ON BASIC HYPERGEOMETRIC SERIES, MOCK THETA FUNCTIONS, AND PARTITIONS (I)

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1. Introduction

In [(1) 66], Bailey defines the basic hypergeometric function as

$${}_r\Phi_s\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_r;\ z;\ q\\ \rho_1,\ldots,\rho_s\end{matrix}\right]=\sum_{n=0}^{\infty} z^n \prod_{j=0}^{n-1}\frac{(1-\alpha_1 q^j)\ldots(1-\alpha_r q^j)}{(1-q^{j+1})(1-\rho_1 q^j)\ldots(1-\rho_s q^j)}.$$

Heine [(3) 97–125] was the first to study this function. He proved the fundamental identity

$${}_2\Phi_1\left[\begin{matrix}a,b;\ z;\ q\\ c\end{matrix}\right]=\prod_{j=0}^{\infty}\frac{(1-bq^j)(1-azq^j)}{(1-cq^j)(1-zq^j)}\,{}_2\Phi_1\left[\begin{matrix}c/b,z;\ b;\ q\\ az\end{matrix}\right].$$

The identities involving the mock theta functions of fifth order can be proved by using special cases of the above identity together with re-arrangement of infinite series [5]. The object of this paper is to extend Watson's method [5] to obtain general identities of basic hypergeometric type. Most of the known identities for the mock theta functions of both third and fifth order are special cases of these general identities.

It turns out that, if

$$F(a,b,c;\ k,l;\ r,s;\ q;\ x;\ t)\equiv\sum_{n=0}^{\infty}\prod_{\lambda=0}^{rn+s-1}\frac{(1-aq^\lambda)}{(1-q^{\lambda+1})}\prod_{\nu=0}^{kn+l-1}\frac{(1-bx^\nu)}{(1-cx^\nu)}t^n,$$

then, with ξ a primitive rth root of unity,

$$F(a,b,c;\ k,l;\ r,s;\ q;\ x;\ t)$$

$$=r^{-1}\prod_{\lambda=0}^{\infty}\frac{(1-bx^\lambda)}{(1-cx^\lambda)}\sum_{j=0}^{r-1}\xi^{-lsj-sj}\sum_{m=0}^{\infty}\prod_{\beta=0}^{m-1}\frac{(1-cb^{-1}x^\beta)}{(1-x^{\beta+1})}\times$$

$$\times\prod_{\gamma=0}^{\infty}\frac{(1-a\xi^{j}t^{1/r}x^{km/r}q^\gamma)}{(1-\xi^{j}t^{1/r}x^{km/r}q^\gamma)}(bx^{l-ks/r})^m \quad (1.1)$$

under suitable conditions.

Utilizing the fact that

$${}_2\Phi_1\left[\begin{matrix}c/b,z;\ b;\ q\\ az\end{matrix}\right]={}_2\Phi_1\left[\begin{matrix}z,c/b;\ b;\ q\\ az\end{matrix}\right], \quad (1.2)$$

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Heine derived a new identity by applying his fundamental identity to ${}_2\Phi_1\left[\begin{matrix} z,\ c/b;\ b;\ q \\ az \end{matrix}\right]$. By utilizing formulae similar to (1.2) for $F(a,b,c;\,k,l;\,r,s;\,q;\,x;\,t)$, we shall prove many new identities of basic hypergeometric type by means of (1.1); most of the known mock-theta-function identities will be special cases of these new formulae.

In § 2, we shall prove (1.1) and derive the basic hypergeometric series identities. In § 3, we shall deduce the mock-theta-function identities and we shall exhibit an infinite number of new mock theta functions.

In a subsequent paper, I plan to examine further the consequences of this development. In particular, partition theorems and identities related to the mock-theta-function identities will be studied.

2. Fundamental theorems

To simplify notation, we use the following abbreviations due to Watson [(5) 275]:

$$\Pi_n(-a,q) \equiv \prod_{j=0}^{n-1}(1-aq^j), \qquad \Pi_\infty(-a,q) \equiv \prod_{j=0}^{\infty}(1-aq^j).$$

We shall need the identity [(1) 66]

$$\frac{\Pi_\infty(-az,q)}{\Pi_\infty(-z,q)} = \sum_{n=0}^{\infty}\frac{\Pi_n(-a,q)}{\Pi_n(-q,q)}z^n. \tag{2.1}$$

We also note that, if ξ is a primitive rth root of unity, then

$$r^{-1}\sum_{j=0}^{r-1}\xi^{js} = \begin{cases} 1 & s \equiv 0 \pmod r, \\ 0 & s \not\equiv 0 \pmod r \end{cases} \quad s \text{ an integer.}$$

Fundamental Lemma (1). *If k, l, r, s are integers, $0 \leqslant l < k$, $0 \leqslant s < r$, $|t| < 1$, $|x| < 1$, $|q| < 1$, $|b| < |x|^{ks/r-l}$, and ξ a primitive rth root of unity, then*

$$F(a,b,c;\,k,l;\,r,s;\,q;\,x;\,t)$$
$$= r^{-1}\frac{\Pi_\infty(-b,x)}{\Pi_\infty(-c,x)}\times$$
$$\times\sum_{j=0}^{r-1}\xi^{-js}t^{-s/r}\sum_{m=0}^{\infty}\frac{\Pi_m(-c/b,x)\Pi_\infty(-a\xi^j t^{1/r}x^{km/r},q)}{\Pi_m(-x,x)\Pi_\infty(-\xi^j t^{1/r}x^{km/r},q)}(bx^{l-ks/r})^m.$$

Proof. $F(a,b,c;k,l;r,s;q;x;t) = \sum_{n=0}^{\infty} \frac{\Pi_{rn+s}(-a,q)\Pi_{kn+l}(-b,x)}{\Pi_{rn+s}(-q,q)\Pi_{kn+l}(-c,x)} t^n$

$$= \frac{\Pi_\infty(-b,x)}{\Pi_\infty(-c,x)} \sum_{n=0}^{\infty} \frac{\Pi_{rn+s}(-a,q)\Pi_\infty(-cx^{kn+l},x)}{\Pi_{rn+s}(-q,q)\Pi_\infty(-bx^{kn+l},x)} t^n$$

$$= r^{-1} \frac{\Pi_\infty(-b,x)}{\Pi_\infty(-c,x)} \sum_{j=0}^{r-1} \xi^{-js} t^{-s/r} \sum_{n=0}^{\infty} \frac{\Pi_n(-a,q)\Pi_\infty(-cx^{k(n-s)/r+l},x)}{\Pi_n(-q,q)\Pi_\infty(-bx^{k(n-s)/r+l},x)} (\xi^j t^{1/r})^n$$

$$= r^{-1} \frac{\Pi_\infty(-b,x)}{\Pi_\infty(-c,x)} \sum_{j=0}^{r-1} \xi^{-js} t^{-s/r} \sum_{n=0}^{\infty} \frac{\Pi_n(-a,q)}{\Pi_n(-q,q)} (\xi^j t^{1/r})^n \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_m(-c/b,x)}{\Pi_m(-x,x)} (bx^{k(n-s)/r+l})^m, \quad \text{by (2.1)},$$

$$= r^{-1} \frac{\Pi_\infty(-b,x)}{\Pi_\infty(-c,x)} \times$$

$$\times \sum_{j=0}^{r-1} \xi^{-js} t^{-s/r} \sum_{m=0}^{\infty} \frac{\Pi_m(-c/b,x)\Pi_\infty(-a\xi^j t^{1/r} x^{km/r},q)}{\Pi_m(-x,x)\Pi_\infty(-\xi^j t^{1/r} x^{km/r},q)} (bx^{l-ks/r})^m,$$

by (2.1).

