

## Partition Identities\*

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### 1. INTRODUCTION

Intuitively a partition of a positive integer  $n$  may be thought of as an unordered representation of  $n$  as a sum of other positive integers. Thus  $3 + 2 + 2$ ,  $2 + 3 + 2$ ,  $2 + 2 + 3$  represent the same partition of 7. For convenience we usually consider that form of the partition in which the parts are arranged in nonincreasing order of magnitude.

We may also completely describe a partition by giving a list that tells the number of appearances of the various parts. Thus we shall sometimes denote a partition  $\Pi$  by  $\{f_i\}_{i=1}^{\infty}$  (or merely  $\{f_i\}$ ) where the  $f_i$  are non-negative integers of which only finitely many are nonzero. The number partitioned is given by

$$n = \sum_{i=1}^{\infty} f_i i.$$

In the above example  $f_1 = 0$ ,  $f_2 = 2$ ,  $f_3 = 1$ , and  $f_n = 0$  for  $n > 3$ .

The first problem in the theory of partitions is the study of the *unrestricted partition function*  $p(n)$ , the total number of partitions of  $n$ . Many important results are known about  $p(n)$ ; in particular, accurate asymptotic formulae have been found [67, Chap. 6], and infinite families of congruences are known for special sequences of  $n$  [67, Chap. 8]. The elementary aspects of the theory of partitions are given in detail in [66, Chap. 19], and the reader is referred there for such things as a proof of the following formula for the generating function related to  $p(n)$ .

$$1 + \sum_{n=1}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}.$$

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In this paper we shall be concerned with what might be called *restricted partition functions*, that is, partition functions that enumerate only those partitions of  $n$  that satisfy certain given conditions.

For example, we consider the number  $Q(n)$  of partitions of  $n$  into odd parts, and the number  $D(n)$  of partitions of  $n$  into distinct parts. In 1748, L. Euler [54, Chap. 16] presented and proved the first known theorem asserting the identity of two apparently different partition functions. Since

$$\begin{aligned} \sum_{n=0}^{\infty} D(n) q^n &= \prod_{n=1}^{\infty} (1 + q^n) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})}{(1 - q^n)} \\ &= \frac{\prod_{n=1}^{\infty} (1 - q^{2n})}{\prod_{n=1}^{\infty} (1 - q^n)} = \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n-1}} = \sum_{n=0}^{\infty} Q(n) q^n, \end{aligned}$$

we see that  $D(n) = Q(n)$  for each integer  $n$ . We may state this result as follows:

(1.1) *The number of partitions of  $n$  in which the summands are odd equals the number of partitions of  $n$  in which the summands are distinct.*

Theorems such as this are generally referred to as partition identities. The fact that such theorems may be quite difficult is not at all obvious from the simple proof of Euler's theorem. Indeed it was not until the early 1900's that substantially deeper partition identities were discovered. In 1917, P. A. MacMahon [71, Chap. 3] describes the following two theorems which he states are unproven but have been verified for  $n \leq 89$ .

(1.2) *The number of partitions of  $n$  in which the minimal difference between summands is at least 2 equals the number of partitions of  $n$  into parts of the forms  $5m + 1$  and  $5m + 4$ .*

(1.3) *The number of partitions of  $n$  in which the minimal difference between summands is at least 2 and 1 does not appear equals the number of partitions of  $n$  into parts of the forms  $5m + 2$  and  $5m + 3$ .*

These theorems are known as the Rogers–Ramanujan identities. In 1894, Rogers [73] proved the following series-product identities:

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2) \cdots (1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \quad (1.4)$$

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \quad (1.5)$$

From these analytic identities it is possible to deduce (1.2) and (1.3) (see [71, Chap. 3]); however, Rogers did not give the combinatorial interpretation and his work was soon forgotten. Around 1913, Ramanujan rediscovered (1.4) and (1.5) but could not find a proof. He communicated these identities to G. H. Hardy who also was baffled by them. It was only in 1917 when Ramanujan was looking through some old volumes of the Proceedings of the London Mathematical Society that he ran onto Rogers's paper containing proofs of (1.4) and (1.5). After that several further proofs of these results were found by Rogers and Ramanujan [74, 75]. Also it should be mentioned that Schur [77] discovered these theorems independently in 1917.

In 1926, Schur [78] discovered the following theorem of a similar type.

(1.6) *The number of partitions of  $n$  in which the minimal difference between summands is at least 3 and at least 6 between summands that are both multiples of 3 equals the number of partitions of  $n$  into parts of the forms  $6m + 1$  and  $6m + 5$ .*

Certain nonexistence theorems for identities like (1.1)–(1.3), and (1.6) were proved by Lehmer and Alder in the late 1940's [2, 70].

It was not until 1960 that two further identities of this nature were discovered by Göllnitz [59] (they were discovered independently by B. Gordon [62] in 1965 and have been called the Göllnitz–Gordon identities).

(1.7) *The number of partitions of  $n$  in which the minimal difference between summands is at least 2 and at least 4 between even summands, equals the number of partitions of  $n$  into parts of the forms  $8m + 1$ ,  $8m + 4$ , and  $8m + 7$ .*

(1.8) *The number of partitions of  $n$  in which the minimal difference between summands is at least 2, at least 4 between even summands, and all summands are greater than 2 equals the number of partitions of  $n$  into parts of the forms  $8m + 3$ ,  $8m + 4$ , and  $8m + 5$ .*

Then in 1961, Gordon [61] made a significant breakthrough in this

study by proving a theorem for an infinite number of identities; the Rogers–Ramanujan identities (1.2) and (1.3) correspond to the cases  $k = 2$ ,  $a = 1, 2$ .

(1.9) *Let  $B_{k,a}(n)$  denote the number of partitions of  $n$  of the form  $n = b_1 + b_2 + \cdots + b_s$  where  $b_i \geq b_{i+1}$ ,  $b_i - b_{i+k-1} \geq 2$  and 1 appears as a summand at most  $a - 1$  times; let  $A_{k,a}(n)$  denote the number of partitions of  $n$  into parts that are not congruent to  $0, \pm a \pmod{2k + 1}$ . Then  $A_{k,a}(n) = B_{k,a}(n)$  for each  $n$ .*

A great deal of subsequent work has related to theorems similar to (1.8). The following seems to be the most general result of this nature known [20].

(1.10) *Let  $B_{\lambda,k,a}(n)$  denote the number of partitions of  $n$  of the form  $n = b_1 + b_2 + \cdots + b_s$ , where  $b_i \geq b_{i+1}$ , only parts divisible by  $\lambda + 1$  may be repeated,  $b_i - b_{i+k-1} \geq \lambda + 1$  with strict inequality if  $(\lambda + 1) \mid b_i$ , and the total number of appearances of summands in the set  $\{1, 2, \dots, \lambda + 1\}$  is at most  $a - 1$ . If  $\lambda$  is even, let  $A_{\lambda,k,a}(n)$  denote the number of partitions of  $n$  into parts such that no part  $\not\equiv 0 \pmod{\lambda + 1}$  may be repeated and no part is congruent to  $0, \pm(a - \frac{1}{2}\lambda)(\lambda + 1) \pmod{(2k - \lambda + 1)(\lambda + 1)}$ ; if  $\lambda$  is odd, let  $A_{\lambda,k,a}(n)$  denote the number of partitions of  $n$  into parts such that no part  $\not\equiv 0 \pmod{\frac{1}{2}(\lambda + 1)}$  may be repeated, no part is congruent to  $\lambda + 1 \pmod{2\lambda + 2}$ , and no part is congruent to*

$$0, \pm(2a - \lambda) \frac{1}{2}(\lambda + 1) \pmod{(2k - \lambda + 1)(\lambda + 1)}.$$

*Then provided  $k \geq 2\lambda - 1$ , and  $k \geq a \geq \lambda$ , we have that  $A_{\lambda,k,a}(n) = B_{\lambda,k,a}(n)$ .*

The condition  $k \geq 2\lambda - 1$  is thought to be unnecessary although the only known proof of (1.10) relies heavily on this inequality.

Identity (1.10) includes many of the previous identities as special cases. In particular (1.2) is the case  $\lambda = 0$ ,  $k = 2 = a$ ; (1.3) is the case  $\lambda = 0$ ,  $k = 2$ ,  $a = 1$ ; (1.6) is essentially equivalent to the case  $\lambda = 2$ ,  $k = 2 = a$ ; (1.7) is the case  $\lambda = 1$ ,  $k = 2 = a$ ; (1.8) is the case  $\lambda = 1$ ,  $k = 2$ ,  $a = 1$ , and (1.10) is the case  $\lambda = 0$ . The case  $\lambda = 1$  yields a previously discovered identity generalizing the Göllnitz–Gordon identities [7]. The case  $\lambda = 2$  yields a generalization of Schur’s identity (1.6), and we shall study this generalization in Section 4. We chose this example because it has not been proved independently before; also it

illustrates some of the complexities that arise in the full proof of (1.10) but is not nearly as troublesome as the general result.

We observe that the partition function  $B_{2,k,a}(n)$  (or in general  $B_{\lambda,k,a}(n)$ ) is related to partitions in which there are different conditions on the differences  $b_i - b_{i+k-1}$  in the partitions  $n = b_1 + b_2 + \cdots + b_s$ . There are, however, at least two general theorems on partitions [17, 18] in which the conditions are on adjacent differences, that is,  $b_i - b_{i+1}$ . To state these theorems we require some further terminology.

Let  $A = \{a_1, a_2, \dots, a_r\}$  be a set of positive integers listed in increasing order of magnitude, and suppose that  $\sum_{j=1}^{i-1} a_j < a_i$  for each  $i$  (this guarantees that the various sums  $a_{i_1} + \cdots + a_{i_j}$  are all distinct). We let  $A'$  denote the  $2^r - 1$  different sums of distinct elements of  $A$ , and we write  $A' = \{\alpha_1, \alpha_2, \dots, \alpha_{2^r-1}\}$  where the  $\alpha_i$  are listed in increasing order of magnitude (and so  $\alpha_{2^i} = a_{i+1}$ ). The letter  $N$  denotes a positive integer not smaller than  $a_1 + a_2 + \cdots + a_r$ . We let  $A_N$  denote the set of all positive integers congruent to some  $a_i$  modulo  $N$ ;  $-A_N$  denotes the set of all positive integers congruent to some  $-a_i$  modulo  $N$ ;  $A_N'$  denotes the set of all positive integers congruent to some  $\alpha_i$  modulo  $N$ , and  $-A_N'$  denotes the set of all positive integers congruent to some  $-\alpha_i$  modulo  $N$ . If  $m \in A'$ , then  $m$  is uniquely representable as  $a_{i_1} + \cdots + a_{i_j}$  with  $a_{i_1} > \cdots > a_{i_j}$ ; we let  $\omega(m) = j$ , and  $\nu(m) = a_{i_j}$ . By  $\beta_N(s)$ , we denote the least positive residue of  $s$  modulo  $N$ .

