SIEVES IN THE THEORY OF PARTITIONS.

By George E. Andrews.[1]

1. Introduction. Many results in the theory of partitions are obtained by identifying two apparently different functions that directly enumerate certain sets of partitions. For example $\prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}$ is the generating function for partitions with odd parts [13; p. 276], and $\prod_{n=1}^{\infty}(1+q^n)$ is the generating function for partitions with distinct parts [13; p. 276]. Following the simple argument of Euler, we see that

$$(1.1)\qquad \prod_{n=1}^{\infty}(1+q^n)=\prod_{n=1}^{\infty}(1-q^{2n})(1-q^n)^{-1}=\prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}.$$

The identity of these two generating functions proves Euler's partition theorem that the partitions of n with odd parts are equinumerous with the partitions of n with distinct parts.

Other theorems rely on different types of counts by the generating function. For example, from Euler's pentagonal number theorem [13; p. 284]

$$(1.2)\qquad \prod_{n=1}^{\infty}(1-q^n)=\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{2}j(3j-1)}$$

one can [13; p. 285] deduce that

$$(1.3)\qquad p^e(n)-p^0(n)=\begin{cases}(-1)^j & \text{if } n=\frac{1}{2}j(3j\pm 1),\\ 0 & \text{otherwise,}\end{cases}$$

where $p^e(n)$ (resp. $p^0(n)$) is the number of partitions of n into an even (resp. odd) number of distinct parts. Here the generating function $\prod_{n=1}^{\infty}(1-q^n)$ counts the excess of one set of partitions over another set.

Further partition theorems have been proved using simple inclusion-exclusion arguments (see [17] and [6]).

In this paper we shall fully develop a new sieve technique for counting partitions. This technique has already been applied in Ref. [8] to prove

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(1.2). As announced in Ref. [8], we shall derive a new partition theorem that contains the Rogers-Ramanujan identities

$$(1.4)\qquad 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\frac{1+\sum_{n=1}^{\infty}(-1)^n q^{\frac{1}{2}n(5n-1)}(1+q^n)}{\prod_{n=1}^{\infty}(1-q^n)}$$

and

$$(1.5)\qquad 1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}=\frac{1+\sum_{n=1}^{\infty}(-1)^n q^{\frac{1}{2}n(5n-3)}(1+q^{3n})}{\prod_{n=1}^{\infty}(1-q^n)}$$

as special cases. Actually our proof will interpret the right-hand expressions in (1.4) and (1.5) as sieves that act on all partitions and that sieve out all partitions except those enumerated by the corresponding left-hand expressions.

Our general partition theorem (Theorem 4.1) will be stated and proved in Section 4. The next section is devoted to a study of successive ranks, a concept introduced by A. O. L. Atkin [9]. In Section 3, we study relevant generating functions related to successive ranks. After proving Theorem 4.1, we conclude by discussing the Rogers-Ramanujan identities and other related results.

2. Successive ranks. As is well-known [13; p. 273], any partition may be represented by a Ferrars graph. For example, $6+6+4+3+1+1+1$ has the representation

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In 1944, F. J. Dyson [11] defined the rank of a partition as the largest part minus the number of parts. Thus the rank of the preceding partition is $6-7=-1$. We remark that Dyson made certain conjectures relating rank to the Ramanujan congruences [11], and Atkin and Swinnerton-Dyer were able to prove these conjectures [10].

Later, Atkin [9] defined the successive ranks of a partition. One can view of the Ferrars graph of a partition as a set of nested right angles of nodes. The partition $6+6+4+3+1+1+1$, may be viewed as 3 such right angles

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If π is a partition, we define $r_i(\pi)$ as the number of nodes in the horizontal part of the i-th right angle in the graph of π minus the number of nodes in the vertical part of the i-th right angle. Thus if π is $6+6+4+3+1+1+1$, then $r_1(\pi)=6-7=-1$, $r_2(\pi)=5-3=2$, $r_3(\pi)=2-2=0$.

Our interest lies in the "oscillation" of the ranks between certain positive and negative bounds.

Definition 1. Let h be the largest integer for which there exists a sequence $j_1<j_2<\cdots<j_h$ such that $r_{j_1}(\pi)>2k-i-1$, $r_{j_2}(\pi)\leqq-(i-1)$, $r_{j_3}(\pi)>2k-i-1$, $r_{j_4}(\pi)\leqq-(i-1)$, and so on. We define h to be the (k,i)-positive oscillation of π.

Definition 2. Let g be the largest integer for which there exists a sequence $j_1<j_2<\cdots<j_g$ such that $r_{j_1}(\pi)\leqq-(i-1)$, $r_{j_2}(\pi)>2k-i-1$, $r_{j_3}\leqq-(i-1)$, $r_{j_4}(\pi)>2k-i-1$, and so on. We define g to be the (k,i)-negative oscillation of π.

If π denotes the partition $6+6+4+3+1+1+1$, then the $(1,1)$-positive oscillation of π is 2, the $(2,1)$-negative oscillation of π is 1.

Our interest centers upon partition functions related to these oscillations.

Definition 3. Let $p_{k,i}(a,b;\mu;N)$ (resp. $m_{k,i}(a,b;\mu;N)$) denote the number of partitions of N with at most b parts, with largest part at most a and with (k,i)-positive (resp. (k,i)-negative) oscillation at least μ.