The conditions of the lemma justify the above interchanges of summation.

We now take a special case of the fundamental lemma which will be of central importance in our theory.

THEOREM A. *Under the conditions of the fundamental lemma,*

$F(a,b,c;k,l;r,s;q^e;q^d;t)$

$$= r^{-1} \frac{\Pi_\infty(-b,q^d)}{\Pi_\infty(-c,q^d)} \sum_{j=0}^{r-1} \xi^{-js} t^{-s/r} \sum_{\mu=0}^{re-1} (bq^{dl-kds/r})^\mu \frac{\Pi_\infty(-a\xi^j t^{1/r} q^{kd\mu/r},q^e)}{\Pi_\infty(-\xi^j t^{1/r} q^{kd\mu/r},q^e)} \times$$

$$\times F(c/b, \xi^j t^{1/r} q^{kd\mu/r}, a\xi^j t^{1/r} q^{kd\mu/r}; kd, 0; re, \mu; q^d; q^e; b^{re} q^{de(rl-ks)})$$

Proof. $F(a,b,c;k,l;r,s;q^e;q^d;t)$

$$= r^{-1} \frac{\Pi_\infty(-b,q^d)}{\Pi_\infty(-c,q^d)} \sum_{j=0}^{r-1} \xi^{-js} t^{-s/r} \sum_{m=0}^{\infty} \frac{\Pi_m(-c/b,q^d)\Pi_\infty(-a\xi^j t^{1/r} q^{kmd/r},q^e)}{\Pi_m(-q^d,q^d)\Pi_\infty(-\xi^j t^{1/r} q^{kmd/r},q^e)} \times$$

$$\times (bq^{dl-kds/r})^m$$

(by the fundamental lemma)

$$= r^{-1}\frac{\Pi_\infty(-b,q^d)}{\Pi_\infty(-c,q^d)}\sum_{j=0}^{r-1}\xi^{-js\eta-sj/r}\sum_{\mu=0}^{re-1}(bq^{dl-kdsj/r})^\mu\times$$

$$\times\sum_{p=0}^{\infty}\frac{\Pi_{rep+\mu}(-c/b,q^d)\,\Pi_\infty(-a\xi^{j}t^{1/r}q^{kd\mu/r+kdep},q^e)(bq^{dl-kdsj/r})^{rep}}{\pi_{rep+\mu}(-q^d,q^d)\,\Pi_\infty(-\xi^{j}t^{1/r}q^{kd\mu/r+kdep},q^e)}$$

(by putting $m = rep+\mu$)

$$= r^{-1}\frac{\Pi_\infty(-b,q^d)}{\Pi_\infty(-c,q^d)}\sum_{j=0}^{r-1}\xi^{-js\eta-sj/r}\sum_{\mu=0}^{re-1}(bq^{dl-kdsj/r})^\mu\,\frac{\Pi_\infty(-a\xi^{j}t^{1/r}q^{kd\mu/r},q^e)}{\Pi_\infty(-\xi^{j}t^{1/r}q^{kd\mu/r},q^e)}\times$$

$$\times\sum_{p=0}^{\infty}\frac{\Pi_{rep+\mu}(-c/b,q^d)\,\Pi_{kdp}(-\xi^{j}t^{1/r}q^{kd\mu/r},q^e)}{\Pi_{rep+\mu}(-q^d,q^d)\,\Pi_{kdp}(-a\xi^{j}t^{1/r}q^{kd\mu/r},q^e)}(b^{re}q^{dlre-kdse})^p.$$

This is the desired result. We now consider four specializations of Theorem A.

THEOREM A_1.
$$\sum_{n=0}^{\infty}\frac{\Pi_n(-a,q^2)\,\Pi_n(-b,q)}{\Pi_n(-q^2,q^2)\Pi_n(-c,q)}t^n$$

$$= \frac{\Pi_\infty(-b,q)\,\Pi_\infty(-at,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(-t,q^2)}\sum_{m=0}^{\infty}\frac{\Pi_{2m}(-c/b,q)\,\Pi_m(-t,q^2)}{\Pi_{2m}(-q,q)\,\Pi_m(-at,q^2)}b^{2m}+$$

$$+\frac{\Pi_\infty(-b,q)\Pi_\infty(-atq,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(-tq,q^2)}\sum_{m=0}^{\infty}\frac{\Pi_{2m+1}(-c/b,q)\,\Pi_m(-tq,q^2)}{\Pi_{2m+1}(-q,q)\Pi_m(-atq,q^2)}b^{2m+1}.$$

Proof. In Theorem A, take $k = r = 1$, $s = l = 0$, $e = 2$, $d = 1$.

THEOREM A_2 (Heine).
$$\sum_{n=0}^{\infty}\frac{\Pi_n(-a,q^2)\,\Pi_n(-b,q^2)}{\Pi_n(-q^2,q^2)\,\Pi_n(-c,q^2)}t^n$$

$$= \frac{\Pi_\infty(-b,q^2)\,\Pi_\infty(-at,q^2)}{\Pi_\infty(-c,q^2)\,\Pi_\infty(-t,q^2)}\sum_{m=0}^{\infty}\frac{\Pi_m(-c/b,q^2)\,\Pi_m(-t,q^2)}{\Pi_m(-q^2,q^2)\,\Pi_m(-at,q^2)}b^m.$$

Proof. In Theorem A, take $k = r = 1$, $s = l = 0$, $e = d = 2$.

THEOREM A_3.
$$\sum_{n=0}^{\infty}\frac{\Pi_n(-a,q^2)\,\Pi_{2n}(-b,q)}{\Pi_n(-q^2,q^2)\,\Pi_{2n}(-c,q)}t^n$$

THEOREM A_2. *If $s = 0$ or 1, then*

$$\sum_{n=0}^{\infty} \frac{\Pi_{2n+s}(-a,q)\,\Pi_n(-b,q)}{\Pi_{2n+s}(-q,q)\,\Pi_n(-c,q)}\,t^n$$

$$= \tfrac{1}{2}t^{-\frac{1}{2}s}\,\frac{\Pi_\infty(-b,q)\,\Pi_\infty(-at^{\frac{1}{2}},q)}{\Pi_\infty(-c,q)\,\Pi_\infty(-t^{\frac{1}{2}},q)} \sum_{m=0}^{\infty} \frac{\Pi_{2m}(-c/b,q)\,\Pi_m(-t^{\frac{1}{2}},q)}{\Pi_{2m}(-q,q)\,\Pi_m(-at^{\frac{1}{2}},q)}\,(b^2q^{-s})^m +$$