(1.11) *Let  $\mathcal{D}(A_N : n)$  denote the number of partitions of  $n$  into distinct parts taken from  $A_N$ . Let  $\mathcal{E}(A_N' : n)$  denote the number of partitions of  $n$  of the form,  $n = b_1 + \cdots + b_s$ , where  $b_i \in A_N'$  and  $b_i - b_{i+1} \geq N\omega(\beta_N(b_{i+1})) + \nu(\beta_N(b_{i+1})) - \beta_N(b_{i+1})$ , for each  $i$ . Then for each  $n$ ,  $\mathcal{D}(A_N : n) = \mathcal{E}(A_N' : n)$ .*

(1.12) *Let  $\mathcal{F}(-A_N : n)$  denote the number of partitions of  $n$  into distinct parts taken from  $-A_N$ . Let  $\mathcal{J}(-A_N' : n)$  denote the number of partitions of  $n$  of the form,  $n = b_1 + \cdots + b_s$ , where  $b_i \in -A_N'$  and*

$$b_i - b_{i+1} \geq N \cdot \omega(\beta_N(-b_i)) + \nu(\beta_N(-b_i)) - \beta_N(-b_i),$$

*for each  $i$ , and furthermore  $b_s \geq N \cdot (\omega(\beta_N(-b_s)) - 1)$ . Then*

$$\mathcal{F}(-A_N : n) = \mathcal{J}(-A_N' : n).$$

If  $N = 3$ ,  $a_1 = 1$ ,  $a_2 = 2$ , then both (1.11) and (1.12) reduce to a result essentially equivalent to Schur's Theorem (1.6). In Section 5,

we shall treat (1.11) in the case  $N = 7$ ,  $a_1 = 1$ ,  $a_2 = 2$ ,  $a_3 = 4$ . Other specializations of these results may be found in [17] and [18]. In [33], (1.11) is used to solve partially a problem posed by Alder [3].

As the number of results for partition identities has increased tremendously recently, it becomes important to construct a general theory for such results so that the relationships between the various identities may be better perceived. Also a general theory allows one to pose the general goals and important problems of the subject in reasonably concise and comprehensible terms. In Section 2, we undertake to formulate the foundations of such a general theory. Our starting point lies in the observation that all the identities we have here discussed concern conditions on the frequency of appearance of parts in partitions according to various rules.

Section 3 is devoted to an extensive study of the simplest type of partition identities—those essentially similar to Euler's identity (1.1). Sections 4–6 provide some examples of the various techniques used in proving partition theorems. In Section 7, we return to the general theory begun in Sections 2 and 3. Finally Section 8 concludes by discussing possible generalizations and extensions of the theory of partition identities.

## 2. THE CENTRAL PROBLEMS

We are going to couch the study of partition identities in the terminology of lattice theory. Lattice theory provides a viewpoint which allows us to unify most of the known results on partition identities, and it provides an especially simple statement of the fundamental problems.

LEMMA 2.1. *Let  $\mathcal{S}$  denote the set of all sequences  $\{f_i\}_{i=1}^{\infty}$  (more briefly  $\{f_i\}$ ) where each  $f_i$  is a nonnegative integer and where only finitely many  $f_i$  are nonzero. Then  $\mathcal{S}$  forms a distributive lattice under the partial ordering*

$$\{f_i\} \leq \{g_i\} \text{ provided } f_i \leq g_i \text{ for each } i.$$

*Proof.* Observing that

$$\{f_i\} \cap \{g_i\} = \{\min(f_i, g_i)\},$$

and

$$\{f_i\} \cup \{g_i\} = \{\max(f_i, g_i)\},$$

we may easily verify that  $\mathcal{S}$  is a distributive lattice [76; Section 9; 44, Chap. 1, Section 6].

LEMMA 2.2. *If  $N$  denotes the set of nonnegative integers, and if*

$$\sigma : \mathcal{S} \rightarrow N$$

where  $\sigma(\{f_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} f_i \cdot i$ , then  $\sigma$  is a positive valuation on  $\mathcal{S}$ .

*Proof.* If  $\Pi = \{f_i\}_{i=1}^{\infty}$  and  $\Pi^1 = \{g_i\}_{i=1}^{\infty}$

$$\begin{aligned} \sigma(\Pi \cap \Pi^1) + \sigma(\Pi \cup \Pi^1) &= \sum_{i=1}^{\infty} (\min(f_i, g_i)i + \max(f_i, g_i)i) \\ &= \sum_{i=1}^{\infty} f_i i + \sum_{i=1}^{\infty} g_i i = \sigma(\Pi) + \sigma(\Pi^1). \end{aligned}$$

If  $\Pi > \Pi^1$ , then there exists  $j$  such that  $f_j > g_j$ . Hence

$$\sigma(\Pi) = \sum_{i=1}^{\infty} f_i \cdot i \geq \sum_{\substack{i=1 \\ i \neq j}}^{\infty} g_i \cdot i + (g_j + 1)j = \sigma(\Pi^1) + j.$$

First let us observe how these concepts relate to the classical study of partitions. The sequences  $\{f_i\}$  making up  $\mathcal{S}$  may be thought of as defining a partition where  $f_i$  gives the number of times the part  $i$  appears. The function  $\sigma$  merely maps each sequence  $\{f_i\}$  onto the number being partitioned.

Recall now that a *semi-ideal*  $J$  in a lattice  $L$  is a subset of  $L$  such that if  $a \in J$ ,  $x \in L$ , and  $x \leq a$ , then  $x \in J$  [44, p. 56].

DEFINITION 1. A semi-ideal in the lattice  $\mathcal{S}$  is called a *partition ideal*.

The concept of a partition ideal has evolved from that of a ‘‘partition condition’’ as described in [28]. Partition ideals are less general than partition conditions; however, we shall see that they describe most all known partition identities.

DEFINITION 2. If  $C$  is a partition ideal in  $\mathcal{S}$ , we say that  $p(C; n)$  is the *C-partition function* if for each  $n$ ,  $p(C; n)$  denotes the cardinality of the set  $\{\Pi \mid \Pi \in C, \sigma(\Pi) = n\}$ .

DEFINITION 3. We say that two partition ideals  $C_1$  and  $C_2$  of  $\mathcal{S}$  are *partition-theoretically equivalent* (more briefly PT-equivalent) if for each nonnegative integer

$$p(C_1; n) = p(C_2; n),$$

and we shall write  $C_1 \sim^{\text{PT}} C_2$ .

DEFINITION 4. We say that two partition ideals  $C_1$  and  $C_2$  of  $S$  are *asymptotically equivalent* (more briefly A-equivalent) if  $p(C_2; n) \neq 0$  for  $n > n_0$  and

$$\lim_{n \rightarrow \infty} [p(C_1; n)/p(C_2; n)] = 1,$$

and we write  $C_1 \sim^{\text{A}} C_2$ .

DEFINITION 5. We say that two partition ideals  $C_1$  and  $C_2$  of  $S$  are *weakly asymptotically equivalent* (more briefly WA-equivalent) if  $p(C_2; n) \neq 0$  for  $n > n_0$  and

$$\lim_{n \rightarrow \infty} \frac{\log p(C_1; n)}{\log p(C_2; n)} = 1,$$

and we write  $C_1 \sim^{\text{WA}} C_2$ .

It is obvious that  $\sim^{\text{PT}}$ ,  $\sim^{\text{A}}$ , and  $\sim^{\text{WA}}$  are equivalence relations in the standard sense.

Before considering some examples to justify this terminology, let us present the fundamental classification problem in the theory of partition identities.

### *First Problem*

*Fully describe the equivalence classes in  $\mathcal{S}$  under the equivalence  $\sim^{\text{PT}}$ .*

Of interest also is the same problem with  $\sim^{\text{PT}}$  replaced by  $\sim^{\text{A}}$  or  $\sim^{\text{WA}}$ . In Sections 3 and 7 we shall prove some general theorems concerning the complete problem. For the remainder of this section we shall endeavor to justify calling this the fundamental classification problem.

The intuitive description of a partition ideal in  $\mathcal{S}$  is just this: if a given partition  $\Pi$  satisfies a certain specified condition then all partitions formed by deleting one or many summands from  $\Pi$  also satisfy the same specified condition. This situation is clearly in evidence in all the theorems described in the introduction as we shall see.



DEFINITION 6. Let  $C(a_1, a_2, \dots, a_r; m)$  denote  $\{\Pi \mid \Pi = \{f_i\} \in \mathcal{S}$  and if  $f_i \neq 0$ , then  $i \equiv a_1, a_2, \dots, \text{ or } a_r \pmod{m}\}$ . Let  $C_d(a_1, a_2, \dots, a_r; m)$  denote  $\{\Pi \mid \Pi = \{f_i\} \in \mathcal{S}, 0 \leq f_i \leq 1, \text{ and if } f_i \neq 0, \text{ then } i \equiv a_1, a_2, \dots, \text{ or } a_r \pmod{m}\}$ .

It is obvious that both  $C(a_1, a_2, \dots, a_r; m)$  and  $C_d(a_1, a_2, \dots, a_r; m)$  are partition ideals in  $\mathcal{S}$ . Furthermore,  $p(C(a_1, a_2, \dots, a_r; m); n)$  is the number of partitions of  $n$  into parts congruent to  $a_1, a_2, \dots, a_r \pmod{m}$  while  $\hat{p}(C_d(a_1, a_2, \dots, a_r; m); n)$  is the number of such partitions in which the parts are distinct.

DEFINITION 7. Let  $D(r; b_1, b_2, \dots, b_m; m)$  denote  $\{\Pi \mid \Pi = \{f_j\} \in \mathcal{S}, f_j = 0 \text{ when } j < r, 0 \leq f_j \leq 1, \text{ and if } f_j = 1 \text{ with } j \equiv k \pmod{m}, \text{ then } f_{j+1} = f_{j+2} = \dots = f_{j+b_k-1} = 0\}$ .

Again it is clear that  $D(b_1, b_2, \dots, b_m; m)$  is an order ideal in  $\mathcal{S}$ . Furthermore,  $p(D(b_1, b_2, \dots, b_m; m); n)$  is the number of partitions of  $n$  of the form  $n = C_1 + C_2 + \dots + C_s$  where  $C_i - C_{i+1} \geq b_h$  if  $C_{i+1} \equiv h \pmod{m}$ . With these definitions we may briefly state (1.1), (1.2), (1.3), (1.6), (1.7) and (1.8) in the following equivalences, respectively:

$$C(1; 2) \stackrel{PT}{\sim} C_d(1; 1) \tag{2.1}$$

$$C(1, 4; 5) \stackrel{PT}{\sim} D(1; 2; 1) \tag{2.2}$$

$$C(2, 3; 5) \stackrel{PT}{\sim} D(2; 2; 1) \tag{2.3}$$

$$C(1, 5; 6) \stackrel{PT}{\sim} D(1; 3, 3, 4; 3) \tag{2.4}$$

$$C(1, 4, 7; 8) \stackrel{PT}{\sim} D(1; 2, 3; 2) \tag{2.5}$$

$$C(3, 4, 5; 8) \stackrel{PT}{\sim} D(3; 2, 3; 2). \tag{2.6}$$

Göllnitz [60] has also proved the following results

$$C_d(1, 2, 4; 4) \stackrel{PT}{\sim} C(1, 5, 6; 8) \stackrel{PT}{\sim} D(1; 3, 2; 2) \tag{2.7}$$

$$C_d(2, 3, 4; 4) \stackrel{PT}{\sim} C(2, 3, 7; 8) \stackrel{PT}{\sim} D(2; 3, 2; 2). \tag{2.8}$$

We point out that (1.9)–(1.12) as well as the theorems in [8–11], and [50] can all be interpreted as equivalences of partition ideals modulo  $\sim^{PT}$ .