To these partition functions we associate the related generating functions

$$P_{k,i}(a,b;\mu;q)=\sum_{N\geqq 0}p_{k,i}(a,b;\mu;N)q^N, \tag{2.1}$$

$$M_{k,i}(a,b;\mu;q)=\sum_{N\geqq 0}m_{k,i}(a,b;\mu;N)q^N. \tag{2.2}$$

In the next section we shall derive recurrences for these functions that will allow us to identify them with expressions involving Gaussian polynomials.

3. Recurrence formulae. We start by noting the obvious close relationship between negative and positive oscillation.

LEMMA 3.1. *The (k,i)-positive oscillation of π is either* 1 *larger or* 1 *smaller than the (k,i)-negative oscillation of π.*

Proof. Let $j_1 < j_2 < \cdots < j_h$ be the sequence related to the (k,i)-positive as given in Definition 1. Either there does exist a $j_0 < j_1$ such that $r_{j_0}(\pi) \leqq -(i-1)$ or there does not. In the first case we see that the (k,i)-negative oscillation is 1 larger and in the second case it is 1 smaller. □

LEMMA 3.2. *The following recurrences hold for $\mu \geqq 1$:*

$$\begin{aligned} &m_{k,i}(a,b;\mu;N) - m_{k,i}(a-1,b;\mu;N) \\ &\qquad - m_{k,i}(a,b-1;\mu;N) + m_{k,i}(a-1,b-1;\mu;N) \end{aligned}$$

$$(3.1)\quad = \begin{cases} m_{k,i}(a-1,b-1;\mu;N-a-b+1), & \text{if } a-b > -(i-1), \\ p_{k,i}(a-1,b-1;\mu-1;N-a-b+1), & \text{if } a-b \leqq -(i-1); \end{cases}$$

$$\begin{aligned} &p_{k,i}(a,b;\mu;N) - p_{k,i}(a-1,b;\mu;N) \\ &\qquad - p_{k,i}(a,b-1;\mu;N) + p_{k,i}(a-1,b-1;\mu;N) \end{aligned}$$

$$(3.2)\quad = \begin{cases} m_{k,i}(a-1,b-1;\mu-1;N-a-b+1), & \text{if } a-b > 2k-i-1, \\ p_{k,i}(a-1,b-1;\mu;N-a-b+1), & \text{if } a-b \leqq 2k-i-1. \end{cases}$$

Proof. Let us start by examining the left side of (3.1). The expression $m_{k,i}(a,b;\mu;N) - m_{k,i}(a-1,b;\mu;N)$ denotes the number of partitions of N with at most b parts, with (k,a)-negative oscillation at least μ, and with largest part exactly equal to a. Therefore, the expression

$$\begin{aligned} &(m_{k,i}(a,b;\mu;N) - m_{k,i}(a-1,b;\mu;N)) \\ &\qquad - (m_{k,i}(a,b-1;\mu;N) - m_{k,i}(a-1,b-1;\mu;N)) \end{aligned}$$

(which is the left-hand side of (3.1)) denotes the number of partitions of N with exactly b parts, with largest part exactly a, and with (k,a)-negative oscillation at least μ.

We now transform these partitions that are enumerated by the left-hand side of (3.1) by deleting the largest part (namely a) and subtracting 1 from each of the $b-1$ remaining parts. In terms of the Ferrars graph, we have removed the outer right angle of nodes from each partition. The transformed partitions are partitions of $N-a-b+1$ into at most $b-1$ parts with largest part at most $a-1$.

Suppose first that $a-b > -(i-1)$; then the removal of the outer right angle of nodes from the Ferrars graph has no effect on the (k,i)-negative oscillation which is still equal to μ. Consequently our transformed partition is of the type enumerated by $m_{k,i}(a-1,b-1;\mu;N-a-b+1)$. Since the above procedure is clearly reversible, we see that if $a-b>-(i-1)$, then

$$\begin{aligned} m_{k,i}(a,b;\mu;N) &- m_{k,i}(a-1,b;\mu;N) \\ &- m_{k,i}(a,b-1;\mu;N) + m_{k,i}(a-1,b-1;\mu;N) \\ &= m_{k,i}(a-1,b-1;\mu;N-a-b+1). \end{aligned}$$

Therefore the top half of (3.1) is established.

Now suppose that $a-b \leqq -(i-1)$; in this case, the removal of the outer right angle of nodes from the Ferrars graph may affect the (k,i)-negative oscillation. In fact, the resulting partition is now characterized by the fact that it has (k,i)-positive oscillation at least $\mu-1$. Thus the transformed partitions in this case are of the type enumerated by $p_{k,i}(a-1,b-1;\mu-1;N-a-b+1)$. Again the transformation used is reversible. Therefore if $a-b \leqq -(i-1)$, then

$$\begin{aligned} m_{k,i}(a,b;\mu;N) &- m_{k,i}(a-1,b;\mu;N) \\ &- m_{k,i}(a,b-1;\mu;N) + m_{k,i}(a-1,b-1;\mu;N) \\ &= p_{k,i}(a-1,b-1;\mu-1;N-a-b+1). \end{aligned}$$

Therefore the bottom half of (3.1) is established.

We omit the proof of (3.2) since it perfectly parallels the proof of (3.1) with the roles of (k,i)-positive and (k,i)-negative oscillation interchanged. □

It is now a simple matter to translate Lemma 3.2 into relations among the generating functions.