$$+\tfrac{1}{2}t^{-\frac{1}{2}s}(-1)^s\,\frac{\Pi_\infty(-b,q)\,\Pi_\infty(at^{\frac{1}{2}},q)}{\Pi_\infty(-c,q)\,\Pi_\infty(t^{\frac{1}{2}},q)} \sum_{m=0}^{\infty} \frac{\Pi_{2m}(-c/b,q)\,\Pi_m(t^{\frac{1}{2}},q)}{\Pi_{2m}(-q,q)\,\Pi_m(at^{\frac{1}{2}},q)}\,(b^2q^{-s})^m +$$

$$+\tfrac{1}{2}t^{-\frac{1}{2}s}\,\frac{\Pi_\infty(-b,q)\,\Pi_\infty(-at^{\frac{1}{2}}q^{\frac{1}{2}},q)}{\Pi_\infty(-c,q)\,\Pi_\infty(-t^{\frac{1}{2}},q)}\times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_{2m+1}(-c/b,q)\,\Pi_m(-t^{\frac{1}{2}}q^{\frac{1}{2}},q)}{\Pi_{2m+1}(-q,q)\,\Pi_m(-at^{\frac{1}{2}}q^{\frac{1}{2}},q)}\,(b^2q^{-s})^{m+\frac{1}{2}} +$$

$$+\tfrac{1}{2}t^{-\frac{1}{2}s}(-1)^s\,\frac{\Pi_\infty(-b,q)\,\Pi_\infty(at^{\frac{1}{2}}q^{\frac{1}{2}},q)}{\Pi_\infty(-c,q)\,\Pi_\infty(t^{\frac{1}{2}}q^{\frac{1}{2}},q)}\times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_{2m+1}(-c/b,q)\,\Pi_m(t^{\frac{1}{2}}q^{\frac{1}{2}},q)}{\Pi_{2m+1}(-q,q)\,\Pi_m(at^{\frac{1}{2}}q^{\frac{1}{2}},q)}\,(b^2q^{-s})^{m+\frac{1}{2}}.$$

Proof. In Theorem A, take $k = 1$, $r = 2$, $l = 0$, $e = d = 1$.

We now combine these theorems to prove identities which will yield the mock-theta-function identities as special cases. In two instances, we shall actually exhibit our results in terms of the basic hypergeometric function ${}_r\Phi_s$.

THEOREM 1.
$$\sum_{n=0}^{\infty} \frac{\Pi_n(-cq/bt,q^2)\,\Pi_n(-b,q)}{\Pi_n(-q^2,q^2)\,\Pi_n(-c,q)}\,t^n$$

$$= \frac{\Pi_\infty(-cq/b,q^2)\,\Pi_\infty(-cb,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(-q,q^2)\,\Pi_\infty(b,q)} \sum_{m=0}^{\infty} \frac{\Pi_m(-q/t,q^2)\,\Pi_m(-b^2,q^2)}{\Pi_m(-q^2,q^2)\,\Pi_m(-cb,q^2)}\,t^m +$$

$$+\frac{b\,\Pi_\infty(-c/b,q^2)\,\Pi_\infty(-cbq,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(-q,q^2)\,\Pi_\infty(b,q)} \sum_{m=0}^{\infty} \frac{\Pi_m(-q^3/t,q^2)\,\Pi_m(-b^2,q^2)}{\Pi_m(-q^2,q^2)\,\Pi_m(-cbq,q^2)}\,t^mq^m.$$

In the notation of basic hypergeometric series, the theorem becomes

$${}_3\Phi_2\begin{bmatrix} (cq/bt)^{\frac{1}{2}}, -(cq/bt)^{\frac{1}{2}}, b; t; q \\ -q, c \end{bmatrix}$$

$$= \frac{\Pi_\infty(-cq/b,q^2)\,\Pi_\infty(-cb,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(-q,q^2)\,\Pi_\infty(b,q)}\,{}_2\Phi_1\begin{bmatrix} q/t, b^2; t; q^2 \\ cb \end{bmatrix} +$$

$$+\frac{\Pi_\infty(-c/b,q^2)\,\Pi_\infty(-cbq,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(-q,q^2)\,\Pi_\infty(b,q)}\,{}_2\Phi_1\begin{bmatrix} q^3/t, b^2; tq; q^2 \\ cbq \end{bmatrix}.$$

Proof.
$$\sum_{n=0}^{\infty} \frac{\Pi_n(-cq/bt, q^2)\,\Pi_n(-b,q)}{\Pi_n(-q^2,q^2)\,\Pi_n(-c,q^2)}\, t^n$$

$$= \frac{\Pi_\infty(-b,q)\,\Pi_\infty(-cq/b,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(-t,q^2)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_m(-c/b,q^2)\,\Pi_m(-cq/b,q^2)\,\Pi_m(-t,q^2)}{\Pi_m(-q^2,q^2)\,\Pi_m(-q,q^2)\,\Pi_m(-cq/b,q^2)}\, b^{2m} -$$

$$- \frac{b\,\Pi_\infty(-b,q)\,\Pi_\infty(-cq^2/b,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(-tq,q^2)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{(1-c/b)\,\Pi_m(-cq^2/b,q^2)\,\Pi_m(-cq/b,q^2)\,\Pi_m(-tq,q^2)}{\Pi_m(-q^2,q^2)(1-q)\,\Pi_m(-q^3,q^2)\,\Pi_m(-cq^2/b,q^2)}\, b^{2m}$$

(by Theorem A_1, with $a = cq/bt$)

$$= \frac{\Pi_\infty(-b,q)\,\Pi_\infty(-cq/b,q^2)\,\Pi_\infty(-t,q^2)\,\Pi_\infty(-cb,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(-t,q^2)\,\Pi_\infty(-q,q^2)\,\Pi_\infty(-b^2,q^2)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_m(-q/t,q^2)\,\Pi_m(-b^2,q^2)}{\Pi_m(-q^2,q^2)\,\Pi_m(-cb,q^2)}\, t^m +$$

$$+ \frac{b(1-c/b)\,\Pi_\infty(-b,q)\,\Pi_\infty(-cq^2/b,q^2)\,\Pi_\infty(-tq,q^2)\,\Pi_\infty(-cbq,q^2)}{(1-q)\,\Pi_\infty(-c,q)\,\Pi_\infty(-tq,q^2)\,\Pi_\infty(-q^3,q^2)\,\Pi_\infty(-b^2,q^2)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_m(-q^2/t,q^2)\,\Pi_m(-b^2,q^2)}{\Pi_m(-q^2,q^2)\,\Pi_m(-cbq,q^2)}\, t^m q^m$$

(by Theorem A_2 applied to the sums in the previous equation). Simplifying this last expression, we obtain the desired result.