From these comments it is clear that several equivalence classes modulo  $\sim^{\text{PT}}$  contain more than one element, and at least two contain at least three elements. Before we begin we introduce two more concepts that will be useful in our future discussions.

DEFINITION 8. We say that  $C$ , a partition ideal, has *order*  $k$  provided  $k$  is the least integer such that whenever  $\{f_i\} \notin C$ , then there exists  $m$  such that the sequence  $\{f'_i\}$  defined by

$$f'_i = \begin{cases} f_i, & i = m, m+1, \dots, m+k-1, \\ 0, & \text{otherwise,} \end{cases}$$

is also not in  $C$ .

Intuitively,  $C$  being of order  $k$  makes explicit the idea that summands of  $\Pi$  at least  $k$  units apart cannot affect whether or not  $\Pi \in C$ .

PROPOSITION 1. *The partition ideals*

$$C(a_1, \dots, a_r; m) \quad \text{and} \quad C_d(a_1, \dots, a_r; m)$$

*each have order 1. The partition ideal  $D(r; b_1, b_2, \dots, b_m; m)$  has order  $\max(b_1, b_2, \dots, b_m)$ .*

*Proof.* If  $\{f_i\} \notin C(a_1, a_2, \dots, a_r; m)$  then there exists  $j$  such that  $f_j \neq 0$  and  $j \not\equiv a_1, a_2, \dots, \text{ or } a_r \pmod{m}$ . Thus the sequence  $\{f'_i\}$  defined by

$$f'_i = \begin{cases} f_j, & \text{if } i = j, \\ 0, & \text{otherwise} \end{cases}$$

is not in  $C(a_1, a_2, \dots, a_r; m)$ . The same reasoning shows that  $C_d(a_1, a_2, \dots, a_r; m)$  has order 1.

Finally we consider  $D(r; b_1, b_2, \dots, b_m; m)$ , and we let

$$M = \max(b_1, b_2, \dots, b_m).$$

If  $\{f_i\} \notin D(r; b_1, b_2, \dots, b_m; m)$ , then there exist  $j$  and  $k$  such that  $f_j = f_k = 1$  and  $0 < k - j < b_r$  where  $r \equiv j \pmod{m}$ . Then the sequence  $\{f'_i\}$  defined by

$$f'_i = \begin{cases} f_i, & i = j, j+1, \dots, k, \dots, j+M-1, \\ 0, & \text{otherwise} \end{cases}$$

is not in  $D(r; b_1, b_2, \dots, b_m; m)$ . To see that  $M$  satisfies the minimality condition we let  $b_t = M$  and consider the sequence  $\{f_i''\}_{i=1}^\infty$  defined by

$$f_i'' = \begin{cases} 1, & i = mM + t, (m+1)M + t, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly  $\{f_i''\}$  is in  $D(r; b_1, b_2, \dots, b_m; m)$ . ■

The special role of partition ideals of order 1 will become obvious in succeeding sections. In Section 3, we shall prove that the generating function for  $C$ -partition functions where  $C$  is of order 1 are infinite products of a very simple sort. Actually as we shall see most of the known results on partition identities concern equivalences of two partition ideals where at least one has order 1. This has certainly been the case with (2.1)–(2.8).

#### Second Problem

*Fully describe those equivalence classes in  $\mathcal{S}$  modulo  $\sim^{\text{PT}}$  that contain a partition ideal of order 1.*

### 3. PARTITION IDEALS OF ORDER 1

**THEOREM 1.** *Let  $C$  be a partition ideal of order 1. Let  $\gamma_l$  be the set of  $\{f_i\} \in C$  such that  $f_i = 0$  for all  $i \neq l$ . Let*

$$d_l = \sup_{\{f_i\} \in \gamma_l} f_l.$$

*Then for  $|q| < 1$*

$$\sum_{n=0}^{\infty} p(C; n) q^n = \frac{\prod_{l=1, d_l < \infty}^{\infty} (1 - q^{l(d_l+1)})}{\prod_{l=1}^{\infty} (1 - q^l)}.$$

*Proof.* We first must prove that  $\Pi = \{f_i\} \in C$  if and only if  $f_i \leq d_i$  for each  $i$ . Suppose that this condition holds for  $\Pi = \{f_i\}$ . If  $\Pi \notin C$ , then there must exist  $\Pi' = \{f_i'\}_{i=1}^\infty \notin C$ , where all  $f_i' = 0$  except for a single  $j$  and  $f_j' = f_j$ . Now by the fact that  $d_j$  is a supremum of a set of nonnegative integers we see that  $d_j$  may be  $+\infty$ ; however, there must exist  $\{f_i''\} \in \gamma_j$  such that  $f_j' = f_j \leq f_j''$  since  $f_j \leq d_j$ . Hence

$$\Pi' = \{f_i'\} \leq \{f_i''\} \in \gamma_j \subset C,$$

and since  $C$  is a partition ideal of  $\mathcal{S}$ , we see that  $\Pi' \in C$ , a contradiction. Hence  $\Pi \in C$ .

On the other hand, if  $\Pi = \{f_i\} \in C$ , then  $\Pi'_j = \{f'_i\}$  defined by

$$f'_i = \begin{cases} f_j, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

is in  $\gamma_j$  for each  $j$ . Hence

$$f_j = f'_j \leq \sup_{\{g_i\} \in \gamma_j} g_j = d_j,$$

and the prescribed condition holds for each  $j$ .

Thus  $p(C; n)$  is the number of partitions of  $n$  in which each  $j$  appears at most  $d_j$  times. Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} p(C; n) q^n &= \prod_{\substack{j=1 \\ d_j \neq \infty}}^{\infty} (1 + q^j + q^{2j} + \cdots + q^{d_j j}) \prod_{\substack{j=1 \\ d_j = \infty}}^{\infty} (1 + q^j + q^{2j} + \cdots) \\ &= \prod_{\substack{j=1 \\ d_j \neq \infty}}^{\infty} \frac{(1 - q^{j(d_j+1)})}{(1 - q^j)} \prod_{\substack{j=1 \\ d_j = \infty}}^{\infty} \frac{1}{(1 - q^j)} \\ &= \frac{\prod_{j=1, d_j < \infty}^{\infty} (1 - q^{j(d_j+1)})}{\prod_{j=1}^{\infty} (1 - q^j)}. \quad \blacksquare \end{aligned}$$

Theorem 1 is the reason that infinite products play such an important role in the study of partition identities. Since most known identities involve a partition ideal of order 1, Theorem 1 shows that the associated identity for the generating functions will involve an infinite product.

Since partition ideals of order 1 have such nice associated generating functions we shall be able to prove some rather general theorems concerning these ideals.

**THEOREM 2.** *The ideals in  $\mathcal{S}$  are the partition ideals of order 1.*

*Proof.* We recall that an ideal in a lattice is a semi-ideal closed under union [44, p. 25].

Suppose  $C$  is of order 1. Then we know that there exist integers  $d_i$  ( $d_i$  may also equal  $+\infty$ ) such that  $\{f_i\} \in C$  if and only if  $f_i \leq d_i$  for each  $i$  (see the proof of Theorem 1). Hence if  $\{f_i\}$  and  $\{f'_i\}$  are both in  $C$ , then for each  $i$

$$\max(f_i, f'_i) \leq d_i;$$

therefore

$$\{\max(f_i, f_i')\} = \{f_i\} \cup \{f_i'\} \in C.$$

Hence  $C$  is an ideal in  $\mathcal{S}$ .

Conversely, suppose that  $C$  is an ideal  $\mathcal{S}$ . Define  $d_j$  by

$$d_j = \sup_{\{f_i\} \in C} f_j;$$

we note that  $d_j$  may be  $+\infty$ .

Suppose now that  $\{f_i\} \in \mathcal{S}$ ; obviously if  $f_i > d_i$  for some  $i$  then  $\{f_i\} \notin C$ . Suppose that  $f_{i_0} \leq d_{i_0}$  for each  $i_0$ . Then by the definition of  $d_i$ , there must exist for each  $i_0$  a  $\{g_i(i_0)\} \in C$  such that  $f_{i_0} \leq g_{i_0}(i_0)$ . Now define

$$h_i(i_0) = \begin{cases} f_i, & \text{if } i = i_0, \\ 0, & \text{otherwise.} \end{cases}$$

Since  $C$  is an ideal and since  $h_i(i_0) \leq g_i(i_0)$  for each  $i$ , we see that  $\{h_i(i_0)\} \leq \{g_i(i_0)\}$  and therefore  $\{h_i(i_0)\} \in C$ .

Now by the definition of elements of  $\mathcal{S}$  we know that there exists  $N$  such that  $f_i = 0$  for  $i > N$ . Consequently,

$$\{f_i\} = \bigcup_{i_0=1}^N \{h_i(i_0)\},$$

and this is in  $C$ , since  $C$  is an ideal. Thus  $\{f_i\} \in C$  if and only if  $f_i \leq d_i$  for each  $i$ . Thus  $C$  is a partition ideal of order 1. ■

**THEOREM 3.** *Suppose  $C$  is a partition ideal of order 1 and let the  $d_j$  be the associated suprema as in Theorem 1. Let  $C'$  be a second partition ideal of order 1 and let  $d_j'$  be the associated suprema. Then  $C \sim^{\text{PT}} C'$  if and only if the two sequences  $\{j(d_j + 1)\}_{d_j < \infty}$  and  $\{j(d_j' + 1)\}_{d_j' < \infty}$  are merely reorderings of one another.*

*Proof.* If the above two sequences are identical except for ordering, then

$$\begin{aligned} \sum_{n=0}^{\infty} p(C; n) q^n &= \frac{\prod_{j=1, d_j < \infty}^{\infty} (1 - q^{j(d_j+1)})}{\prod_{j=1}^{\infty} (1 - q^j)} \\ &= \frac{\prod_{j=1, d_j' < \infty}^{\infty} (1 - q^{j(d_j'+1)})}{\prod_{j=1}^{\infty} (1 - q^j)} = \sum_{n=0}^{\infty} p(C'; n) q^n. \end{aligned}$$

Comparing coefficients of  $q^n$  in the above sums, we see that  $p(C; n) = p(C'; n)$  for each  $n$ . Thus  $C \sim^{\text{PT}} C'$ .