COROLLARY 3.1. *For each* $\mu \geqq 1$,

$$\begin{aligned} M_{k,i}(a,b;\mu;q) &- M_{k,i}(a-1,b;\mu;q) \\ &- M_{k,i}(a,b-1;\mu;q) + M_{k,i}(a-1,b-1;\mu;q) \end{aligned}$$

$$= q^{a+b-1} \begin{cases} M_{k,i}(a-1,b-1;\mu;q), & \text{if } a-b>-(i-1), \\ P_{k,i}(a-1,\ b-1;\mu-1;q), & \text{if } a-b \leqq -(i-1); \end{cases} \tag{3.3}$$

$$\begin{aligned} P_{k,i}(a,b;\mu;q) &- P_{k,i}(a-1,b;\mu;q) \\ &- P_{k,i}(a,b-1;\mu;q) + P_{k,i}(a-1,b-1;\mu;q) \end{aligned}$$

$$= q^{a+b-1} \begin{cases} M_{k,i}(a-1,b-1;\mu-1;q), & \text{if } a-b>2k-i-1, \\ P_{k,i}(a-1,b-1;\mu;q), & \text{if } a-b \leqq 2k-i-1. \end{cases} \tag{3.4}$$

Proof. Comparing coefficients of q^N in these identities, we can establish these results directly from Lemma 3.2. □

LEMMA 3.3. *The following relations hold:*

$$P_{k,i}(a,b;0;q)=M_{k,i}(a,b;0;q)=\binom{a+b}{a}_q, \tag{3.5}$$

and for $\mu\geqq 1$

$$P_{k,i}(0,b;\mu;q)=P_{k,i}(a,0;\mu;q)=M_{k,i}(0,b;\mu;q) \tag{3.6}$$

$$=M_{k,i}(a,0;\mu;q)=0,$$

where

$$\binom{N}{M}_q=\begin{cases}\dfrac{(1-q^N)\cdots(1-q^{N-M+1})}{(1-q^M)\cdots(1-q)}, & \text{if } 0<M<N\\ 1, & \text{if } M=0 \text{ or } N,\\ 0, & \text{otherwise.}\end{cases} \tag{3.7}$$

Proof. For (3.5), we observe that all partitions have (k,i)-positive and (k,i)-negative oscillation at least 0. Hence $P_{k,i}(a,b;0;q)$ and $M_{k,i}(a,b;0;q)$ are each the generating function for partitions with largest part at most a and with at most b parts. It is well-known [16; p. 269] that this generating function is $\binom{a+b}{a}_q$; thus (3.5) is established.

As for (3.6), we observe that only the empty partition of 0 has either no parts or no largest part. Since the empty partition of 0 has 0 for all of its (k,i)-oscillations, we see that the generating functions appearing in (3.6) must all be identically 0. □

We shall now prove two recurrence formulae for the Gaussian polynomials (3.7). These will be central in the proof of Theorem 3.1.

Our proofs of these lemmas will rely on the following well-known relations among Gaussian polynomials [14; pp. 84-85]

$$\binom{N}{M}_q=\binom{N-1}{M-1}_q+q^M\binom{N-1}{M}_q, \tag{3.8}$$

$$\binom{N}{M}_q=\binom{N-1}{M}_q+q^{N-M}\binom{N-1}{M-1}_q, \tag{3.9}$$

$$\binom{N}{M}_q=\binom{N}{N-M}_q. \tag{3.10}$$

LEMMA 3.4.

$$q^L\binom{A+B}{B-X}_q-q^L\binom{A+B-1}{B-X}_q-q^L\binom{A+B-1}{B-X-1}_q+q^L\binom{A+B-2}{B-X-1}_q$$

$$=q^{A+B-1}q^L\binom{A+B-2}{B-X-1}_q.$$

Proof. By (3.9), we see that

$$q^{L}\binom{A+B}{B-X}_q-q^{L}\binom{A+B-1}{B-X}_q-q^{L}\binom{A+B-1}{B-X-1}_q+q^{L}\binom{A+B-2}{B-X-1}_q$$

$$=q^{L+A+X}\binom{A+B-1}{B-X-1}_q-q^{L+A+X}\binom{A+B-2}{B-X-2}_q$$

$$=q^{L+A+B-1}\binom{A+B-2}{B-X-1}_q \qquad \text{(by (3.8))}. \qquad \square$$

LEMMA 3.5. *For each integer $r\geqq 0$, let*

$$f(A,r;X,Y,L)=q^{L}\frac{(1-q^{(r+1)(A+Y+L)})}{(1-q^{A+Y+1})}\binom{2A+X}{A+Y}_q+\sum_{j=0}^{r-2}q^{L+A+Y+j+2}\frac{(1-q^{(r-j-1)(A+Y+1)})}{(1-q^{A+Y+1})}\binom{2A+X+j}{A+Y-1}_q.$$

Then for each integer $r\geqq 1$,

$$f(A,r;X,Y,L)-f(A,r-1;X,Y,L)-f(A-1,r+1;X,Y,L)+f(A-1,r;X,Y,L)$$

$$=q^{L+2A+2Y+r}\binom{2A+X+r-2}{A+Y}_q.$$

Proof.

$$f(A,r;X,Y,L)-f(A,r-1;X,Y,L)$$

$$=q^{L+r(A+Y+1)}\binom{2A+X}{A+Y}_q+\sum_{j=0}^{r-2}q^{L+A+Y+j+2+(r-j-2)(A+Y+1)}\binom{2A+X+j}{A+Y-1}_q.$$