THEOREM 2.
$$\sum_{n=0}^{\infty} \frac{\Pi_n(q/t,q)\,\Pi_n(-b,q)}{\Pi_n(-q,q)\,\Pi_n(-c,q)}\, t^n$$

$$= \frac{\Pi_\infty(-bq,q^2)\,\Pi_\infty(q,q)\,\Pi_\infty(-c^2/b,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(c/b,q)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_{2m}(c/bt,q)\,\Pi_m(-b,q^2)}{\Pi_{2m}(-q,q)\,\Pi_m(-c^2/b,q^2)}\, t^{2m} +$$

$$+ \frac{\Pi_\infty(-b,q^2)\,\Pi_\infty(q,q)\,\Pi_\infty(-c^2q/b,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(c/b,q)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_{2m+1}(c/bt,q)\,\Pi_m(-bq,q^2)}{\Pi_{2m+1}(-q,q)\,\Pi_m(-c^2q/b,q^2)}\, t^{2m+1}.$$

In the notation of basic hypergeometric series, the theorem becomes

$$
{}_2\Phi_1\left[\begin{matrix}-q/t, b; t; q\\ c\end{matrix}\right]
$$

$$
= \frac{\Pi_\infty(-bq,q^2)\,\Pi_\infty(q,q)\,\Pi_\infty(-c^2/b,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(c/b,q)}\;{}_3\Phi_2\left[\begin{matrix}-c/bt, -cq/bt, -b; t^2; q^2\\ q, c^2/b\end{matrix}\right]+
$$

$$
+\frac{t\,\Pi_\infty(-b,q^2)\,\Pi_\infty(q,q)\,\Pi_\infty(-c^2q/b,q^2)(1-c/bt)}{\Pi_\infty(-c,q)\,\Pi_\infty(c/b,q)(1-q)}\times
$$

$$
\times\,{}_3\Phi_2\left[\begin{matrix}-cq^2/bt, -cq^3/bt, bq; t^2; q^2\\ q^3, c^2q/b\end{matrix}\right].
$$

Proof.

$$
\sum_{n=0}^{\infty}\frac{\Pi_n(q/t,q)\,\Pi_n(-b,q)}{\Pi_n(-q,q)\,\Pi_n(-c,q)}t^n
$$

$$
= \frac{\Pi_\infty(-b,q)\,\Pi_\infty(q,q)}{\Pi_\infty(-c,q)\,\Pi_\infty(-t,q)}\sum_{m=0}^{\infty}\frac{\Pi_m(-c/b,q)\,\Pi_m(-t,q)}{\Pi_m(-q,q)\,\Pi_m(q,q)}b^m
$$

(by Theorem A_2, first replacing q^2 by q, and then setting $a = -q/t$)

$$
= \frac{\Pi_\infty(-b,q)\,\Pi_\infty(q,q)}{\Pi_\infty(-c,q)\,\Pi_\infty(-t,q)}\sum_{m=0}^{\infty}\frac{\Pi_m(-c^2/b^2,q^2)\Pi_m(-t,q)}{\Pi_m(-q^2,q^2)\,\Pi_m(c/b,q)}b^m
$$

$$
= \frac{\Pi_\infty(-b,q)\,\Pi_\infty(q,q)\,\Pi_\infty(-t,q)\,\Pi_\infty(-c^2/b,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(-t,q)\,\Pi_\infty(c/b,q)\,\Pi_\infty(-b,q^2)}\times
$$

$$
\times\sum_{m=0}^{\infty}\frac{\Pi_{2m}(c/bt,q)\,\Pi_m(-b,q^2)}{\Pi_{2m}(-q,q)\,\Pi_m(-c^2/b,q^2)}t^{2m}+
$$

$$
+\frac{\Pi_\infty(-b,q)\,\Pi_\infty(q,q)\,\Pi_\infty(-t,q)\,\Pi_\infty(-c^2q/b,q^2)}{\Pi_\infty(-c,q)\,\Pi_\infty(-t,q)\,\Pi_\infty(c/b,q)\,\Pi_\infty(-bq,q^2)}\times
$$

$$
\times\sum_{m=0}^{\infty}\frac{\Pi_{2m+1}(c/bt,q)\,\Pi_m(-bq,q^2)}{\Pi_{2m+1}(-q,q)\,\Pi_m(-c^2q/b,q^2)}t^{2m+1}
$$

(by Theorem A_1).

Simplifying this last expression, we obtain the desired result.

THEOREM 3. *If $l = 0$, or 1, then*

$$\sum_{n=0}^{\infty} \frac{\Pi_n(-b, q^2) q^{\frac{1}{2}n(n+1)+ln}}{\Pi_n(-q, q)\, \Pi_n(-c, q^2)}$$

$$= \frac{\Pi_\infty(-bq^2, q^4)\, \Pi_\infty(q^2, q^2)\, \Pi_\infty(-c/b, q^2)}{\Pi_\infty(-c, q^2)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_{2m}(q^{1+2l}b/c, q^2)\, \Pi_m(-b, q^4)}{\Pi_{2m}(-q^2, q^2)} (c/b)^{2m} +$$

$$+ \frac{\Pi_\infty(-b, q^4)\, \Pi_\infty(q^2, q^2)\, \Pi_\infty(-c/b, q^2)}{\Pi_\infty(-c, q^2)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_{2m+1}(q^{1+2l}b/c, q^2)\, \Pi_m(-bq^2, q^4)}{\Pi_{2m+1}(-q^2, q^2)} (c/b)^{2m+1}.$$

Proof. In Theorem A_2, interchange b and t, then replace c by $-q^{1+l}$; then replace a by c/b and let $t \to 0$. We thus obtain

$$\sum_{n=0}^{\infty} \frac{\Pi_n(-b, q^2) q^{\frac{1}{2}n(n+1)+ln}}{\Pi_n(-q, q)\, \Pi_n(-c, q^2)}$$

$$= \frac{\Pi_\infty(-b, q^2)\, \Pi_\infty(q^{1+l}, q)}{\Pi_\infty(-c, q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-c/b, q^2)}{\Pi_m(-q^2, q^2)\, \Pi_{2m}(q^{1+l}, q)} b^m$$

$$= \frac{\Pi_\infty(-b, q^2)\Pi_\infty(q^{1+l}, q)}{\Pi_\infty(-c, q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-c/b, q^2)}{\Pi_m(-q^4, q^4)\Pi_m(q^{1+2l}, q^2)} b^m$$

$$= \frac{\Pi_\infty(-b, q^2)\, \Pi_\infty(q^{1+l}, q)\, \Pi_\infty(-c/b, q^2)}{\Pi_\infty(-c, q^2)\, \Pi_\infty(q^{1+2l}, q^2)\, \Pi_\infty(-b, q^4)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_{2m}(q^{1+2l}b/c, q^2)\, \Pi_m(-b, q^4)}{\Pi_{2m}(-q^2, q^2)} (c/b)^{2m} +$$

$$+ \frac{\Pi_\infty(-b, q^2)\, \Pi_\infty(q^{1+l}, q)\, \Pi_\infty(-c/b, q^2)}{\Pi_\infty(-c, q^2)\, \Pi_\infty(q^{1+2l}, q^2)\, \Pi_\infty(-bq^2, q^4)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_{2m+1}(q^{1+2l}b/c, q^2)\, \Pi_m(-bq^2, q^4)}{\Pi_{2m+1}(-q^2, q^2)} (c/b)^{2m+1}$$

(by Theorem A_1).

Simplifying this last expression we obtain the desired result.

