}\frac{(\alpha;q)_n(\frac{\alpha\tau q}{\beta};q)_n\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta;q)_n(\tau;q)_{n+1}} \tag{2.1}$$

Hence

$$\begin{aligned}\sum_{n\geq 0}p^{eu}_{ou}(n)q^n&=\sum_{n\geq 0}\frac{q^{2n+1}}{(q;q^2)_{n+1}(q^{2n+2};q^2)_\infty} \\ &=\frac{q}{(1-q)(q^2;q^2)_\infty}\sum_{n\geq 0}\frac{(q^2;q^2)_nq^{2n}}{(q^3;q^2)_n} \\ &=\frac{q}{(1-q)(q^2;q^2)_n}\sum_{n\geq 0}(1+q^{2n+2})q^{2n^2+3n} \\ &\qquad\text{(by (2.1) with } q\to q^2,\ \alpha=q^2,\ \beta=q^3,\ \tau=q^2) \\ &=\frac{1}{(1-q)(q^2;q^2)_\infty}\left(\sum_{n=-\infty}^{\infty}q^{(2n+1)(n+1)}-1\right) \\ &=\frac{1}{(1-q)(q^2;q^2)_\infty}\left(\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}-1\right)\quad\text{(by [1, p. 23])} \\ &=\frac{1}{1-q}\left(\frac{1}{(q;q^2)_\infty}-\frac{1}{(q^2;q^2)_\infty}\right)\end{aligned} \tag{2.2}$$

Hence if we exclude the appearance of 1 from the partitions enumerated by $p^{eu}_{ou}(n)$, we find

$$\sum_{n\geq 0}\frac{q^{2n+1}}{(q^3;q^2)_n(q^{2n+2};q^2)_\infty}=\frac{1}{(q;q^2)_\infty}-\frac{1}{(q^2;q^2)_\infty}. \tag{2.3}$$

Finally if we subtract $q/(q^2;q^2)_\infty$, we obtain

$$\sum_{n\geq 1}\frac{q^{2n+1}}{(q^3;q^2)_n(q^{2n+2};q^2)_\infty}=\frac{1}{(q;q^2)_\infty}-\frac{(1+q)}{(q^2;q^2)_\infty} \tag{2.4}$$

$$= \sum_{n\geq 0} (p_o(n) - p_e(n) - p_e(n-1))q^n,$$

and comparing coefficient of q^n on both sides of (2.4) we obtain Theorem 1.

3 Proof of Theorem 2

$$\begin{aligned}
&\sum_{n\geq 0} p_{eu}^{od}(n) = F_{eu}^{od}(q) \\
&= \sum_{n\geq 0} \frac{q^{2n}(-q^{2n+1};q^2)_\infty}{(q^2;q^2)_n} \\
&= (-q;q^2)_\infty \sum_{n\geq 0} \frac{q^{2n}}{(q^2;q^2)_n(-q;q^2)_n}.
\end{aligned}$$

Hence

$$\begin{aligned}
F_{eu}^{od}(-q) &= \frac{1}{2}(q;q^2)_\infty \sum_{n\geq 0} \frac{q^n(1+(-1)^n)}{(q;q)_n} \\
&= \frac{1}{2}(q;q^2)_\infty \left(\frac{1}{(q;q)_\infty} + \frac{1}{(-q;q)_\infty} \right) \quad \text{(by [1, p. 19])} \\
&= \frac{1}{2} \left(\frac{1}{(q^2;q^2)_\infty} + (q;q^2)_\infty^2 \right) \quad \text{(by [1, p. 5])}.
\end{aligned}$$

Therefore

$$\begin{aligned}
F_{eu}^{od}(q) &= \frac{1}{2(q^2;q^2)_\infty} \left(1 + \sum_{n=-\infty}^{\infty} q^{n^2} \right) \quad \text{(by [1, p. 23])} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{n^2} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{1+3+\cdots+(2n-1)},
\end{aligned} \tag{3.1}$$

and now comparing coefficients of q^n on both sides of (3.1) we obtain Theorem 2. □

4 Proof of Theorem 3

$$F_{od}^{eu} = \sum_{n\geq 0} \frac{(-q;q^2)_n q^{2n+1}}{(q^{2n+2};q^2)_\infty} \tag{4.1}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} (-q;-q)_{2n} q^{2n+1}.$$

Hence

$$F_{od}^{eu}(-q) = \frac{-q}{2(q^2;q^2)_\infty} \sum_{n\geq 0} (q;q)_n q^n (1+(-1)^n). \tag{4.2}$$

We now require two identities from the literature. First, by (2.1) with $\beta = 0$, $\alpha = \tau = q$,

$$\sum_{n\geq 0} (q;q)_n q^n = q^{-1} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n-1)/2} (1+q^n), \tag{4.3}$$

and, by [8, p. 162] and [9, p. 153]

$$1+q\sum_{n=0}^{\infty} (-1)^n q^n (q;q)_n = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n} \tag{4.4}$$

$$= \sum_{\substack{n\geq 0 \\ |j|\leq n}} (-1)^{n+j} q^{n(3n+1)/2-j^2} (1-q^{2n+1}) \tag{4.5}$$

Now substitute the relevant expressions from (4.3) and (4.4) into (4.2), and, after simplification, we obtain Theorem 3.