Also by repeated application of (3.8), we see that

$$q^{L+2A+2Y+r}\binom{2A+X+r-2}{A+Y}_q$$

$$=\sum_{j=0}^{r-3}q^{j(A+Y)+L+2A+2Y+r}\binom{2A+X+r-3-j}{A+Y-1}_q+q^{L+2A+2Y+r+(r-2)(A+Y)}\binom{2A+X}{A+Y}_q$$

$$=\sum_{j=0}^{r-3}q^{(r-1-j)(A+Y)+L+r}\binom{2A+X+j}{A+Y-1}_q+q^{L+r(A+Y+1)}\binom{2A+X}{A+Y}_q$$

$$=\sum_{j=0}^{r-3}q^{L+A+Y+j+2+(r-j-2)(A+Y+1)}\binom{2A+X+j}{A+Y-1}_q+q^{L+r(A+Y+2)}\binom{2A+X}{A+Y}_q.$$

Therefore, combining these two results, we derive that

$$A,r;X,Y,L)-f(A,r-1;X,Y,L)-q^{L+2A+2Y+r}\binom{2A+X+r-2}{A+Y}_q+f(A-1,r;X,Y,L)$$

$$=q^{L+A+Y+r}\binom{2A+X+r-2}{A+Y-1}_q+f(A-1,r;X,Y,L)$$

$$=q^{L+A+Y+r}\{\sum_{j=0}^{r-1}q^{j(A+Y-1)}\binom{2A+X+r-3-j}{A+Y-2}_q$$

$$+q^{r(A+Y+1)}\binom{2A+X-2}{A+Y-1}_q\}+f(A-1,r;X,Y,L)$$

(by repeated application of (3.8))

$$=\sum_{j=0}^{r-1}q^{L+A+Y+r+(r-1-j)(A+Y-1)}\binom{2A+X-2+j}{A+Y-2}_q$$

$$+q^{L+(r+1)(A+Y)}\binom{2A+X-2}{A+Y-1}_q+f(A-1,r;X,Y,L)$$

$$=q^{L}\frac{(1-q^{(r+2)(A+Y)})}{(1-q^{A+Y})}\binom{2(A-1)+X}{A-1+Y}_q$$

$$+\sum_{j=0}^{r-2}q^{L+A+Y+j+1}\frac{(1-q^{(r-j)(A+Y)})}{(1-q^{A+Y})}\binom{2(A-1)+X+j}{A-1+Y-1}_q$$

$$=f(A-1,r+1;X,Y,L). \qquad \square$$

We are now prepared to prove the main theorem on generating functions that will be essential to our sieve.

THEOREM 3.1. *If $b=a$ or $a-1$ and $k\geqq i>0$, then*

$$M_{k,i}(a,b;2\mu;q)=q^{\mu((4k+2)\mu+(2k-2i+1))}\binom{a+b}{b-(2k+1)\mu}_q, \tag{3.11}$$

$$M_{k,i}(a,b;2\mu-1;q)=q^{(2\mu-1)((2k+1)\mu-(2k-i+1))}\binom{a+b}{b-(2k+1)\mu+2k-i+1}_q, \tag{3.12}$$

$$P_{k,i}(a,b;2\mu;q)=q^{\mu((4k+2)\mu-(2k-2i+1))}\binom{a+b}{b+(2k+1)\mu}_q, \tag{3.13}$$

$$P_{k,i}(a,b;2\mu-1;q)=q^{(2\mu-1)((2k+1)\mu-i)}\binom{a+b}{a-(2k+1)\mu+i}_q. \tag{3.14}$$

Proof. We first note that the recurrence relations (3.3) and (3.4) together with the initial conditions (3.5) and (3.6) uniquely determine $M_{k,i}(a,b;\mu;q)$ and $P_{k,i}(a,b;\mu;q)$. We now define new functions $M^*_{k,i}(a,b;\mu;q)$ and $P^*_{k,i}(a,b;\mu;q)$ in terms of Gaussian polynomials. Our object is to identify these new functions with the generating functions being considered.

$$M^*_{k,i}(a,b;2\mu;q)=q^{\mu((4k+2)\mu+(2k-2i+1))}\binom{a+b}{b-(2k+1)\mu}_q \tag{3.15}$$

for $a-b \geqq -(i-1)$;

(3.16) $M^*_{k,i}(a,b;2\mu-1;q)$

$$= q^{(2\mu-1)((2k+1)\mu-(2k-i+1))}\binom{a+b}{b-(2k+1)\mu+2k-i+1}_q$$

for $a-b \geqq -(i-1)$;

(3.17) $P^*_{k,i}(a,b;2\mu;q) = q^{\mu((4k+2)\mu-(2k-2i+1))}\binom{a+b}{a-(2k+1)\mu}_q$

for $a-b \leqq 2k-i$;

(3.18) $P^*_{k,i}(a,b;2\mu-1;q) = q^{(2\mu-1)((2k+1)\mu-i)}\binom{a+b}{a-(2k+1)\mu+i}_q$

for $a-b \leqq 2k-i$;