THEOREM 4. *If $l = 0$, or 1, then*

$$\sum_{n=0}^{\infty} \frac{\Pi_n(q^{1+l}/t, q^2)\,\Pi_n(-b, q)}{\Pi_n(-q^2, q^2)} t^n$$

$$= \frac{b^l \Pi_\infty(-tb^2q^l, q^2)}{\Pi_\infty(-tq^l, q^2)\,\Pi_\infty(-q, q^2)\,\Pi_\infty(b, q)} \times$$

$$\times \sum_{m=0}^{\infty} (-1)^m \frac{\Pi_{2m}(-b^2, q^2)}{\Pi_m(-q^4, q^4)\,\Pi_{2m}(-tq^l b^2, q^2)} q^{2m^2+4lm} +$$

$$+ \frac{b^{1-l}\Pi_\infty(q^2, q^2)\,\Pi_\infty(-b, q)}{\Pi_\infty(-q, q^2)\,\Pi_\infty(-b^2, q^4)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_{2m}(-q^{2-l}/t, q^2)\,\Pi_m(-b^2, q^4)}{\Pi_{2m}(-q^2, q^2)} t^{2m} q^{(1-l)2m} +$$

$$+ \frac{b^{1-l}\Pi_\infty(q^2, q^2)\,\Pi_\infty(-b, q)}{\Pi_\infty(-q, q^2)\,\Pi_\infty(-b^2q^2, q^4)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_{2m+1}(-q^{2-l}/t, q^2)\,\Pi_m(-b^2q^2, q^4)}{\Pi_{2m+1}(-q^2, q^2)} t^{2m+1} q^{(1-l)(2m+1)}.$$

Proof.

$$\sum_{n=0}^{\infty} \frac{\Pi_n(q^{1+l}/t, q^2)\,\Pi_n(-b, q)}{\Pi_n(-q^2, q^2)} t^n$$

$$= \frac{\Pi_\infty(-b, q)\,\Pi_\infty(q^{1+2l}, q^4)}{(-tq^l, q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-tq^l, q^2)}{\Pi_{2m+l}(-q, q)\,\Pi_m(q^{1+2l}, q^2)} b^{2m+l} +$$

$$+ \frac{\Pi_\infty(-b, q)\,\Pi_\infty(q^3, q^4)}{\Pi_\infty(-tq^{1-l}, q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-tq^{1-l}, q^2)}{\Pi_{2m+1-l}(-q, q)\,\Pi_m(q^3, q^2)} b^{2m+1-l}$$

(by putting $c = 0$ and $a = q^{1+l}/t$ in Theorem A_1 and noting that the set $\{l, 1-l\}$ is the set $\{0, 1\}$)

$$= \frac{\Pi_\infty(-b, q)\,\Pi_\infty(q^{1+2l}, q^4)}{\Pi_\infty(-tq^l, q^2)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_\infty(-tq^l, q^2)}{\Pi_m(-q^2, q^2)\,\Pi_{m+l}(-q, q^2)\,\Pi_{m+l}(q, q^2)} b^{2m+l} +$$

$$+ \frac{\Pi_\infty(-b, q)\,\Pi_\infty(q^3, q^4)}{\Pi_\infty(-tq^{1-l}, q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-tq^{1-l}, q^2)}{\Pi_m(-q^2, q^2)\,\Pi_{m+1-l}(-q, q^2)\,\Pi_m(q^2, q^2)} b^{2m+1-l}$$

$$= \frac{\Pi_\infty(-b,q)\Pi_\infty(q,q^2)\Pi_\infty(-q^{2+4l},q^4)}{\Pi_\infty(-tq^l,q^2)\Pi_\infty(-q^2,q^4)} \sum_{m=0}^{\infty} \frac{\Pi_m(-tq^l,q^2)\,b^{2m+l}}{\Pi_m(-q^2,q^2)\Pi_m(-q^{2+4l},q^4)} +$$

$$+ \frac{\Pi_\infty(-b,q)\Pi_\infty(q^2,q^2)\Pi_\infty(-q^{3-2l},q^2)}{\Pi_\infty(-tq^{1-l},q^2)\Pi_\infty(-q,q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-tq^{1-l},q^2)\,b^{2m+1-l}}{\Pi_m(-q^4,q^4)\Pi_m(-q^{3-2l},q^2)}$$

$$= \frac{b^l\Pi_\infty(-b,q)\Pi_\infty(-q^{2+4l},q^4)\Pi_\infty(-tb^2q^l,q^2)}{\Pi_\infty(-tq^l,q^2)\Pi_\infty(-q,q^2)\Pi_\infty(-q^{2+4l},q^4)\Pi_\infty(-b^2,q^4)} \times$$

$$\times \sum_{m=0}^{\infty} (-1)^m \frac{\Pi_{2m}(-b^2,q^2)}{\Pi_m(-q^4,q^4)\Pi_{2m}(-tq^lb^2,q^2)} q^{2m^2+4lm} +$$

$$+ \frac{b^{1-l}\Pi_\infty(-b,q)\Pi_\infty(q^2,q^2)\Pi_\infty(-q^{3-2l},q^2)\Pi_\infty(-tq^{1-l},q^2)}{\Pi_\infty(-tq^{1-l},q^2)\Pi_\infty(-q,q^2)\Pi_\infty(-q^{3-2l},q^2)\Pi_\infty(-b^2,q^4)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_{2m}(-q^{1-l}/t,q^2)\Pi_m(-b^2,q^4)}{\Pi_{2m}(-q^2,q^2)} t^{2m} q^{(1-l)2m} +$$

$$+ \frac{b^{1-l}\Pi_\infty(-b,q)\Pi_\infty(q^2,q^2)\Pi_\infty(-q^{3-2l},q^2)\Pi_\infty(-tq^{1-l},q^2)}{\Pi_\infty(-tq^{1-l},q^2)\Pi_\infty(-q,q^2)\Pi_\infty(-q^{3-2l},q^2)\Pi_\infty(-b^2q^2,q^4)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_{2m+1}(-q^{1-l}/t,q^2)\Pi_m(-b^2q^2,q^4)}{\Pi_{2m+1}(-q^2,q^2)} t^{2m+1} q^{(1-l)(2m+1)},$$

by applying Theorem A_2 to the first summand and Theorem A_1 to the second summand in the previous equation.

Simplifying this last expression, we obtain the desired result.

THEOREM 5. *If $s = 0$, or 1, then*

$$\sum_{n=0}^{\infty} \frac{\Pi_n(-b,q)q^n}{\Pi_{2n+s}(-q,q)}$$

$$= \frac{1}{2} \frac{q^{-\frac{1}{2}s}(1+(-1)^s)\Pi_\infty(-b,q)}{\Pi_\infty(-q,q^2)\Pi_\infty(-b^2q^{-s},q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-b^2q^{-s},q^2)}{\Pi_{2m}(-q,q)} q^{2m^2} +$$

$$+ \frac{1}{2} \frac{q^{\frac{1}{2}(1-s)}(1-(-1)^s)\Pi_\infty(-b,q)}{\Pi_\infty(-q,q^2)\Pi_\infty(-b^2q^{1-s},q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-b^2q^{1-s},q^2)}{\Pi_{2m+1}(-q,q)} q^{2m^2+2m} +$$

$$+ \frac{1}{2} \frac{bq^{-s}\Pi_\infty(-b,q)}{\Pi_\infty(-q,q)} \sum_{m=0}^{\infty} \frac{(b^2q^{-s})^m}{\Pi_{m+1}(-q^{m+1},q)} +$$

$$+ (-1)^s \frac{1}{2} \frac{bq^{-s}\Pi_\infty(-b,q)}{\Pi_\infty(-b^2q^{-s},q)} \sum_{m=0}^{\infty} (-1)^m \frac{\Pi_{2m}(-b^2q^{-s},q)}{\Pi_m(-q^2,q^2)} q^{m^2+2m}.$$