5 An Anti-telescoping, Combinatorial Proof of Theorem 1

Theorem 1 implies that if $n > 1$, then the number of partitions of n into odd parts is always greater than the number of partitions of n into even parts. It is worth noting that Theorem 1 has a very direct proof following the anti-telescoping method of [5], and the crucial step in the anti-telescoping method has an almost immediate bijective proof.

This latter assertion is, in fact,

$$\frac{1}{(1-q^{2j})(1-q)} + \frac{q^{2j-1}}{(1-q^{2j})(1-q^{2j-1})} = \frac{1}{(1-q^{2j-1})(1-q)}, \tag{5.1}$$

which may be rewritten as

$$\frac{1}{(1-q^{(2j-1)+1})(1-q)} + \frac{q^{2j-1}}{(1-q^{(2j-1)+1})(1-q^{2j-1})} = \frac{1}{(1-q^{2j-1})(1-q)}. \tag{5.2}$$

the first term on the left of (5.2) is the generating function for partition into 1's and $(2j-1)$'s with at least as many 1's as $(2j-1)$'s. The second term on the left of (5.2) is the generating function for partition into 1's and $(2j-1)$'s with more $(2j-1)$'s than 1's. Consequently the two terms taken together generate all partitions into 1's and $(2j-1)$'s , and that is precisely what is generated by the right side of (5.2) (of course, (5.2) is immediately proved algebraically).

We now multiply both sides of (5.1) by $1/((q^{2j+2};q^2)_\infty(q^3;q^2)_{j-2})$ and isolate on the left the term with q^{2j-1} in the numerator. Hence

$$\frac{q^{2j-1}}{(q^3;q^2)_{j-1}(q^{2j};q^2)_\infty} = \frac{1}{(q;q^2)_j(q^{2j+2};q^2)_\infty} - \frac{1}{(q;q^2)_{j-1}(q^{2j};q^2)_\infty}. \tag{5.3}$$

We sum (5.3) for $1 \le j \le N$, to obtain (via telescoping the right hand side)

$$\sum_{j=1}^{N} \frac{q^{2j-1}}{(q^3;q^2)_{j-1}(q^{2j};q^2)_\infty} = \frac{1}{(q;q^2)_N(q^{2N+2};q^2)_\infty} - \frac{1}{(q^2;q^2)_\infty}, \tag{5.4}$$

and letting $N \to \infty$, we obtain (2.3) which is equivalent to Theorem 1.

6 The Remaining Four Functions

Of the remaining four functions, both $F_{ed}^{od}(q)$ and $F_{od}^{ed}(q)$ can be transformed by (2.1), the Rogers-Fine identity, but the resulting formulas are not noticeably simpler than the original generating functions.

As for $F_{ed}^{ou}(q)$, we find

$$\begin{aligned} F_{ed}^{ou} &= \sum_{n\ge 0} \frac{(-q^2;q^2)_n q^{2n}}{(q^{2n+1};q^2)_\infty} \\ &= \frac{1}{(q;q^2)_\infty} \sum_{n\ge 0} (q;-q)_{2n} q^{2n} \end{aligned} \tag{6.1}$$

hence by (6.1),

$$\begin{aligned} F_{ed}^{eu}(-q) &= \frac{1}{(-q;q^2)_\infty} \sum_{n\ge 0} (-q;q)_{2n} q^{2n} \\ &= \frac{1}{2(-q;q^2)_\infty} \sum_{n\ge 0} (-q;q)_n q^n (1+(-1)^n) \end{aligned} \tag{6.2}$$

Now expanding $(-q;q)_\infty$ by considering the largest of the partitions generated, we see that

$$1 + \sum_{n\ge 0} (-q;q)_n q^{n+1} = (-q;q)_\infty, \tag{6.3}$$

and by (2.1) with $\alpha = q$, $\beta = 0$ and $\tau = -q$, we find that

$$\sum_{n\geq 0}(-q)_n(-q)^{n+1} = \sum_{n\geq 0} q^{n(3n-1)/2}(1-q^n). \tag{6.4}$$