If, however, the two sequences are not identical, then there exists a least integer  $h$  that appears a different number of times in each sequence. Without loss of generality, we may assume that  $h$  appears  $r$  times in the first sequence,  $s$  times in the second sequence and that  $r > s$ . Consequently,

$$\prod_{\substack{j=1 \\ d_j < \infty \\ j(d_j+1) \geq h}}^{\infty} (1 - q^{j(d_j+1)}) \neq \prod_{\substack{j=1 \\ d'_j < \infty \\ j(d'_j+1) \geq h}}^{\infty} (1 - q^{j(d'_j+1)})$$

since on the left hand side the coefficient of  $q^h$  arises from  $(1 - q^h)^r = 1 - rq^h + \dots$  and is therefore  $-r$ , while on the right hand side the coefficient of  $q^h$  arises from  $(1 - q^h)^s = 1 - sq^h + \dots$  and is therefore  $-s$  which is not  $-r$ . From the minimality of  $h$ , we see directly that

$$\prod_{\substack{d_j < \infty \\ j(d_j+1) < h}} (1 - q^{j(d_j+1)}) = \prod_{\substack{d'_j < \infty \\ j(d'_j+1) < h}} (1 - q^{j(d'_j+1)}).$$

Therefore combining this equality with our previous inequality, we see that

$$\begin{aligned} \sum_{n=0}^{\infty} p(C; n) q^n &= \frac{\prod_{j=1, d_j < \infty}^{\infty} (1 - q^{j(d_j+1)})}{\prod_{j=1}^{\infty} (1 - q^j)} \\ &\neq \frac{\prod_{j=1, d'_j < \infty}^{\infty} (1 - q^{j(d'_j+1)})}{\prod_{j=1}^{\infty} (1 - q^j)} = \sum_{n=0}^{\infty} p(C'; n) q^n. \end{aligned}$$

Examining the extremes of the above inequality, we see that  $p(C; n) \neq p(C'; n)$  for some  $n$ . Consequently  $C \not\sim^{\text{PT}} C'$ . ■

Theorem 3 is quite useful in the study of what are known as ‘‘Euler pairs’’ (see [23, 85]). To discuss fully Euler pairs we shall require some further notation.

**DEFINITION 9.** If  $T$  is a set of nonnegative integers, then

$$\begin{aligned} C(T) &= \{\{f_i\} \mid \{f_i\} \in \mathcal{S} \text{ and if } f_j > 0 \text{ then } j \in T\}, \\ C_d(T) &= \{\{f_i\} \mid \{f_i\} \in \mathcal{S} \text{ } 0 \leq f_j \leq 1, \text{ and if } f_j > 0 \text{ then } j \in T\}. \end{aligned}$$

We remark that  $C(T)$  and  $C_d(T)$  are partition ideals in  $\mathcal{S}$  of order 1. Furthermore,  $p(C(T); n)$  is the number of partitions of  $n$  into parts taken

from the set  $T$ , and  $p(C_d(T); n)$  is the number of partitions of  $n$  into distinct parts taken from the set  $T$ .

DEFINITION 10 (see [23]). We say that two sets of positive integers  $T_1$  and  $T_2$  form an *Euler pair*  $(T_1, T_2)$  if

$$C_d(T_1) \stackrel{PT}{\sim} C(T_2).$$

The central theorem in [23] characterizes the set of all Euler pairs. In [85], Subbarao extends this work by defining Euler pairs of higher order and giving a much simpler proof of his more general result. As we shall see Theorem 3 also allows us to give a simple proof of the characterization of Euler pairs.

THEOREM 4.  $(S_1, S_2)$  is an Euler pair if and only if  $2S_1 \subseteq S_1$  and  $S_2 = S_1 - 2S_1$ .

*Proof.* We shall apply Theorem 2 to  $C_d(S_1)$  and  $C(S_2)$ . The associated sequence for  $C_d(S_1)$  is  $\{n \mid n = 2k, k \in S_1\}$  together with  $\{n \mid n \notin S_1\}$ ; the associated sequence for  $C(S_2)$  is  $\{n \mid n \notin S_2\}$ .

We first observe that if  $2S_1 \not\subseteq S_1$  then the first associated sequence involves at least one repetition; this means that the two sequences are unequal since the second involves no repetitions.

If  $2S_1 \subseteq S_1$ , then for the two sequences to be equal we see that it is necessary and sufficient that  $S_2'$  the complement of  $S_2$  in  $N$  be  $2S_1 \cup S_1'$ . Hence  $S_2 = S_1 - 2S_1$ . ■

This result on Euler pairs allows us to exhibit an infinite number of equivalence classes modulo  $\sim^{PT}$  that contain an infinity of partition ideals. Our approach is suggested by some results of Guy [63].

THEOREM 5. Let  $\mathcal{C}$  be an equivalence class modulo  $\sim^{PT}$  that contains  $C_d(S_1)$  and  $C(S_2)$  where  $(S_1, S_2)$  is an Euler pair. If  $S_2$  is an infinite set, then  $\mathcal{C}$  is infinite.

*Proof.* Suppose

$$S_2 = \{\sigma_1, \sigma_2, \sigma_3, \dots\},$$

where the  $\sigma_i$  are positive integers arranged in increasing size. Define the order ideal  $C_j$  by  $C_j = \{\{f_i\} \mid \text{if } i = 2^s \sigma_k \text{ for some } s \geq 0 \text{ and } k < j \text{ then } 0 \leq f_i \leq 1, \text{ otherwise } f_i = 0 \text{ unless } i = \sigma_r \text{ for } r \geq j \text{ in which case the } f_i \text{ is unrestricted}\}$ .

First we comment on whether  $C_j$  is well-defined. There is clearly a possible conflict if  $2^s\sigma_k = \sigma_r$  for some  $s \geq 0$ ,  $k < j$ ,  $r \geq j$ . However, this is impossible for since  $\sigma_k \in S_2 \subset S_1$ ,  $2\sigma_k \in S_1$ ,  $2^2\sigma_k \in S_1$ , ...,  $2^s\sigma_k \in S_1$  because  $2S_1 \subseteq S_1$  by Theorem 3. Now  $s > 0$  since  $k < j \leq r$ . Therefore  $2^s\sigma_k \in 2S_1$ . Hence  $2^s\sigma_k \notin S_2 = S_1 - 2S_1$  while  $\sigma_r \in S_2$ . Thus  $C_j$  is always well-defined.

Clearly the  $C_j$  are partition ideals of order 1 and each is different from the others. To conclude our proof, we show that

$$\begin{aligned}
 C_1 &\stackrel{PT}{\sim} C_2 \stackrel{PT}{\sim} C_3 \stackrel{PT}{\sim} C_4 \stackrel{PT}{\sim} \dots \\
 \sum_{n=0}^{\infty} p(C_j; n) q^n &= \frac{\prod_{i=2^s\sigma_k, s \geq 0, k < j} (1 + q^i)}{\prod_{l=j}^{\infty} (1 - q^{\sigma_l})} \\
 &= \frac{1}{(1 - q^{\sigma_j})} \cdot \frac{\prod_{i=2^s\sigma_k, s \geq 0, k < j} (1 + q^i)}{\prod_{l=j+1}^{\infty} (1 - q^{\sigma_l})} \\
 &= \frac{\prod_{l=0}^{\infty} (1 + q^{2^l\sigma_j}) \prod_{i=2^s\sigma_k, s \geq 0, k < j} (1 + q^i)}{\prod_{l=j+1}^{\infty} (1 - q^{\sigma_l})} \\
 &= \frac{\prod_{i=2^s\sigma_k, s \geq 0, k < j+1} (1 + q^i)}{\prod_{l=j+1}^{\infty} (1 - q^{\sigma_l})} = \sum_{n=0}^{\infty} p(C_{j+1}; n) q^n.
 \end{aligned}$$

Hence comparing coefficients of  $q^n$  in the two above sums, we see that  $p(C_j; n) = p(C_{j+1}; n)$  for each  $n$ . Therefore  $C_j \stackrel{PT}{\sim} C_{j+1}$  for each  $j$ . Since  $C_1 = C(S_2)$ , we see that  $\mathcal{C}$  contains an infinite number of elements. ■

It is not difficult to deduce from Theorem 5 that there exist infinitely many such  $\mathcal{C}$  with infinitely many elements. Indeed if  $S_2$  is any set of odd positive integers and if  $S_1$  is the set of all positive integers whose largest odd factor lies in  $S_2$ , then clearly by Theorem 3,  $(S_1, S_2)$  is an Euler pair. Furthermore if  $S_2 \neq S_2^*$ , then  $C(S_2) \not\stackrel{PT}{\sim} C(S_2^*)$  (for a proof of this see the Lemma in [23]). Thus each different infinite set of odd integers  $S_2$  gives rise to a different Euler pair and a new equivalence class containing the partition ideals associated with that Euler pair.

The results in this section have described some of the partition ideals of order 1 and have given some idea of the nature of some of the equivalence classes modulo  $\sim^{PT}$ . As the Introduction and equivalences (2.1)–(2.8) indicate, most of the more important theorems concerning



partition identities arise from equivalences of the sort  $C_1 \sim^{\text{PT}} C_2$  where  $C_1$  is of order 1 and  $C_2$  is of higher order. In the next three sections we shall examine some of the techniques used in proving these deeper results. Later in Section 7, we shall return to the study of the structure of the equivalence classes of  $\mathcal{S}$ .

#### 4. WELL-POISED THEOREMS

We refer to the family of partition identities given in (1.10) as well-poised since the generating function for the case  $a = k$  is an infinite product times the well-poised basic hypergeometric series:

$${}_{\lambda+3}\Phi_{\lambda+2} \left[ \begin{matrix} x, q^{\lambda+1}\sqrt{x}, -q^{\lambda+1}\sqrt{x}, -q, -q^2, \dots, -q^\lambda; q^{\lambda+1}, -x^k q^{\frac{1}{2}(\lambda+1)(2k-\lambda+1)n + \frac{1}{2}(\lambda-1)} \\ -\sqrt{x}, -\sqrt{x}, -xq^\lambda, -xq^{\lambda-1}, \dots, -xq \end{matrix} \right] \quad (4.1)$$

(for more details concerning well-poised basic, hypergeometric series, see [81, 21]). We have chosen to here present a complete proof of (1.10) in the case  $\lambda = 2$ . This choice allows us to observe some of the difficulties that arise in the proof of (1.10) without embroiling us in the intricate details that accompany proof of the full result [20]. Also Theorem 6 has independent interest since it is a generalization of Schur's Theorem (1.6).

We shall study the following functions:

$$C_{k,a}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{kn} q^{\frac{3}{2}(2k-1)n^2 + \frac{3}{2}n - 3an} (1 - x^a q^{6na}) (x, q^3)_n (-q; q^3)_n (-q^2; q^3)_n}{(q^3; q^3)_n (1-x) (-xq; q^3)_n (-xq^2; q^3)_n}, \quad (4.2)$$

$$D_{k,a}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{kn} q^{\frac{3}{2}(2k-1)n^2 + \frac{3}{2}n + 3(k-a)n} (xq^3; q^3)_n (-q; q^3)_n (-q^2; q^3)_n}{(q^3; q^3)_n (-xq; q^3)_n (-xq^2; q^3)_n} \times \left\{ 1 - \frac{x^a q^{(6n+3)(a-1)} (1 + q^{3n+1}) (1 + q^{3n+2})}{(1 + xq^{3n+1}) (1 + xq^{3n+2})} \right\}, \quad (4.3)$$

where  $|q| < 1$ ,  $x \neq q^{-3n-1}, q^{-3n-2}$  for any integer  $n \geq 0$ , and

$$(a; q)_n = (a)_n = (1-a)(1-aq) \cdots (1-aq^{n-1}),$$

$$(a; q)_\infty = (a)_\infty = \lim_{n \rightarrow \infty} (a; q)_n.$$

$$H_{k,a}(x) = \frac{(-xq; q^3)_\infty (-xq^2; q^3)_\infty}{(xq^3; q^3)_\infty} C_{k,a}(x), \quad (4.4)$$

$$J_{k,a}(x) = \frac{(-xq; q^2)_\infty (-xq^2; q^3)_\infty}{(xq^3; q^3)_\infty} D_{k,a}(x). \quad (4.5)$$

From these four equations, we see directly that

$$C_{k,0}(x) = H_{k,0}(x) = 0, \quad (4.6)$$

and

$$H_{k,a}(0) = J_{k,a}(0) = 0 \quad \text{for } 1 \leq a \leq k. \quad (4.7)$$

We now must prove some lemmas.