(3.19) $M^*_{k,i}(a-1,a+i-1+r;2\mu;q)$

$$= q^{\mu((4k+2)\mu+(2k-2i+1))}\frac{(1-q^{(r+2)(a+i-1-(2k+1)\mu)})}{1-q^{a+i-1-(2k+1)\mu}}\binom{2a+i-3}{a+i-2-(2k+1)\mu}_q$$
$$+\sum_{j=0}^{r-1} q^{\mu((4k+2)\mu+(2k-2i+1))+a+i-(2k+1)\mu+j}\cdot$$
$$\frac{(1-q^{(r-j)(a+i-1-(2k+1)\mu)})}{1-q^{a+i-1-(2k+1)\mu}}\binom{2a+i-3+j}{a+i-3-(2k+1)\mu}_q$$

for $r \geqq -1$, $a \geqq 1$, $\mu \geqq 1$;

(3.20) $M^*_{k,i}(a-1,a+i-1+r;2\mu-1;q)$

$$= q^{(2\mu-1)((2k+1)\mu-(2k-i+1))}\frac{(1-q^{(r+2)(a-(2k+1)\mu+2k)})}{(1-q^{a-(2k+1)\mu+2k})}\binom{2a+i-3}{a-(2k+1)\mu+2k-1}_q$$
$$+\sum_{j=0}^{r-1} q^{(2\mu-1)((2k+1)\mu-(2k-i+1))+a-(2k+1)\mu+2k+1+j}\cdot$$
$$\frac{(1-q^{(r-j)(a-(2k+1)\mu+2k)})}{(1-q^{a-(2k+1)\mu+2k})}\binom{2a+i-3+j}{a-(2k+1)\mu+2k-2}_q$$

for $r \geqq -1$, $a \geqq 1$, $\mu \geqq 1$;

(3.21) $P^*_{k,i}(b+2k-i-1+r,b-1;2\mu;q)$

$$= q^{\mu((4k+2)\mu-(2k-2i+1))}\frac{(1-q^{(r+1)(b+2k-i-(2k+1)\mu)})}{1-q^{b+2k-i-(2k+1)\mu}}\binom{2b+2k-i-2}{b+2k-i-1-(2k+1)\mu}_q$$
$$+\sum_{j=0}^{r-2} q^{\mu((4k+2)\mu-(2k-2i+1))-(2k+1)\mu+b+j+2k-i+1}\cdot$$
$$\frac{(1-q^{(r-j-1)(b+2k-i-(2k+1)\mu)})}{1-q^{b+2k-i-(2k+1)\mu}}\binom{2b+2k-i-2+j}{b+2k-i-2-(2k+1)\mu}_q$$

for $r \geqq 0,\ b \geqq 1,\ \mu \geqq 1$;

(3.22) $P^*_{k,i}(b+2k-i-1+r, b-1; 2\mu-1; q)$

$$= q^{(2\mu-1)((2k+1)\mu-i)} \frac{(1-q^{(r+1)(b+2k-(2k+1)\mu)})}{1-q^{b+2k-(2k+1)\mu}} \binom{2b+2k-i-2}{b+2k-1-(2k+1)\mu}_q$$

$$+ \sum_{j=0}^{r-2} q^{(2\mu-1)((2k+1)\mu-i)-(2k+1)\mu+b+j+2k+1} \cdot$$

$$\frac{(1-q^{(r-j-1)(b+2k-(2k+1)\mu)})}{1-q^{b+2k-(2k+1)\mu}} \binom{2b+2k-i-2+j}{b+2k-2-(2k+1)\mu}_q$$

for $r \geqq 0,\ b \geqq 1,\ \mu \geqq 1$.

Finally

(3.23) $$M^*_{k,i}(a,b;0;q) = P^*_{k,i}(a,b;0;q) = \binom{a+b}{a}_q.$$

We first observe that the initial condition (3.5) is prescribed in (3.23). Furthermore if either a or b is set equal to zero in $M^*_{k,i}(a,b;\mu;q)$ or $P^*_{k,i}(a,b;\mu;q)$ with $\mu \geqq 1$; then inspection of the appropriate defining equation among (3.15)-(3.22) shows that the resulting function is identically zero. Hence the initial conditions prescribed in our Lemma 3.3 are satisfied by $M^*_{k,i}(a,b;\mu;q)$ and $P^*_{k,i}(a,b;\mu;q)$.

We now shall examine the recurrence relations (3.3) and (3.4). The top line of (3.3) and the bottom line of (3.4) follow in each case by applying Lemma 3.4 to the appropriate expressions among (3.15)-(3.19). Finally in the notation of Lemma 3.5, we see that

(3.24) $M^*_{k,i}(a-1, a+i-1+r; 2\mu; q)$

$$= f(a, r+1; i-1, i-2-(2k+1)\mu, (4k+2)\mu^2+(2k-2i+1)\mu)$$

for $a \geqq 1, r \geqq -1, \mu > 0$;

(3.25) $M^*_{k,i}(a-1, a+i-1+r; 2\mu-1; q)$

$$= f(a, r+1; i-3, 2k-1-(2k+1)\mu, (2\mu-1)((2k+1)\mu-2k+i-1))$$

for $a \geqq 1, r \geqq -1, \mu > 0$;

(3.26) $P^*_{k,i}(b+2k-i-1+r, b-1; 2\mu; q)$

$$= f(b, r; 2k-i-2, 2k-i-1-(2k+1)\mu, \quad (4k+2)\mu^2-(2k-2i+1)\mu), \quad \text{for } b \geqq 1, r \geqq 0, \mu > 0;$$

$$(3.27) \quad P^*_{k,i}(b+2k-i-1+r, b-1; 2\mu-1; q) = f(b, r; 2k-i-2, 2k-1-(2k+1)\mu, \quad (2\mu-1)((2k+1)\mu-i)), \quad \text{for } b \geqq 1, r \geqq 0, \mu > 0$$

It is now simply a matter of inspection to verify that Lemma 3.5 implies that the bottom line of (3.3) and the top line of (3.4) hold for $M^*_{k,i}(a, b; \mu; q)$ and $P^*_{k,i}(a, b; \mu; q)$ if $\mu > 0$.