Proof.

$$\sum_{n=0}^{\infty}\frac{\Pi_n(-b,q)q^n}{\Pi_{2n+s}(-q,q)}=\frac{1}{2}\frac{q^{-\frac{1}{2}s}\Pi_\infty(-b,q)}{\Pi_\infty(-q^{\frac12},q)}\sum_{m=0}^{\infty}\frac{(b^2q^{-s})^m}{\Pi_m(-q^2,q^2)\Pi_m(q^{\frac12},q)}+$$

$$+(-1)^s\frac{1}{2}\frac{q^{-\frac{1}{2}s}\Pi_\infty(-b,q)}{\Pi_\infty(q^{\frac12},q)}\sum_{m=0}^{\infty}\frac{(b^2q^{-s})^m}{\Pi_m(-q^2,q^2)\Pi_m(-q^{\frac12},q)}+$$

$$+\frac{1}{2}\frac{q^{-\frac{1}{2}s}\Pi_\infty(-b,q)}{\Pi_\infty(-q,q)}\sum_{m=0}^{\infty}\frac{(b^2q^{-s})^{m+\frac12}}{\Pi_{m+1}(-q^{m+1},q)}+$$

$$+(-1)^s\frac{1}{2}\frac{q^{-\frac{1}{2}s}\Pi_\infty(-b,q)}{(1-q)\Pi_\infty(q,q)}\sum_{m=0}^{\infty}\frac{(b^2q^{-s})^{m+\frac12}}{\Pi_m(-q,q)\Pi_m(-q^2,q^2)}$$

(by Theorem A_4, setting $a=c=0$, $t=q$ and simplifying)

$$=\frac{1}{2}\frac{q^{-\frac{1}{2}s}\Pi_\infty(-b,q)}{\Pi_\infty(-q^{\frac12},q)\Pi_\infty(q^{\frac12},q)\Pi_\infty(-b^2q^{-s},q^2)}\sum_{m=0}^{\infty}\frac{\Pi_m(-b^2q^{-s},q^2)}{\Pi_{2m}(-q,q)}q^{2m^2}+$$

$$+\frac{1}{2}\frac{q^{-\frac{1}{2}s}\Pi_\infty(-b,q)}{\Pi_\infty(-q^{\frac12},q)\Pi_\infty(q^{\frac12},q)\Pi_\infty(-b^2q^{1-s},q^2)}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\Pi_m(-b^2q^{1-s},q^2)}{\Pi_{2m+1}(-q,q)}q^{2m^2+2m+\frac12}+$$

$$+\frac{1}{2}\frac{q^{-\frac{1}{2}s}(-1)^s\Pi_\infty(-b,q)}{\Pi_\infty(q^{\frac12},q)\Pi_\infty(-q^{\frac12},q)\Pi_\infty(-b^2q^{-s},q^2)}\sum_{m=0}^{\infty}\frac{\Pi_m(-b^2q^{-s},q^2)}{\Pi_{2m}(-q,q)}q^{2m^2}-$$

$$-\frac{1}{2}\frac{q^{-\frac{1}{2}s}(-1)^s\Pi_\infty(-b,q)}{\Pi_\infty(q^{\frac12},q)\Pi_\infty(-q^{\frac12},q)\Pi_\infty(-b^2q^{-s},q^2)}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\Pi_m(-b^2q^{1-s},q^2)}{\Pi_{2m+1}(-q,q)}q^{2m^2+2m+\frac12}+$$

$$+\frac{1}{2}\frac{q^{-\frac{1}{2}s}\Pi_\infty(-b,q)}{\Pi_\infty(-q,q)}\sum_{m=0}^{\infty}\frac{(b^2q^{-s})^{m+\frac12}}{\Pi_{m+1}(-q^{m+1},q)}+$$

$$+(-1)^s\frac{1}{2}\frac{q^{-\frac{1}{2}s}(b^2q^{-s})^{\frac12}\Pi_\infty(-b,q)}{(1-q)\Pi_\infty(q,q)\Pi_\infty(-q^2,q^2)\Pi_\infty(-b^2q^{-s},q)}\times$$

$$\times\sum_{m=0}^{\infty}(-1)^m\frac{\Pi_{2m}(-b^2q^{-s},q)}{\Pi_m(-q^2,q^2)}q^{m^2+2m},$$

by applying Theorem A_1 to the first and second summands and Theorem A_3 to the fourth summand in the previous equation.

Simplifying this last expression and combining the first summand with the third and the second with the fourth, we obtain the desired result.

Theorem 6. *If $s = 0$, or 1, then*

$$\sum_{n=0}^{\infty} \frac{\Pi_n(-b,q)q^{2n}}{\Pi_{2n+s}(-q,q)} = \frac{1}{2}\frac{q^{-s}\Pi_\infty(-b,q)}{\Pi_\infty(-q,q)} \sum_{m=0}^{\infty} \frac{(b^2q^{-s})^m}{\Pi_m(-q^{m+1},q)} +$$

$$+(-1)^s\frac{1}{2}\frac{q^{-s}\Pi_\infty(-b,q)}{\Pi_\infty(-b^2q^{-s},q)} \sum_{m=0}^{\infty} (-1)^m \frac{\Pi_{2m}(-b^2q^{-s},q)}{\Pi_m(-q^2,q^2)} q^{m^2} +$$

$$+\frac{1}{2}\frac{bq^{-\frac{1}{2}s}(1+(-1)^s)\Pi_\infty(-b,q)}{\Pi_\infty(-q,q^2)\Pi_\infty(-b^2q^{-s},q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-b^2q^{-s},q^2)}{\Pi_{2m}(-q,q)} q^{2m^2+2m} +$$

$$+\frac{1}{2}\frac{bq^{+\frac{1}{2}(1-s)}(1-(-1)^s)\Pi_\infty(-b,q)}{\Pi_\infty(-q,q^2)\Pi_\infty(-b^2q^{1-s},q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-b^2q^{1-s},q^2)}{\Pi_{2m+1}(-q,q)} q^{2m^2+4m}.$$

Proof.

$$\sum_{n=0}^{\infty} \frac{\Pi_n(-b,q)q^{2n}}{\Pi_{2n+s}(-q,q)} = \frac{1}{2}\frac{q^{-s}\Pi_\infty(-b,q)}{\Pi_\infty(-q,q)} \sum_{m=0}^{\infty} \frac{(b^2q^{-s})^m}{\Pi_m(-q^{m+1},q)} +$$

$$+(-1)^s\frac{1}{2}\frac{q^{-s}\Pi_\infty(-b,q)}{\Pi_\infty(q,q)} \sum_{m=0}^{\infty} \frac{(b^2q^{-s})^m}{\Pi_m(-q,q)\Pi_m(-q,q^2)} +$$

$$+\frac{1}{2}\frac{q^{-s}\Pi_\infty(-b,q)}{(1-q)\Pi_\infty(-q^{\frac{1}{2}},q)} \sum_{m=0}^{\infty} \frac{(b^2q^{-s})^{m+\frac{1}{2}}}{\Pi_m(-q^3,q^2)\Pi_m(q^{\frac{1}{2}},q)} +$$

$$+(-1)^s\frac{1}{2}\frac{q^{-s}\Pi_\infty(-b,q)}{(1-q)\Pi_\infty(q^{\frac{1}{2}},q)} \sum_{m=0}^{\infty} \frac{(b^2q)^{m+\frac{1}{2}}}{\Pi_m(-q^2,q^2)\Pi_m(-q^{\frac{1}{2}},q)}$$