LEMMA 4.1.  $H_{k,a}(x) - H_{k,a-1}(x) = x^{a-1} J_{k,k-a+1}(x)$ .

*Proof.* We shall prove the equivalent identity

$$C_{k,a}(x) - C_{k,a-1}(x) = x^{a-1} D_{k,k-a+1}(x). \quad (4.8)$$

we see that

$$\begin{aligned} C_{k,a}(x) - C_{k,a-1}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{kn} q^{\frac{3}{2}(2k-1)n^2 + \frac{3}{2}n} (x; q^3)_n (-q; q^3)_n (-q^2; q^3)_n}{(q^3; q^3)_n (1-x) (-xq; q^3)_n (-xq^2; q^3)_n} \\ &\quad \times \{q^{-3an}(1 - q^{3n}) + x^{a-1} q^{3n(a-1)}(1 - xq^{3n})\} \end{aligned}$$

We split this sum into two separate parts one for each of the main terms inside the curly brackets, and we replace  $n$  by  $n + 1$  in the first sum. Hence we see that

$$\begin{aligned} &C_{k,a}(x) - C_{k,a-1}(x) \\ &= - \sum_{n=0}^{\infty} \frac{(-1)^n x^{k(n+1)} q^{\frac{3}{2}(2k-1)(n+1)^2 + \frac{3}{2}(n+1)} (x; q^3)_{n+1} (-q; q^3)_{n+1} (-q^2; q^3)_{n+1}}{(q^3; q^3)_n (1-x) (-xq; q^3)_{n+1} (-xq^2; q^3)_{n+1}} \\ &\quad + x^{a-1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{kn} q^{\frac{3}{2}(2k-1)n^2 + \frac{3}{2}n + 3n(a-1)} (x; q^3)_{n+1} (-q; q^3)_n (-q^2; q^3)_n}{(q^3; q^3)_n (1-x) (-xq; q^3)_n (-xq^2; q^3)_n} \\ &= x^{a-1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{kn} q^{\frac{3}{2}(2k-1)n^2 + \frac{3}{2}n + 3n(a-1)} (xq^3; q^3)_n (-q; q^3)_n (-q^2; q^3)_n}{(q^3; q^3)_n (-xq; q^3)_n (-xq^2; q^3)_n} \\ &\quad \times \left\{ 1 - \frac{x^{k-a+1} q^{6kn - 6an + 3(k-a)} (1 + q^{3n+1})(1 + q^{3n+2})}{(1 + xq^{3n+1})(1 + xq^{3n+2})} \right\} \\ &= x^{a-1} D_{k,k-a+1}(x). \end{aligned}$$

LEMMA 4.2.

$$J_{k,a} = H_{k,a}(xq^3) + (xq + xq^2) H_{k,a-1}(xq^3) + x^2q^3 H_{k,a-2}(xq^3).$$

*Proof.* We shall prove the equivalent identity:

$$\begin{aligned} D_{k,a}(x) &= \frac{(1 - xq^3)}{(1 + xq)(1 + xq^2)} \\ &\quad \times \{C_{k,a}(xq^3) + (xq + xq^2) C_{k,a-1}(xq^3) + x^2q^3 C_{k,a-2}(xq^3)\} \\ D_{k,a}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{kn} q^{\frac{1}{2}(2k-1)n^2 + \frac{1}{2}n + 3(k-a)n} (xq^3; q^3)_n (-q; q^3)_n (-q^2; q^3)_n}{(q^3; q^3)_n (-xq; q^3)_n (-xq^2; q^3)_n} \\ &\quad \times \left\{ \frac{(1 + xq^{3n+1})(1 + xq^{3n+2}) - x^a q^{(6n+3)(a-1)} (1 + q^{3n+1})(1 + q^{3n+2})}{(1 + xq^{3n+1})(1 + xq^{3n+2})} \right\} \\ &= \frac{(1 - xq^3)}{(1 + xq)(1 + xq^2)} \\ &\quad \times \sum_{n=0}^{\infty} \frac{(-1)^n (xq^3)^{kn} q^{\frac{3}{2}(2k-1)n^2 + \frac{3}{2}n} (xq^3; q^3)_n (-q; q^3)_n (-q^2; q^3)_n}{(1 - xq^3)(q^3; q^3)_n (-xq^4; q^3)_n (-xq^5; q^3)_n} \quad (4.9) \\ &\quad \times \{q^{-3an}(1 - (xq^3)^a q^{6an}) + x(q + q^2) q^{-3(a-1)n} (1 - (xq^3)^{a-1} q^{6n(a-1)}) \\ &\quad + x^2q^3 q^{-3(a-2)n} (1 - (xq^3)^{a-2} q^{6n(a-2)})\} \\ &= \frac{(1 - xq^3)}{(1 + xq)(1 + xq^2)} \\ &\quad \times \{C_{k,a}(xq^3) + x(q + q^2) C_{k,a-1}(xq^3) + x^2q^3 C_{k,a-2}(xq^3)\}. \quad \blacksquare \end{aligned}$$

LEMMA 4.3.  $H_{k,1}(x) = J_{k,k}(x)$ .

*Proof.* Set  $a = 1$  in Lemma 4.1 and recall that  $H_{k,0}(x)$  is identically 0.  $\blacksquare$

LEMMA 4.4.  $J_{k,2}(x) = (1 + xq + xq^2) J_{k,k}(xq^3) + xq^3 J_{k,k-1}(xq^3)$ .

*Proof.* By Lemmas 4.1–4.3, we have that

$$\begin{aligned} J_{k,2}(x) &= H_{k,2}(xq^3) + x(q + q^2) H_{k,1}(xq^3) \\ &= H_{k,1}(xq^3) + xq^3 J_{k,k-1}(xq^3) + x(q + q^2) J_{k,k}(xq^3) \\ &= (1 + xq + xq^2) J_{k,k}(xq^3) + xq^3 J_{k,k-1}(xq^3). \quad \blacksquare \end{aligned}$$

LEMMA 4.5.

$$\begin{aligned} J_{k,a}(x) - J_{k,a-1}(x) \\ = x^{a-1}q^{3a-6}(q^3 J_{k,k-a+1}(xq^3) + (q^2 + q) J_{k,k-a+2}(xq^3) + J_{k,k-a+3}(xq^3)). \end{aligned}$$

*Proof.* By Lemmas 4.1, and 4.2, we see that

$$\begin{aligned} J_{k,a}(x) - J_{k,a-1}(x) \\ = (H_{k,a}(xq^3) - H_{k,a-1}(xq^3)) + x(q + q^2)(H_{k,a-1}(xq^3) - H_{k,a-2}(xq^3)) \\ + x^2q^3(H_{k,a-2}(xq^3) - H_{k,a-3}(xq^3)) \\ = x^{a-1}q^{3a-6}(q^3 J_{k,k-a+1}(xq^3) + (q + q^2) J_{k,k-a+2}(xq^3) + J_{k,k-a+3}(xq^3)). \quad \blacksquare \end{aligned}$$

LEMMA 4.6.  $H_{k,-a}(x) = -x^{-a}H_{k,a}(x)$ .

*Proof.* This follows directly from (4.2), (4.4) and the fact that

$$q^{3an}(1 - x^{-a}q^{-6na}) = -x^{-a}q^{-3an}(1 - x^aq^{6an}). \quad \blacksquare$$

LEMMA 4.7.  $J_{k,1}(x) = (1 - x) J_{k,k}(xq^3)$ .

*Proof.* By Lemmas 4.2, 4.3, and 4.6,

$$\begin{aligned} J_{k,1}(x) &= H_{k,1}(xq^3) + x^2q^3H_{k,-1}(xq^3) \\ &= J_{k,k}(xq^3) - xH_{k,1}(xq^3) \\ &= (1 - x) J_{k,k}(xq^3). \quad \blacksquare \end{aligned}$$

LEMMA 4.8.

$$\begin{aligned} J_{k,k}(x) - J_{k,k-1}(x) \\ = x^{k-1}q^{3k-6}((1 - xq^3) q^3 J_{k,k}(xq^6) + (q + q^2) J_{k,2}(xq^3) + J_{k,3}(xq^3)). \end{aligned}$$

*Proof.* Set  $a = k$  in Lemma 4.5, and apply Lemma 4.7 to  $J_{k,1}(xq^3)$ . \blacksquare

We are now prepared to prove the following result.

**THEOREM 6.** *Let  $A_{2,k,a}(n)$  denote the number of partitions of  $n$  in which only multiples of 3 may be repeated and no part is  $\equiv 0, \pm(3a - 3) \pmod{6k - 3}$ . Let  $B_{2,k,a}(n)$  denote the number of partitions of  $n$  of the*

form  $n = b_1 + b_2 + \cdots + b_s$ , where  $b_i \geq b_{i+1}$ , only parts divisible by 3 may be repeated,  $b_i - b_{i+k-1} \geq 3$  with strict inequality if  $3 \mid b_i$ , and the total number of appearances of the parts 1, 2, and 3 is at most  $a - 1$ . Let  $A_{2,k,a}^*(n)$  denote the number of partitions of  $n$  into parts that are  $\equiv 0, 1, 3, 5 \pmod{6}$  but  $\not\equiv 0, \pm(3a-3) \pmod{6k-3}$ . Then for  $2 \leq a \leq k$  and  $k \geq 3$ , we have

$$A_{2,k,a}(n) = B_{2,k,a}(n) = A_{2,k,a}^*(n).$$

*Remark.* The case  $k = a = \lambda = 2$  reduces to Schur's theorem (1.6). The restriction  $k \geq 3$  is unnecessary, but when  $k = 2$  the details are somewhat different [15].

*Proof.* Let us consider the following expansion of  $J_{k,a}(x)$ .

$$J_{k,a}(x) = \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} j_{k,a}(M, N) x^M q^N,$$

where the variables  $x$  and  $q$  are subject to the conditions listed after (4.2).

We shall now substitute this series into the identities we have obtained in the Lemmas. This will allow us to compute certain recursion relations for the  $j_{k,a}(M, N)$ .

By (4.2), (4.4), and (4.7), we see that

$$\begin{aligned} j_{k,a}(M, N) &= 1 && \text{if } M = N = 0 \\ &= 0 && \text{if either } M \leq 0 \text{ or } N \leq 0, \text{ but } M^2 + N^2 \neq 0 \end{aligned} \quad (4.10)$$

By Lemma 4.4,

$$\begin{aligned} j_{k,2}(M, N) &= j_{k,k}(M, N - 3M) + j_{k,k}(M - 1, N - 3M + 2) \\ &\quad + j_{k,k}(M - 1, N - 3M + 1) + j_{k,k-1}(M - 1, N - 3M). \end{aligned} \quad (4.11)$$

By Lemma 4.5 (with  $3 \leq a \leq k - 1$ ),

$$\begin{aligned} j_{k,a}(M, N) - j_{k,a-1}(M, N) &= j_{k,k-a+1}(M - a + 1, N - 3M) + j_{k,k-a+2}(M - a + 1, N - 3M + 1) \\ &\quad + j_{k,k-a+2}(M - a + 1, N - 3M + 2) \\ &\quad + j_{k,k-a+3}(M - a - 1, N - 3M + 3). \end{aligned} \quad (4.12)$$

By Lemma 4.8,

$$\begin{aligned}
& j_{k,k}(M, N) - j_{k,k-1}(M, N) \\
&= j_{k,k}(M - k + 1, N - 6M + 3k - 3) \\
&\quad - j_{k,k}(M - k, N - 6M + 3k) + j_{k,2}(M - k + 1, N - 3M + 1) \\
&\quad + j_{k,2}(M - k + 1, N - 3M + 2) + j_{k,2}(M - k + 1, N - 3M + 3).
\end{aligned} \tag{4.13}$$

We now observe that Eqs. (4.10)–(4.13) uniquely define  $j_{k,a}(M, N)$  for all  $M$  and  $N$  with  $2 \leq a \leq k$ : Eq. (4.10) provides the initial conditions; then one proceeds by a triple mathematical induction first on  $N$ , then  $M$ , then  $a$  (for  $2 \leq a \leq k$ ) to verify the uniqueness of  $j_{k,a}(M, N)$ .