Thus we have observed that $M^*_{k,i}(a, b; \mu; q)$ and $P^*_{k,i}(a, b; \mu; q)$ satisfy (3.3), (3.4), (3.5), and (3.6); and since these four equations uniquely define $M_{k,i}(a, b; \mu; q)$ and $P_{k,i}(a, b; \mu; q)$, we see that

$$(3.28) \qquad P_{k,i}(a, b; \mu; q) = P^*_{k,i}(a, b; \mu; q),$$

and

$$(3.29) \qquad M_{k,i}(a, b; \mu; q) = M^*_{k,i}(a, b; \mu; q).$$

Equation (3.11) now follows from (3.15) and (3.29); (3.12) follows from (3.16) and (3.29); (3.13) follows from (3.17) and (3.28); finally (3.14) follows from (3.18) and (3.28). □

Definition 4. Let $p_{k,i}(\mu; N)$ (resp. $m_{k,i}(\mu; N)$ denote the number of partitions of N with (k, i)-positive (resp. (k, i)-negative) oscillation at least μ.

Definition 5. Let

$$P_{k,i}(\mu; q) = \sum_{N \geqq 0} p_{k,i}(\mu; N) q^N,$$

$$M_{k,i}(\mu; q) = \sum_{N \geqq 0} m_{k,i}(\mu; N) q^N.$$

THEOREM 3.2. *The following relations hold for* $|q| < 1$.

$$(3.29) \qquad M_{k,i}(2\mu; q) = \frac{q^{\mu((4k+2)\mu+2k-2i+1)}}{(q)_\infty}, \quad \mu \geqq 0;$$

$$(3.30) \qquad M_{k,i}(2\mu-1; q) = \frac{q^{(2\mu-1)((2k+1)\mu-2k+i-1)}}{(q)_\infty}, \quad \mu > 0;$$

$$(3.31)\qquad P_{k,i}(2\mu;q)=\frac{q^{\mu((4k+2)\mu-2k+2i-1)}}{(q)_\infty},\ \mu\geqq 0;$$

$$(3.32)\qquad P_{k,i}(2\mu-1;q)=\frac{q^{(2\mu-1)((2k+1)\mu-i)}}{(q)_\infty},\ \mu>0,$$

where $(q)_\infty=\prod_{n=1}^{\infty}(1-q^n)$.

Proof. First we observe that for $|q|<1$,

$$|P_{k,i}(\mu;q)-P_{k,i}(a,a;\mu;q)|\leqq\sum_{n=a+1}^{\infty}p(n)|q|^n\to 0 \text{ as } a\to\infty,$$

and

$$|M_{k,i}(\mu;q)-M_{k,i}(a,a;\mu;q)|\leqq\sum_{n=a+1}^{\infty}p(n)|q|^n\to 0 \text{ as } a\to\infty,$$

where $p(n)$ is the ordinary partition function. Finally

$$\lim_{a\to\infty}q^x\binom{2a+Y}{a-Z}=\frac{q^x}{(q)_\infty}.$$

Consequently (3.29) follows from (3.11), (3.30) from (3.12), (3.31) from (3.13), and (3.32) from (3.14). □

4. The sieve. First we shall introduce the partition function that arises from our sieve technique.

Definition 6. Let $Q_{k,i}(n)$ denote the number of partitions π of n such that $-(i-2)\leqq r_i(\pi)\leqq 2k-i-1$ for each of the successive ranks of π.

The following lemma is the inclusion-exclusion aspect of our sieve.

LEMMA 4.1. *For each integer* $n\geqq 0$,

$$(4.1)\qquad Q_{k,i}(n)=p_{k,i}(0;n)+\sum_{\mu=1}^{\infty}(-1)^\mu m_{k,i}(\mu;n)+\sum_{\mu=1}^{\infty}(-1)^\mu p_{k,i}(\mu;n).$$

Proof. First we remark that $Q_{k,i}(n)$ counts the set of all partitions of n that have 0 as (k,i)-positive oscillation *and* 0 as (k,i)-negative oscillation.

On the other hand, the right-hand side of (4.1) is a weighted count of the partitions of n. First suppose π is a partition of n with 0 as (k,i)-positive and (k,i)-negative oscillation. Then π is counted once by $p_{k,i}(0;n)$ and not at all by each of $m_{k,i}(\mu;n)$ and $p_{k,i}(\mu;n)$ for each $\mu>0$. Next suppose that π is a partition of n with (k,i)-positive oscillation $r>0$.