(by Theorem A_4, setting $a = c = 0, t = q^2$ and simplifying)

$$= \frac{1}{2}\frac{q^{-s}\Pi_\infty(-b,q)}{\Pi_\infty(-q,q)} \sum_{m=0}^{\infty} \frac{(b^2q^{-s})^m}{\Pi_m(-q^{m+1},q)} +$$

$$+(-1)^s\frac{1}{2}\frac{q^{-s}\Pi_\infty(-b,q)}{\Pi_\infty(q,q)\Pi_\infty(-q,q^2)\Pi_\infty(-b^2q^{-s},q)} \times$$

$$\times \sum_{m=0}^{\infty} (-1)^m \frac{\Pi_{2m}(-b^2q^{-s},q)}{\Pi_m(-q^2,q^2)} q^{m^2} +$$

$$+\frac{1}{2}\frac{q^{-s}(b^2q^{-s})^{\frac{1}{2}}}{(1-q)\Pi_\infty(-q^{\frac{1}{2}},q)\Pi_\infty(q^{\frac{1}{2}},q)\Pi_\infty(-b^2q^{-s},q^2)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_m(-b^2q^{-s},q^2)}{\Pi_{2m}(-q,q)} q^{2m^2+2m} +$$

$$+\frac{1}{2}\frac{q^{-s}(b^2q^{-s})^{\frac{1}{2}}\Pi_\infty(-b,q)}{(1-q)\Pi_\infty(-q,q)\Pi_\infty(q,q)\Pi_\infty(-b^2q^{1-s},q^2)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi_m(-b^2q^{1-s},q^2)}{\Pi_{2m+1}(-q,q)} q^{2m^2+4m+1} +$$

$$+(-1)^s\frac{1}{2}\frac{q^{-s}(b^2q^{-s})^{\frac12}\,\Pi_\infty(-b,q)}{(1-q)\,\Pi_\infty(q^{\frac12},q)\,\Pi_\infty(-q^{\frac12},q)\,\Pi_\infty(-b^2q^{-s},q^2)}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\Pi_m(-b^2q^{-s},q^2)}{\Pi_{2m}(-q,q)}q^{2m^2+2m}-$$

$$-\frac{1}{2}\frac{q^{-s}(-1)^s(b^2q^{-s})^{\frac12}\,\Pi_\infty(-b,q)}{(1-q)\,\Pi_\infty(q^{\frac12},q)\,\Pi_\infty(-q^{\frac12},q)\,\Pi_\infty(-b^2q^{1-s},q^2)}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\Pi_m(-b^2q^{1-s},q^2)}{\Pi_{2m+1}(-q,q)}q^{2m^2+4m+1},$$

by applying Theorem A_1 to the third and fourth summands and Theorem A_3 to the second summand in the previous equation.

Simplifying this last expression and combining the third summand with the fifth and the fourth with the sixth, we obtain the desired result.

3. The mock-theta-function identities

It is now a simple matter to derive most of the known mock-theta-function identities. First we shall derive all of the fifth-order identities [(5) 277–9]. Secondly we shall define and give identities for an infinite number of third-order functions; five of the original third-order functions will be among these new functions.

The fifth-order functions of Ramanujan are set out below.

$$f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{\Pi_n(q,q)},\qquad \phi_0(q)=\sum_{n=0}^{\infty}q^{n^2}\Pi_n(q,q^2),$$

$$\psi_0(q)=\sum_{n=0}^{\infty}q^{\frac12(n+1)(n+2)}\Pi_n(q,q),\qquad F_0(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{\Pi_n(-q,q^2)},$$

$$\chi_0(q)=\sum_{n=0}^{\infty}\frac{q^n}{\Pi_n(-q^{n+1},q)},\qquad \bar{\chi}_0(q)=1+\sum_{n=0}^{\infty}\frac{q^{2n+1}}{\Pi_{n+1}(-q^{n+1},q)},$$

$$f_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{\Pi_n(q,q)},\qquad \phi_1(q)=\sum_{n=0}^{\infty}q^{(n+1)^2}\Pi_n(q,q^2),$$

$$\psi_1(q)=\sum_{n=0}^{\infty}q^{\frac12 n(n+1)}\Pi_n(q,q),\qquad F_1(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{\Pi_{n+1}(-q,q^2)},$$

$$\chi_1(q)=\sum_{n=0}^{\infty}\frac{q^n}{\Pi_{n+1}(-q^{n+1},q)}.$$

Related functions appearing in the fifth-order identities are:

$$\vartheta_4(0,q) = \frac{\Pi_\infty(-q,q)}{\Pi_\infty(q,q)}, \qquad \vartheta_2(0,q) = 2q^{\frac{1}{4}}\frac{\Pi_\infty(-q^4,q^4)}{\Pi_\infty(-q^2,q^4)},$$

$$G(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{\Pi_n(-q,q)} = \{\Pi_\infty(-q,q^5)\,\Pi_\infty(-q^4,q^5)\}^{-1},$$

$$H(q) = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{\Pi_n(-q,q)} = \{\Pi_\infty(-q^2,q^5)\,\Pi_\infty(-q^3,q^5)\}^{-1}.$$

We now give the identities.

(1a) $\quad f_0(q) = \phi_0(-q^2)+\frac{1}{2}q^{\frac{1}{4}}\vartheta_2(0,q)H(q^4)-(F_0(q^2)-1)$

(in Theorem 4, take $l = 0$, $b = q$ and let $t \to 0$);

(1b) $\quad \vartheta_4(0,q)G(q) = \phi_0(-q^2)-\frac{1}{2}q^{\frac{1}{4}}\vartheta_2(0,q)H(q^4)+(F_0(q^2)-1)$

(in Theorem 4, take $l = 0$, $b = -q$ and let $t \to 0$);

(1c) $\quad \psi_0(q) = \frac{1}{2}q^{\frac{1}{4}}\vartheta_2(0,q)H(q^4)+(F_0(q^2)-1)$

(in Theorem 3, take $l = 1$, $b = q^2$ and let $c \to 0$);

(1d) $\quad \chi_0(q) = F_0(q)-\frac{1}{2}\phi_0(-q)+\frac{1}{2}\tilde{\chi}_0(q)$

(in Theorem 5, take $b = q$, $s = 0$);

(1e) $\quad \tilde{\chi}_0(q) = F_0(q)-\frac{1}{2}\phi_0(-q)+\frac{1}{2}\chi_0(q)$

(in Theorem 6, take $b = q$, $s = 1$).

From (1d) and (1e) we deduce that $\chi_0(q) = \tilde{\chi}_0(q)$.