Let  $P_{k,a}(M, N)$  denote the number of partitions of  $N$  of the type enumerated by  $B_{2,k,a}(N)$  with the added condition that there are exactly  $M$  parts. We shall show that  $P_{k,a}(M, N)$  also satisfies (4.10)–(4.13).

The fact that  $P_{k,a}(M, N)$  satisfies (4.10) is obvious; note that  $P_{k,a}(0, 0) = 1$  since the empty partition of 0 is counted here.

Next we prove that (4.12) is valid with  $P_{k,a}(M, N)$  replacing  $j_{k,a}(M, N)$ . First we observe that  $P_{k,a}(M, N) - P_{k,a-1}(M, N)$  counts those partitions of the type enumerated by  $P_{k,a}(M, N)$  with the condition that the total number of appearances of the summands 1, 2, and 3 is *exactly*  $a - 1$ . We split these partitions into 4 disjoint classes: (1) those in which neither 1 nor 2 appear; (2) those in which 1 appears but 2 does not; (3) those in which 2 appears but 1 does not, and (4) those in which both 1 and 2 appear. We now transform all our partitions by deleting all summands that are smaller than 4 and subtracting 3 from each of the other summands.

The number of parts of each partition in each class is now reduced to  $M - a + 1$ . The number being partitioned is reduced to  $N - 3M$  in the first class,  $N - 3(M - 1) - 1 = N - 3M + 2$  in the second class,  $N - 3(M - 1) - 2 = N - 3M + 1$  in the third class, and  $N - 3(M - 2) - 3 = N - 3M + 3$  in the fourth class.

Since originally for partitions in the first class,  $f_3 = a - 1$ , and  $f_3 + f_4 + f_5 + f_6 \leq k - 1$ , we see that  $f_4 + f_5 + f_6 \leq (k - a + 1) - 1$ ; consequently the transformed partitions from the first class are of the type enumerated by  $P_{k,k-a+1}(M - a + 1, N - 3M)$ . Reversal of the above argument shows that there are exactly

$$P_{k,k-a+1}(M - a + 1, N - 3M)$$

elements of the first class. Exactly similar arguments show that the other three classes produce the corresponding partition functions appearing in (4.12).

The argument to prove (4.11) is quite similar except that now the four classes are: (1) 3 appears while 2 and 1 do not; (2) 1 appears while 2 and 3 do not; (3) 2 appears while 1 and 3 do not, and (4) none of 1, 2, or 3 appears. One now proceeds in the manner used to verify (4.12).

Equation (4.13) seems unnecessary; it should be a special case of (4.12). Unfortunately, the function  $J_{k,1}(x)$  is *not* the generating function for  $P_{k,1}(M, N)$  and so there is no hope of proving that it is. Consequently to avoid the appearance of  $J_{k,1}(x)$  and  $P_{k,1}(M, N)$ , we must treat (4.13).

Equation (4.13) is treated at the outset exactly as is (4.12). Indeed the entire left-hand side and the last three terms on the right side are produced exactly as before. In the case where  $f_3 = k - 1$ , we subtract 6 from all summands larger than 3 and delete all summands that equal 3. This produces partitions of the type enumerated by

$$P_{k,k}(M - a + 1, N - 6M - 3k + 3).$$

Examining (4.13), we notice that there is still a term

$$-P_{k,k}(M - k, N - 6M + 3k)$$

left over. This arises from the fact that by including the complete third term on the right-side of (4.13) we counted too much. Namely, if  $f_1 = 0$ ,  $f_2 = 1$ , and  $f_3 = k - 2$ , then  $f_2 + f_3 + f_4 \leq k - 1$  implies  $f_4 = 0$  a condition separate from and *not* implied by  $f_4 + f_5 + f_6 \leq 1$ . Thus we must subtract off the number of those partitions for which  $f_1 = 0$ ,  $f_2 = 1$ ,  $f_3 = k - 2$ ,  $f_4 = 1$ ,  $f_5 = 0$ ,  $f_6 = 0$  but which otherwise are partitions of the type enumerated by  $P_{k,k}(M, N)$ . To count such partitions, we delete all summands  $\leq 6$  and subtract 6 from each of the remaining summands. In this way we obtain just those partitions enumerated by  $P_{k,k}(M - k, N - 6M + 3k)$ , and this accounts for the remaining term on the right hand side of (4.13).

Thus we see that  $P_{k,a}(M, N)$  satisfies (4.10)–(4.13); therefore since the  $j_{k,a}(M, N)$  are the unique solutions of these equations, we have proved that

$$P_{k,a}(M, N) = j_{k,a}(M, N) \tag{4.14}$$

for each  $M$  and  $N$  with  $2 \leq a \leq k$ .

Equation (4.14) is the crucial matter in the proof. The rest is fairly simple. We observe that

$$\begin{aligned}
B_{2,k,a}(N) &= \sum_{M=0}^{\infty} P_{k,a}(M, N). \\
\sum_{N=0}^{\infty} B_{2,k,a}(N) q^N &= \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} P_{k,a}(M, N) q^N \\
&= J_{k,a}(1) \\
&= \frac{(-q; q^3)_{\infty} (-q^2; q^3)_{\infty}}{(q^2; q^3)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3}{2}(2k-1)n^2 + \frac{3}{2}n + 3(k-a)n} \\
&\quad \times (1 - q^{(6n+3)(a-1)}) \\
&= \frac{(-q; q^3)_{\infty} (-q^2; q^3)_{\infty}}{(q^3; q^3)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3}{2}(2k-1)n^2 + \frac{3}{2}n + 3(k-a)n} \\
&= \frac{(-q; q^3)_{\infty} (-q^2; q^3)_{\infty}}{(q^3; q^3)_{\infty}} (q^{6k-3}; q^{6k-3})_{\infty} (q^{6k-3-3(a-1)}; q^{6k-3}) \\
&\quad \times (q^{3a-3}; q^{6k-3})_{\infty} \\
&= \sum_{N=0}^{\infty} A_{2,k,a}(N) q^N,
\end{aligned}$$

where the penultimate expression follows by Jacobi's identity [66, p. 282]. Thus comparing coefficients of  $q^N$  we see that  $B_{2,k,a}(N) = A_{2,k,a}(N)$ . To obtain the result for  $A_{2,k,a}^*(n)$  we merely substitute  $(q; q^6)^{-1} (q^5; q^6)_{\infty}^{-1}$  for  $(-q; q^3)_{\infty} (-q^2; q^3)_{\infty}$  in the penultimate expression above. Thus Theorem 6 is proved. ■

Several closing comments are in order. First of all we see that this result provides an example of two partition ideals that are equivalent modulo  $\sim^{\text{PT}}$  even though the partition ideals related to  $A_{2,k,a}(n)$  and to  $A_{2,k,a}^*(n)$  have order 1 while the partition ideal related to  $B_{2,k,a}(n)$  has order 4.

With regard to the general Eq. (1.10), we see easily that the partition ideal related to  $A_{\lambda,k,a}(n)$  has order 1 while the one related to  $B_{\lambda,k,a}(n)$  has order  $\lambda + 2$ . Thus we see clearly that order is *not* a property of equivalence classes in general.



Finally it should be remarked that the snag of proving (4.13) for  $P_{k,a}(M, N)$  becomes worse and worse for larger and larger  $\lambda$ . This accounts for the complex special partition functions studied in detail in Sections 2 and 4 of [20].

### 5. A THEOREM FOR THE MODULUS 7

The result of Section 4 (a special case of (1.10)) relied on a combinatorial interpretation of the  $q$ -difference equations associated with certain well-poised basic hypergeometric series; also the proof utilized the fact that these series reduced to an instance of the Jacobi triple product identity when  $x = 1$ .

Here we are concerned with a special case of (1.12). The general proofs of (1.11) and (1.12) both rely on Abel's lemma concerning the limiting value of  $\sum a_n x^n$  as  $x \rightarrow 1^-$ ; however, (1.11) also uses  $q$ -difference Eqs. [17] while (1.12) is concerned with what might be termed  $q$ -recurrent sequences [18]. We have chosen for our study a special case of (1.11) which illustrates fairly clearly the techniques related to the general proofs of (1.11) and (1.12).

THEOREM 7.

$$C(1, 9, 11; 14) \stackrel{PT}{\sim} C_a(1, 2, 4; 7) \stackrel{PT}{\sim} D(1; 7, 7, 12, 7, 10, 10, 15; 7).$$

*Remark.* To simplify notation we shall write  $D$  for  $D(1; 7, 7, 12, 7, 10, 10, 15; 7)$ .

*Proof.* The equivalence

$$C(1, 9, 11; 14) \stackrel{PT}{\sim} C_a(1, 2, 4; 7)$$

follows directly from our theorem on Euler pairs, Theorem 4.

To prove the second equivalence we start by defining  $p_a(m, n)$  to be the cardinality of the following set:

$$\left\{ \{f_i\} \mid \{f_i\} \in D, \sum_{i=1}^{\infty} f_i i = n, \sum_{i=1}^{\infty} f_i = m, \text{ and } f_i = 0 \text{ if } i \leq a \right\}.$$

We must now establish the following identities:

$$p_0(m, n) = p_1(m, n) + p_0(m-1, n-7m+6) \quad (5.1)$$

$$p_1(m, n) = p_2(m, n) + p_1(m-1, n-7m+5) \quad (5.2)$$

$$p_2(m, n) = p_3(m, n) + p_0(m-1, n-14m+11) \quad (5.3)$$

$$p_3(m, n) = p_4(m, n) + p_3(m-1, n-7m+3) \quad (5.4)$$

$$p_4(m, n) = p_5(m, n) + p_0(m-1, n-14m+9) \quad (5.5)$$

$$p_5(m, n) = p_6(m, n) + p_1(m-1, n-14m+8) \quad (5.6)$$

$$p_6(m, n) = p_7(m, n) + p_0(m-1, n-21m+14) \quad (5.7)$$

$$p_7(m, n) = p_0(m, n-7m). \quad (5.8)$$

The proofs of these eight identities all resemble one another. We therefore choose (5.7) to present in detail. First we see that  $p_6(m, n) - p_7(m, n)$  is *exactly* the number of partitions  $\pi$  of  $n$  with  $m$  parts such that  $\pi \in D$  and 7 is a summand of  $\pi$ . By the requirements on  $D$  we see that all other summands besides the 7 must be at least as large as 22. We now transform these partitions by deleting the 7 and subtracting 21 from every other part. This leaves us with a partition of the type enumerated by  $p_0(m-1, n-21m+14)$ . Clearly the above process is reversible, and so we have established a one-to-one correspondence between the partitions enumerated by  $p_6(m, n) - p_7(m, n)$  and those enumerated by  $p_0(m-1, n-21m+14)$ . Thus (5.7) is established.