By Lemma 3.1, the (k,i)-negative oscillation of π is either $r-1$ or $r+1$; if $r-1$, then the weighted count for π is

$$1+\sum_{\mu=1}^{r-1}(-1)^{\mu}+\sum_{\mu=1}^{r}(-1)^{\mu}=1+\sum_{\mu=0}^{r-1}(-1)^{\mu}-\sum_{\mu=1}^{r-1}(-1)^{\mu}=0;$$

if $r+1$, then the weighted count for π is

$$1+\sum_{\mu=1}^{r+1}(-1)^{\mu}+\sum_{\mu=1}^{r}(-1)^{\mu}=1-\sum_{\mu=0}^{r}(-1)^{\mu}+\sum_{\mu=1}^{r}(-1)^{\mu}=0.$$

Finally if the (k,i)-positive oscillation of π is zero and the (k,i)-negative oscillation is 1, then the weighted count of π is $1-1=0$.

Thus we see that the right-hand expression in (4.1) counts once each partition of n with 0 as (k,i)-positive and (k,i)-negative oscillation while it counts 0 for each of the other partitions of n. Consequently the right-hand expression in (4.1) is just equal to $Q_{k,i}(n)$. □

THEOREM 4.1. *Let $A_{k,i}(n)$ denote the number of partitions of n into parts that are not congruent to $0, \pm i \pmod{2k+1}$.* Then for $0<i\leqq k$ *and for each $n\geqq 0$,*

$$A_{k,i}(n)=Q_{k,i}(n).$$

Proof. By Lemma 4.1, we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty}Q_{k,i}(n)q^{n}\\
&=\sum_{n=0}^{\infty}p_{k,i}(0;n)q^{\mu}+\sum_{\mu=1}^{\infty}(-1)^{\mu}\sum_{n=0}^{\infty}m_{k,i}(\mu;n)q^{n}+\sum_{\mu=1}^{\infty}(-1)^{\mu}\sum_{n=0}^{\infty}p_{k,i}(\mu;n)q^{n}\\
&=P_{k,i}(0;q)+\sum_{\mu=1}^{\infty}(-1)^{\mu}M_{k,i}(\mu;q)+\sum_{\mu=1}^{\infty}(-1)^{\mu}P_{k,i}(\mu;q)\\
&=\sum_{\mu=1}^{\infty}M_{k,i}(2\mu;q)-\sum_{\mu=1}^{\infty}M_{k,i}(2\mu-1;q)+\sum_{\mu=0}^{\infty}P_{k,i}(2\mu;q)-\sum_{\mu=1}^{\infty}P_{k,i}(2\mu-1;q)\\
&=(q)_{\infty}^{-1}\Big\{\sum_{\mu=1}^{\infty}q^{\mu((4k+2)\mu+2k-2i+1)}-\sum_{\mu=1}^{\infty}q^{(2\mu-1)((2k+1)\mu-2k+i-1)}\\
&\qquad+\sum_{\mu=0}^{\infty}q^{\mu((4k+2)\mu-2k+2i-1)}-\sum_{\mu=1}^{\infty}q^{(2\mu-1)((2k+1)\mu-i)}\Big\}\\
&=(q)_{\infty}^{-1}\Big\{\sum_{\mu=-\infty}^{\infty}q^{\frac{1}{2}2\mu((2k+1)2\mu-2k+2i-1)}-\sum_{\mu=-\infty}^{\infty}q^{\frac{1}{2}(2\mu-1)((2k+1)(2\mu-1)-2k+2i-1)}\Big\}\\
&=(q)_{\infty}^{-1}\sum_{n=-\infty}^{\infty}(-1)^{n}q^{\frac{1}{2}n((2k+1)n-2k+2i-1)}
\end{aligned}$$

$$= (q)_\infty^{-1} \prod_{m=0}^{\infty} (1 - q^{(m+1)(2k+1)})(1 - q^{(2k+1)m+i})(1 - q^{(2k+1)(m+1)-i})$$

(by Jacobi's identity [13; p. 282])

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm i (\mathrm{mod}\ 2k+1)}}^{\infty} (1 - q^n)^{-1}$$

$$= \sum_{n=0}^{\infty} A_{k,i}(n) q^n.$$

Hence comparing coefficients of q^n in the above identities, we see that

$$A_{k,i}(n) = Q_{k,i}(n)$$

for each $n \geqq 0$. □

5. The Rogers-Ramanujan identities. In Section 1, we mentioned that (1.4) and (1.5) (the Rogers-Ramanujan identities) are special cases of Theorem 4.1. In Ref. [13; pp. 290-295] Hardy and Wright show that (1.4) and (1.5) are equivalent to the following partition identities. Namely

$$B(n) = A_{2,2}(n), \tag{5.1}$$

and

$$C(n) = A_{2,1}(n), \tag{5.2}$$

where $B(n)$ is the number of partitions of n with minimal difference at least 2 between summands while $C(n)$ is the number of such partitions in which 1 does not appear as a part.

It is very easy to show that $Q_{2,2}(n) = B(n)$ and $Q_{2,1}(n) = C(n)$, and (5.1) and (5.2) then follows directly from Theorem 4.1.

To prove that $Q_{2,2}(n) = B(n)$, let us consider a partitition π of the type enumerated by $B(n)$, say $n = c_1 + c_2 + \cdots + c_t$, where $c_i - c_{i+1} \geqq 2$. We form a graphical representation of π as follows. The i-th part of π is represented as the i-th right angle in a Ferrars graph where if $c_i = 2s + 1$, the angle is

$$s+1\left\{\begin{array}{l} \overbrace{\cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot}^{s+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array}\right.$$

and if $c_i = 2s$, the angle is

$$s\left\{\begin{array}{l}\overbrace{\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot}^{s+1}\\ \cdot\\ \cdot\\ \cdot\end{array}\right.$$

Since $c_i - c_{i+1} \geqq 2$, these angles do indeed form the successive angles of a Ferrars graph, and clearly the successive ranks are either zero or one. Thus a partition of the type enumerated by $Q_{2,2}(n)$ is produced. The above procedure clearly establishes a one-to-one correspondence between the two types of partitions, and therefore $B(n) = Q_{2,2}(n)$.