Substituting (6.3) and (6.4) into the final expression in (6.2), we obtain

$$F_{ed}^{ou}(-q) = \frac{q^{-1}}{2(-q;q^2)_\infty}\left((-q;q)_\infty - 1 + \sum_{n\geq 1} q^{n(3n-1)/2}(1-q^n)\right). \tag{6.5}$$

Finally, for $F_{ou}^{ed}(q)$, we see that

$$\begin{aligned} F_{ou}^{ed}(q) &= \sum_{n\geq 0}\frac{q^{2n+1}(-q^{2n+2};q^2)_\infty}{(q;q^2)_{n+1}} \\ &= (-q^2;q^2)_\infty \sum_{n\geq 0}\frac{q^{2n+1}}{(q;-q)_{2n+1}}. \end{aligned} \tag{6.6}$$

Therefore

$$F_{ou}^{ed}(-q) = \frac{1}{2}(-q^2;q^2)_\infty \sum_{n\geq 0}\frac{q^n}{(-q;q)_n}(1-(-1)^n). \tag{6.7}$$

By Heine's transformation [1, p. 19]

$$\sum_{n\geq 0}\frac{q^n}{(-q;q)^n} = 2 - \frac{1}{(-q;q)_\infty}, \tag{6.8}$$

and by (2.1) with $\alpha = 0$, $\beta = -q$ and $\tau = -q$,

$$\sum_{n\geq 0}\frac{(-q)^n}{(-q;1)_n} = \sum_{n\geq 0}\frac{q^{n^2+n}}{(-q;q)_n^2(1+q^{n+1})}, \tag{6.9}$$

Thus

$$F_{ou}^{ed}(-q) = \frac{1}{2}\left(2 - \frac{1}{(-q;q)_\infty} - \sum_{n\geq 0}\frac{q^{n^2+n}}{(-q;q)_n^2(1+q^{n+1})}\right) \tag{6.10}$$

7 Conclusion

Our discoveries plus those in [6] suggest that partitions with separated parity are worthy of further study. In particular, in [6], the really interesting class of partition was a subset of those enumerated by $p_{eu}^{ou}(n)$. This points especially to possible subclasses of the eight different type of partitions considered in this paper.

Of course, Theorem 2 cries out for a bijective proof.

References

[1] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, 1976 (Reissued: Cambridge University Press, Cambridge, 1998)

[2] G. E. Andrews, *Questions and conjectures in partition theory*, Amer. Math. Monthly, 93(1986), 708-711.

[3] G. E. Andrews, *Ramanujan's "lost" notebook, V: Euler's partition identity*, Adv. in Math., 61(1986), 156-164.

[4] G. E. Andrews, *Parity in partition identities*, Ramanujan J., 23(2010), 45-90.

[5] G. E. Andrews, *Differences of partition functions: The anti-telescoping method*. In From Fourier Analysis and Number Theory to Radon Transformations and Geometry- In Memory of Leon Ehrenpreis, Developments in Mathematics, Vol. 28, H. M. Farkas, R. C. Gunning, M. I. Knopp & A. Taylor, eds., pp. 1-20. Springer, 2013.

[6] G. E. Andrews, *Integer partitions with even parts below odd parts and mock theta functions*, Annals of Combinatorics, (to appear).

[7] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Part I*, Springer, New York, 2005.

[8] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Part II*, Springer, New York, 2009

[9] G. E. Andrews, F. J. Dyson and D. R. Hickson, *Partitions and indefinite quadratic forms*, Inventiones Math., 91(1988), 391-407.

[10] H. Cohen, *q-identities for Mauss waveforms*, Inventiones Math., 91(1988), 409-422.

[11] N. J. Fine, *Basic Hypergeometric Series and Applications*, Amer. Math. Soc., Providence, RI, 1988.

[12] F. Franklin, *Sur le dévelopment du produit infini* $(1-x)(1-x^2)(1-x^3)\ldots$, Comptes Rendus, 82(1881), 448-450.

[13] K. Kursungoz, *Parity considerations in Andrews-Gordon identities*, European J. Comb., 31(2010), 976-1000.

[14] A. M. Legendre, *Théore des nombres*, Fŕmin Didot frères, Paris, 1830.

[15] Y. Li, H. Ngo, R. C. Rhoades, *Renormalization and quantum modular forms, I, Mauss wave forms*, (to appear).

[16] Y. Li, H. Ngo, R. C. Rhoades, *Renormalization and quantum forms, II, mock theta functions*, (to appear).

The Pennsylvania State University
University Park, PA 16802
gea1@psu.edu