We now define

$$f_a(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_a(m, n) x^n q^n.$$

Then directly from our eight identities, we deduce that

$$f_0(x) = f_1(x) + xq^1 f_0(xq^7), \quad (5.9)$$

$$f_1(x) = f_2(x) + xq^2 f_1(xq^7), \quad (5.10)$$

$$f_2(x) = f_3(x) + xq^3 f_0(xq^{14}), \quad (5.11)$$

$$f_3(x) = f_4(x) + xq^4 f_3(xq^7), \quad (5.12)$$

$$f_4(x) = f_5(x) + xq^5 f_0(xq^{14}), \quad (5.13)$$

$$f_5(x) = f_6(x) + xq^6 f_1(xq^{14}), \quad (5.14)$$

$$f_6(x) = f_7(x) + xq^7 f_0(xq^{21}), \quad (5.15)$$

$$f_7(x) = f_0(xq^7). \quad (5.16)$$

We wish now to solve these equations to obtain an identity in  $f_0(x)$ ,  $f_0(xq^7)$ ,  $f_0(xq^{14})$ , ... . To do this we note that adding all these equations produces an identity  $f_0(x) = E_1$  where  $E_1$  involves only  $f_0, f_1$ , and  $f_3$  (with various arguments). Adding the first three equations produces an identity  $f_3(x) = E_2$ , where  $E_2$  involves only  $f_0$  and  $f_1$ . Furthermore (5.9) may be transformed to produce an identity

$$f_1(x) = f_0(x) - xqf_0(xq^7) = E_3,$$

an expression involving only  $f_0$ . Thus we may use  $E_3$  for substitutions into  $E_2$  to produce an identity  $f_3(x) = E_4$  where  $E_4$  involves only  $f_0$ . Finally we may use  $E_3$  and  $E_4$  for substitutions into  $E_1$  to produce an identity involving only  $f_0$  with various arguments. The final result is

$$\begin{aligned} f_0(x) = & (1 + xq + xq^2 + xq^4)f_0(xq^7) + x(q^3 + q^5 + q^6)(1 - xq^7)f_0(xq^{14}) \\ & + xq^7(1 - xq^7)(1 - xq^{14})f_0(xq^{21}). \end{aligned} \quad (5.17)$$

If we define  $G(x) = f_0(x); (x; q^7)_\infty$ , then by dividing (5.17) by  $(xq^7; q^7)_\infty$  we obtain

$$\begin{aligned} (1 - x)G(x) = & (1 + xq + xq^2 + xq^4)G(xq^7) \\ & + x(q^3q^5 + q^6)G(xq^{14}) + xq^7G(xq^{21}). \end{aligned} \quad (5.18)$$

We now consider the MacLaurin series for  $G(x)$ , namely

$$G(x) = \sum_{n=0}^{\infty} \gamma_n x^n,$$

and we substitute into (5.18). Thus from the coefficient of  $x^n$  after the substitution we see that

$$\begin{aligned} \gamma_n - \gamma_{n-1} = & q^{7n}\gamma_n + q^{7n-6}\gamma_{n-1} + q^{7n-5}\gamma_{n-1} + q^{7n-3}\gamma_{n-1} \\ & + q^{14n-11}\gamma_{n-1} + q^{14-1}\gamma_{n-1} + q^{14n-1}\gamma_{n-1} + q^{21n-14}\gamma_{n-1}. \end{aligned} \quad (5.19)$$

Therefore

$$(1 - q^{7n})\gamma_n = (1 + q^{7n-6})(1 + q^{7n-5})(1 + q^{7n-3})\gamma_{n-1}. \quad (5.20)$$

Since  $\gamma_0 = G(0) = f_0(0) = 1$ , we may easily solve the recurrence (5.20) by iteration.

Therefore,

$$\gamma_n = (-q; q^7)_n (-q^2; q^7)_n (-q^4; q^7)_n / (q^7; q^7)_n \quad (5.21)$$

Hence

$$f_0(x) = (x; q^7)_\infty G(x) = (x; q^7)_\infty \sum_{n=0}^{\infty} \frac{(-q; q^7)_n (-q^2; q^7)_n (-q^4; q^7)_n x^n}{(q^7; q^7)_n}.$$

Letting  $x \rightarrow 1^-$  we deduce using Abel's lemma (or the more general Appell Comparison theorem [52, p. 101]) that

$$\begin{aligned} \sum_{n=0}^{\infty} p_0(D; n) q^n &= f_0(1) \\ &= \lim_{x \rightarrow 1^-} f_0(x) \\ &= (q^7; q^7)_\infty \lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} \frac{(-q; q^7)_n (-q^2; q^7)_n (-q^4; q^7)_n x^n}{(q^7; q^7)_n} \\ &= (-q; q^7)_\infty (-q^2; q^7)_\infty (-q^4; q^7)_\infty \\ &= \sum_{n=0}^{\infty} p(C_d(1, 2, 4; 7); n) q^n. \end{aligned} \quad (5.22)$$

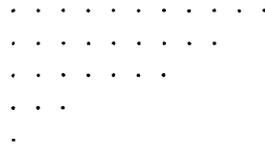
Thus  $D \sim^{\text{PT}} C_d(1, 2, 4; 7)$ , and Theorem 7 is established.  $\blacksquare$

## 6. ELEMENTARY TECHNIQUES

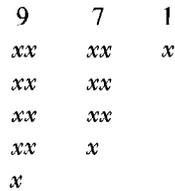
In Sections 4 and 5, we have studied the important interrelationship between partition identities and  $q$ -difference equations. The results obtained were always very general yielding families of identities rather than just single equivalences. In this section we shall discuss how more elementary techniques may be utilized. We choose, for our example, the identity of Göllnitz and Gordon, namely, (1.7) [or equivalently (2.5)]. It is especially appropriate for our purposes since the proof will involve both work with infinite series and a study of the graphical representation of partitions.

**THEOREM 8.**  $p(D(1; 2, 3, 2); n)$  equals the number of partitions of  $n$  of the form  $\sum_{i=1}^s a_i + \sum_{i=1}^t b_i$  where  $a_i$  and  $b_i$  are odd,  $a_i > a_{i+1}$ ,  $b_i > b_{i+1}$ , and  $a_i \leq 2t - 1$ .

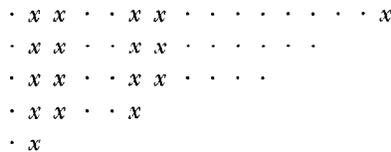
*Proof.* Let us consider the second type of partitions first. As an example, we take the partition of 48 in which the  $a$ -sum is  $9 + 7 + 1$  and the  $b$ -sum is  $11 + 9 + 7 + 3 + 1$ . The  $b$ -sum is now represented as an ordinary Ferrars graph [66, Ch. 19]



The  $a$ -sum is represented in a modified graphical form, namely, each part  $2j + 1$  is represented by two vertical columns of  $j + 1$  and  $j$  nodes respectively. Thus we have



These columns are now inserted in the obvious places in the Ferrars graph for the  $b$ -sum (there is only one place each pair of columns will fit except for the 1 which is put on the far right):



This merged graph (read rows horizontally again) now represents  $16 + 13 + 11 + 6 + 2$ .

In general, the above procedure establishes a one-to-one correspondence between the two types of partitions mentioned in our theorem. This is because the  $b$ -sum has only distinct odd parts; then the  $a$ -sum is joined into the Ferrars graph of the  $b$  sum in such a way that the distance between parts is not decreased and if an even part arises (which occurs due to the lower left hand node in the representation of an  $a_i$ ) then it is at least 3 units larger than the next smallest part. Conversely,

given a partition of the type enumerated by  $p(D(1; 2, 3; 2); n)$  we form the corresponding partition by removing for the  $a$ -sum those pairs of columns the lower left hand node of which is the right-hand-most node of an even part. ■

COROLLARY.

$$\sum_{n=0}^{\infty} p(D(1, 2, 3; 2); n) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q^2)_n}{(q^2; q^2)_n}.$$

*Proof.* By Theorem 8, we know that  $p(D(1; 2, 3; 2); n)$  equals the number of partitions of  $n$  of the form  $\sum_{i=1}^s a_i + \sum_{i=1}^t b_i$  where  $a_i$  and  $b_i$  are odd,  $a_i > a_{i+1}$ ,  $b_i > b_{i+1}$ , and  $a_i \leq 2t - 1$ . For a given  $t$ , the generating function for the  $b$ -sum is

$$\frac{q^{1+3+\dots+(2t-1)}}{(1-q^2)(1-q^4)\dots(1-q^{2t})} = \frac{q^{t^2}}{(q^2; q^2)_t}$$

while the generating function for the  $a$ -sum is

$$(1+q)(1+q^3)\dots(1+q^{2t-1}) = (-q; q^2)_t.$$

Consequently for given  $t$  the generating function for the two sums taken together is

$$q^{t^2}(-q; q^2)_t / (q^2; q^2)_t.$$

Summing over all  $t$ , we obtain the desired result. ■

We shall now consider this generating function (actually we replace  $q$  by  $-q$  for convenience) in the following theorem.

THEOREM 9.

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(q^2; q^2)_n} = (-q; q^8)_{\infty}^{-1} (q^4; q^8)_{\infty}^{-1} (-q^7; q^8)_{\infty}^{-1}.$$

*Proof.* For the proof of this identity we shall need the two elementary identities due to Euler [66, Chap. 19]:

$$\sum_{n=0}^{\infty} \frac{Z^n}{(q)_n} = (Z)_{\infty}^{-1}, \quad (6.1)$$

and

$$\sum_{n=0}^{\infty} \frac{Z^n q^{\frac{1}{2}n(n-1)}}{(q)_n} = (-Z)_{\infty}, \quad (6.2)$$

as well as the famous triple product identity due to Jacobi [66, Chap. 19]:

$$\sum_{n=-\infty}^{\infty} q^{n^2} Z^n = (q^2; q^2)_{\infty} (-Zq; q^2)_{\infty} (-Z^{-1}q; q^2)_{\infty}. \quad (6.3)$$

Although it was long believed that (6.3) was a much deeper identity than (6.1) and (6.2), it has been shown [6] that (6.3) follows rather easily from (6.1) and (6.2). Thus our Theorem is deducible from (6.1) and (6.2) above.