To prove that $C(n) = Q_{2,1}(n)$ we follow the same procedure except that now for parts $c_i = 2s + 1$, the angle is

$$s\left\{\begin{array}{l}\overbrace{\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot}^{s+2}\\ \cdot\\ \cdot\end{array}\right.$$

The argument proceeds as before with the only difference being that now an angle with only one node in it cannot arise, and this is precisely the difference between $B(n)$ and $C(n)$. Thus $C(n) = Q_{2,1}(n)$ and identities (5.1) and (5.2) are seen to be corollaries of Theorem 4.1.

6. Conclusion. There are several questions of interest that arise naturally from this work.

Question 1. B. Gordon [12] (see also [1]) has shown that if $B_{k,a}(n)$ denotes the number of partitions of n of the form $n = b_1 + b_2 + \cdots + b_s$ where $b_i - b_{i+k-1} \geqq 2$ and $b_{s-a+1} > 1$, then

$$B_{k,a}(n) = A_{k,a}(n)$$

for each n. Now $B_{2,2}(n) = B(n)$ and $B_{2,1}(n) = C(n)$, and in these two instances it was easy to establish that $B_{k,a}(n) = Q_{k,a}(n)$. Can direct Ferrars graph type proof be found to show that $B_{k,a}(n) = Q_{k,a}(n)$ in general?

Question 2. Are there sieves that can be used in studying some or all of the partition functions found in Refs. [2], [3], [4], and [5]?

Question 3. One can directly deduce from Theorem 3.2 that

$$(6.1)\quad m_{k,i}(2\mu; n) = p(n - \mu((4k+2)\mu + 2k - 2i + 1)),$$

(6.2) $m_{k,i}(2\mu-1;n)=p(n-(2\mu-1)((2k+1)\mu-2k+i-1))$,

(6.3) $p_{k,i}(2\mu;n)=p(n-\mu((4k+2)\mu-2k+2i-1))$,

(6.4) $p_{k,i}(2\mu-1;n)=p(n-(2\mu-1)((2k+1)\mu-i))$.

Are there direct combinatorial proofs of (6.1)-(6.4).

Finally we remark that the results in Theorem 3.1 can be used to prove the main theorem in Ref. [7] which extends the work of Schur [15].

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REFERENCES.

[1] G. E. Andrews, "An analytic proof of the Rogers-Ramanujan-Gordon identities," *American Journal of Mathematics*, vol. 88 (1966), pp. 844-846.

[2] ———, "Partition theorems related to the Rogers-Ramanujan identities," *Journal of Combinatorial Theory*, vol. 2 (1967), pp. 422-430.

[3] ———, "Some new partition theorems," *Journal of Combinatorial Theory*, vol. 2 (1967), pp. 431-436.

[4] ———, "A generalization of the Göllnitz-Gordon partition theorems," *Proceedings of the American Mathematical Society*, vol. 18 (1967), pp. 945-952.

[5] ———, "A generalization of the classical partition theorems," *Transactions of the American Mathematical Society*, vol. 145 (1969), pp. 205-221.

[6] ———, "On a calculus of partition functions," *Pacific Journal of Mathematics*, vol. 31 (1969), pp. 555-562.

[7] ———, "A polynomial identity which implies the Rogers-Ramanujan identities," *Scripta Math.*, vol 28 (1970), pp. 297-305.

[8] ———, "Sieves for theorems of Euler, Rogers, and Ramanujan," *Proceedings of the Western Michigan Conference on Arithmetic Functions*, Lecture Notes in Mathematics, No. 251, Springer, New York, 1972.

[9] A. O. L. Atkin, "A note on ranks and conjugacy of partitions," *Quarterly Journal of Mathematics* (2), vol. 17 (1966), pp. 335-338.

[10] A. O. L. Atkin and H. P. F. Swinnerton-Dyer, "Some properties of partitions," *Proceedings of the London Mathematical Society* (3), vol. 4 (1954), pp. 84-106.

[11] F. J. Dyson, "Some guesses in the theory of partitions," *Eureka*, vol. 8 (1944), pp. 10-15.

[12] B. Gordon, "A combinatorial generalization of the Rogers-Ramanujan identities," *American Journal of Mathematics*, vol. 83 (1961), pp. 393-399.

[13] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed., Oxford University Press, Oxford, 1960.

[14] H. Rademacher, *Lectures on Elementary Number Theory*, Blaisdell, New York, 1964.

[15] I. J. Schur, "Ein Bertrag zur additiven Zahlentheorie," *Sitzungsber. Akad. Wissensch.*, Berlin, Phys.-Math. Kl., 1917, pp. 302-321.

[16] J. J. Sylvester, "A constructive theory of partitions, arranged in three acts, an interact, and an exodion," *American Journal of Mathematics*, vol. 5 (1882), pp. 251-330.

[17] K. T. Vahlen, "Beiträge zu einer additiven Zahlentheories," *J. für Math.*, vol. 112 (1893), pp. 1-36.