(2a) $\quad f_1(q) = \frac{1}{2}q^{-\frac{1}{4}}\vartheta_2(0,q)G(q^4)-qF_1(q^2)-q^{-1}\phi_1(-q^2)$

(in Theorem 4, take $l = 1$, $b = q$ and let $t \to 0$);

(2b) $\quad \vartheta_4(0,q)H(q) = \frac{1}{2}q^{-\frac{1}{4}}\vartheta_2(0,q)G(q^4)-qF_1(q^2)+q^{-1}\phi_1(-q^2)$

(in Theorem 4, take $l = 1$, $b = -q$ and let $t \to 0$);

(2c) $\quad \psi_1(q) = \frac{1}{2}q^{-\frac{1}{4}}\vartheta_2(0,q)G(q^4)+qF_1(q^2)$

(in Theorem 3, take $l = 0$, $b = q^2$ and let $c \to 0$);

(2d) $\quad \chi_1(q) = 2F_1(q)+q^{-1}\phi_1(-q)$

(in Theorem 5, take $b = q$, $s = 1$).

We now turn to the third-order mock theta functions. We shall study the following five functions.

$$f(\alpha;q)=\sum_{n=0}^{\infty}\frac{q^{n^2-n}\alpha^n}{\Pi_n(q,q)\,\Pi_n(\alpha,q)},\qquad \phi(\alpha;q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{\Pi_m(\alpha q,q^2)},$$

$$\psi(\alpha;q)=\sum_{m=1}^{\infty}\frac{q^{m^2}}{\Pi_m(-\alpha,q^2)},\qquad \nu(\alpha;q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{\Pi_{n+1}(\alpha^2q^{-1},q^2)},$$

$$\omega(\alpha;q)=\sum_{m=0}^{\infty}\frac{q^{2m^2}\alpha^{2m}}{\Pi_{m+1}(-q,q^2)\,\Pi_{m+1}(-\alpha^2q^{-1},q^2)}.$$

When $\alpha=q$, these functions reduce to five of the seven mock theta functions given by Watson [(4) 62]. We shall eventually show that for $\alpha=q^r$ (r any positive integer), the above five functions are mock theta functions. First we give identities relating these functions; to obtain our results we shall utilize the following identity [(2) 34]

$$[\Pi_\infty(-\alpha,q)]^{-1}=\sum_{n=0}^{\infty}\frac{q^{n^2-n}\alpha^n}{\Pi_n(-q,q)\,\Pi_n(-\alpha,q)}. \tag{3.1}$$

We have

(3a) $$f(\alpha;q)=\phi(-\alpha;-q)-(1+\alpha q^{-1})\psi(-\alpha;-q)$$

(in Theorem 1, take $c=-\alpha$, $b=q$ and let $t\to 0$);

(3b) $$\vartheta_4(0,q)[\Pi_\infty(\alpha,q)]^{-1}=\phi(-\alpha;-q)+(1+\alpha q^{-1})\psi(-\alpha;-q)$$

(in Theorem 1, take $c=\alpha$, $b=-q$ and let $t\to 0$; then simplify using (3.1));

(3c) $$\nu(-\alpha;-q)=\tfrac{1}{2}q^{-1}\vartheta_2(0,q)\,[\Pi_\infty(-\alpha^4q^{-2},q^4)]^{-1}+\alpha^2q^{-1}\omega(\alpha^2;q^2)$$

(in Theorem 2, replace q throughout by q^2, then take $b=q^2$, $c=\alpha^2q$ and let $t\to 0$).

We shall now consider these functions with $\alpha=q^r$.

LEMMA 3.1. $\nu(\alpha q;q)=(1-\alpha^4q^{-2})\nu(\alpha,q)+\alpha^2q^{-1}$.

Proof. $(1+\alpha^2q^{-1})\nu(\alpha;q)$

$$=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{\Pi_n(\alpha^2q,q^2)}=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(1+\alpha^2q^{2n+1})}{\Pi_{n+1}(\alpha^2q,q^2)}$$

$$=\nu(\alpha q;q)+\alpha^2q^{-1}\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)}}{\Pi_{n+1}(\alpha^2q,q^2)}$$

$$=\nu(\alpha q;q)+\alpha^2q^{-1}(1+\alpha^2q^{-1})\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{\Pi_{n+1}(\alpha^2q^{-1},q^2)}$$

$$=\nu(\alpha q;q)+\alpha^2q^{-1}(1+\alpha^2q^{-1})(\nu(\alpha;q)-(1+\alpha^2q^{-1})^{-1}).$$

Simplifying we obtain the lemma.

LEMMA 3.2. $f(\alpha q;q) = (1+\alpha)\{2-f(\alpha;q)\}$.

Proof.

$$f(\alpha;q)+\vartheta_4(0,q)\{\Pi_\infty(\alpha,q)\}^{-1} = 2\phi(-\alpha;-q) \tag{3.2.1}$$

(by 3a and 3b),

$$f(\alpha q;q)-\vartheta_4(0,q)\{\Pi_\infty(\alpha,q)\}^{-1}(1+\alpha) = -2(1+\alpha)\psi(-\alpha q;-q) \tag{3.2.2}$$

(by 3a and 3b).

Therefore,

$$\begin{aligned} f(\alpha;q)+f(\alpha q;q)(1+\alpha)^{-1} &= 2\{\phi(-\alpha;-q)-\psi(-\alpha q;-q)\} \\ &= 2\left\{\sum_{m=0}^{\infty}\frac{(-1)^m q^{m^2}}{\Pi_m(\alpha q,q^2)}-\sum_{m=1}^{\infty}\frac{(-1)^m q^{m^2}}{\Pi_m(\alpha q,q^2)}\right\} \\ &= 2. \end{aligned}$$

Simplifying we obtain the lemma.

THEOREM. *For any positive integer r, $f(q^r;q)$, $\phi(-q^r;-q)$, $\psi(-q^r;-q)$, $\nu(q^r;q)$, $\omega(q^r;q)$ are mock theta functions.*

Proof. A mock theta function is a function defined by a q-series, convergent when $|q|<1$, for which we can calculate asymptotic formulae, when q tends to a 'rational point' $\exp(2r\pi i/s)$ of the unit circle, of the same degree of precision as those furnished for the ordinary theta functions by the theory of linear transformation [(4) 57]. Thus, if we show that a function $M(q)$ defined by a q-series, convergent when $|q|<1$, is a linear combination over $\mathscr{C}[q]$ (the ring of polynomials in q with complex coefficients) of theta functions and mock theta functions, then we have proved that $M(q)$ is a mock theta function.

We proceed by mathematical induction. The theorem has been proved for $r=1$ by Watson [4]. Assume true for $r-1$.

Since, by Lemma 3.2,

$$f(q^r;q) = (1+q^{r-1})(2-f(q^{r-1};q)),$$

$f(q^r;q)$ is a mock theta function.

Hence, by (3.2.1), $\phi(-q^r;-q)$ is a mock theta function. Since $\psi(-q^r;-q) = \phi(-q^{r-1};-q)-1$ (directly from the definitions), $\psi(-q^r;-q)$ is a mock theta function.

By Lemma 3.1,

$$v(q^r; q) = (1-q^{4r-4})v(q^{r-1}; q)+q^{2r-2};$$

hence $v(q^r; q)$ is a mock theta function. We conclude from (3c) that $\omega(q^r; q)$ is a mock theta function. Thus our theorem is proved.

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