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(q^2; q^2)_n} \\ &= (q; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2; q^2)_n (q^{2n+1}; q^2)_{\infty}} \\ &= (q; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2; q^2)_n} \sum_{m=0}^{\infty} \frac{q^{2mn+m}}{(q^2; q^2)_m} \quad (\text{by (6.1)}) \\ &= (q; q^2)_{\infty} \sum_{m=0}^{\infty} \frac{q^m}{(q^2; q^2)_m} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2mn}}{(q^2; q^2)_n} \\ &= (q; q^2)_{\infty} \sum_{m=0}^{\infty} \frac{q^m}{(q^2; q^2)_m} (q^{2m+1}; q^2)_{\infty} \quad (\text{by (6.2)}) \\ &= (q; q^2)_{\infty}^2 \sum_{m=0}^{\infty} \frac{q^m}{(q^2; q^2)_m (q; q^2)_m} \\ &= (q; q^2)_{\infty}^2 \sum_{m=0}^{\infty} \frac{q^m}{(q; q)_{2m}} \\ &= \frac{1}{2} (q; q^2)_{\infty}^2 \sum_{m=0}^{\infty} \frac{q^{m/2} (1 + (-1)^m)}{(q; q)_m} \\ &= \frac{1}{2} (q; q^2)_{\infty}^2 ((q^{1/2})_{\infty}^{-1} + (-q^{1/2})_{\infty}^{-1}) \quad (\text{by 6.1}) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} (q; q^2)_\infty (q^{1/2})_\infty (-q^{1/2})_\infty ((q^{1/2})_\infty^{-1} + (-q^{1/2})_\infty^{-1}) \\
&= \frac{1}{2} (q; q^2)_\infty ((-q^{1/2})_\infty + (q^{1/2})_\infty) \\
&= \frac{1}{2} \frac{(q; q^2)_\infty}{(q^2; q^2)_\infty} ((q^2; q^2)_\infty (-q^{1/2}; q^2)_\infty (-q^{3/2}; q^2)_\infty \\
&\quad + (q^2; q^2)_\infty (q^{1/2}; q^2)_\infty (q^{3/2}; q^2)_\infty) \\
&= \frac{1}{2} \frac{(q; q^2)_\infty}{(q^2; q^2)_\infty} \left( \sum_{n=-\infty}^{\infty} q^{n^2+n/2} + \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+n/2} \right) \quad (\text{by (6.3)}) \\
&= \frac{(q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{4n^2+n} \\
&= \frac{(q; q^2)_\infty}{(q^2; q^2)_\infty} (q^8; q^8)_\infty (-q^3; q^8)_\infty (-q^5; q^8)_\infty \quad (\text{by (6.3)}) \\
&= \frac{(q^2; q^4)_\infty}{(-q; -q)_\infty} (q^8; q^8)_\infty (-q^3; q^8)_\infty (-q^5; q^8)_\infty \\
&= \frac{(q^2; q^8)_\infty (q^6; q^8)_\infty (q^8; q^8)_\infty (-q^3; q^8)_\infty (-q^5; q^8)_\infty}{(-q; -q)_\infty} \\
&= (-q; q^8)_\infty^{-1} (q^4; q^8)_\infty^{-1} (-q^7; q^8)_\infty^{-1}. \quad \blacksquare
\end{aligned}$$

**COROLLARY.**  $C(1, 4, 7; 8) \sim^{\text{PT}} D(1; 2, 3; 2)$ .

*Proof.*

$$\begin{aligned}
&\sum_{n=0}^{\infty} p(D(1; 2, 3; 2); n) q^n \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q^2)_n}{(q^2; q^2)_n} \quad (\text{by the Corollary to Theorem 8}) \\
&= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{8n+1})(1 - q^{8n+4})(1 - q^{8n+7})} \quad (\text{by Theorem 9}) \\
&= \sum_{n=0}^{\infty} p(C(1, 4, 7; 8); n) q^n.
\end{aligned}$$



Hence comparing coefficients of  $q^n$  in the extreme sums above, we see that

$$p(D(1; 2, 3; 2); n) = p(C(1, 4, 7; 8); n)$$

for each  $n$ . Hence our result follows. ■

It should be emphasized that the techniques used in this chapter are merely a sample of many that are used in this subject. For further analytic theorems like Theorem 9, we refer the reader to [87, 12–14, 31, 1, 79–81]. The elementary study of partitions via graphical and other arithmetical techniques has been extensively studied. There is an excellent introduction to this aspect in [66; Chap. 19], and in [28] it is shown that many theorems can be proved purely using such means.

## 7. PARTITION IDEALS OF HIGHER ORDER

In Sections 2 and 3 we considered general questions on the structure of the lattice  $\mathcal{S}$  relative to  $\sim^{\text{PT}}$ . In Section 3, we proved that there were infinitely many equivalence classes modulo  $\sim^{\text{PT}}$  that each contained infinitely many partition ideals of order 1. In later sections, we saw that the deeper aspects of this theory concern theorems of the form  $C_1 \sim^{\text{PT}} C_2$  where  $C_1$  is of order 1 and  $C_2$  is of higher order. A natural question arises now, namely: are there equivalence classes modulo  $\sim^{\text{PT}}$  that each contain infinitely many partition ideals of order greater than 1? We can settle this question with the following theorem although it still leaves many questions unanswered.

**THEOREM 10.** *Suppose  $r \geq 5$ . Then there exists an equivalence class modulo  $\sim^{\text{PT}}$  that contains infinitely many partition ideals of order  $r$ .*

*Proof.* Let  $k = r - 2$  so that  $k \geq 3$ .

Define the partition ideal  $D_j$  by letting it be those  $\{f_i\}$  associated with partitions  $\pi$  of the form  $b_1 + b_2 + \cdots + b_s$  where 1) if  $b_i \leq jk$ , then  $b_i - b_{i+1} \geq k$  and if  $k \mid b_i$ ,  $b_i - b_{i+1} > k$ ; 2) if  $b_i > jk$  then  $b_i - b_{i+1} \geq k$  if  $b_i \not\equiv 0, -1 \pmod{k}$ , when  $b_i = kl - 1$ ,  $b_i - b_{i+1} \geq k - 1$ , when  $b_i = kl$ ,  $b_i - b_{i+1} \geq k + 2$ . We also consider the case  $j = \infty$ , where now the parts of the partitions are subject always to condition 1. We shall prove that

$$D_0 \stackrel{\text{PT}}{\sim} D_1 \stackrel{\text{PT}}{\sim} D_2 \stackrel{\text{PT}}{\sim} D_3 \stackrel{\text{PT}}{\sim} \cdots$$

To prove this we define  $p_m(D_j; n)$  as the number of partitions of the type enumerated by  $p(D_j; n)$  with the additional restriction that all summands do not exceed  $m$ . We also consider the related generating functions

$$f_m(j; q) = \sum_{n=0}^{\infty} p_m(D_j; n) q^n.$$

It is clear that  $f_m(j; q)$  is a polynomial in  $q$ , and that

$$\lim_{m \rightarrow \infty} f_m(j; q) = \sum_{n=0}^{\infty} p(D_j; n) q^n.$$

The crux of the proof now lies in establishing the following identities.

$$f_m(j; q) = f_m(\infty; q) \quad \text{if } k \nmid (m+1) \text{ or } m \leq jk, \quad (7.1)$$

$$f_{lk-1}(j; q) = f_{lk-1}(\infty; q) + q^{2k(l-1)+k-1} f_{(l-2)k-1}(\infty; q) \quad \text{for } l > j. \quad (7.2)$$

We prove this by observing that the difference conditions used in defining  $D_{\infty}$  and  $D_j$  imply the following recursive relations.

$$f_m(\infty; q) = f_{m-1}(\infty; q) + q^m f_{m-k}(\infty; q), \quad \text{if } k \nmid m; \quad (7.3)$$

$$f_{lk}(\infty; q) = f_{lk-1}(\infty; q) + q^{lk} f_{lk-k-1}(\infty; q); \quad (7.4)$$

$$f_m(j; q) = f_{m-1}(j; q) + q^m f_{m-k}(j; q), \quad \text{if } k \nmid m \text{ for } m \leq jk \\ \text{and for } m > jk \quad \text{if } k \nmid m \text{ or } m-1; \quad (7.5)$$

$$f_{lk}(j; q) = f_{lk-1}(j; q) + q^{lk} f_{lk-k-1}(j; q) \quad \text{for } l \leq j; \quad (7.6)$$

$$f_{lk-1}(j; q) = f_{lk-2}(j; q) + q^{lk-1} f_{lk-k}(j; q), \quad \text{if } l > j; \quad (7.7)$$

$$f_{lk}(j; q) = f_{lk-1}(j; q) + q^{lk} f_{lk-k-2}(j; q), \quad \text{if } l > j. \quad (7.8)$$

By (7.3)–(7.5) and (7.7) and the fact that  $f_0(\infty; q) = f_0(j; q) = 1$ ,  $f_m(0; q) = f_m(j; q) = 0$  for  $m < 0$ , we obtain using mathematical induction that (7.1) is valid for all  $m \leq jk$ . We now assume that the theorem has been verified for all  $m \leq rk$ , where  $r \geq j$ . Then for  $1 \leq i \leq k-2$ , we have (using a second finite induction on  $i$ )

$$\begin{aligned} f_{rk+i}(j; q) &= f_{rk+i-1}(j; q) + q^{rk+i} f_{rk-k+i}(j; q) \\ &= f_{rk+i-1}(\infty; q) + q^{rk+i} f_{rk-k+i}(\infty; q) \\ &= f_{rk+i}(\infty; q). \end{aligned}$$

$$\begin{aligned}
f_{rk+k-1}(j; q) &= f_{rk+k-2}(j; q) + q^{rk+k-1}f_{rk}(j; q) \\
&= f_{rk+k-2}(\infty; q) + q^{rk+k-1}f_{rk}(\infty; q) \\
&= f_{rk+k-2}(\infty; q) + q^{rk+k-1}f_{rk-1}(\infty; q) \\
&\quad + q^{rk+k-1}(f_{rk}(\infty; q) - f_{rk-1}(\infty; q)) \\
&= f_{rk+k-2}(\infty; q) + q^{rk+k-1}f_{rk-1}(\infty; q) + q^{rk+k-1}f_{rk-k-1}(\infty; q) q^{rk} \\
&= f_{rk+k-1}(\infty; q) + q^{2rk+k-1}f_{rk-k-1}(\infty; q), \\
\\
f_{rk+k}(j; q) &= f_{rk+k-1}(j; q) + q^{rk+k}f_{rk-2}(j; q) \\
&= f_{rk+k-1}(\infty; q) + q^{2rk+k-1}f_{rk-k-1}(\infty; q) + q^{rk+k}f_{rk-2}(\infty; q) \\
&= f_{rk+k-1}(\infty; q) + q^{rk+k}f_{rk-1}(\infty; q) \\
&\quad + q^{rk+k}(f_{rk-2}(\infty; q) - f_{rk-1}(\infty; q) + q^{rk-1}f_{rk-k-1}(\infty; q)) \\
&= f_{rk+k-1}(\infty; q) + q^{rk+k}f_{rk-1}(\infty; q) \\
&= f_{rk+k}(\infty; q).
\end{aligned}$$

Thus, in general, we see that if Eqs. (7.1) and (7.2) are valid for  $m \leq rk$ , then they are valid for  $m \leq rk + k$ . Consequently, they are valid for all  $m$ . Clearly then we see that for  $|q| < 1$ ,

$$\lim_{m \rightarrow \infty} f_m(j; q) = \lim_{m \rightarrow \infty} f_m(\infty; q).$$

Therefore,

$$\sum_{n=0}^{\infty} p(D_j; n) q^n = \sum_{n=0}^{\infty} p(D_\infty; n) q^n,$$

and so for each  $n$

$$p(D_j; n) = p(D_\infty; n).$$

Therefore  $D_j \sim^{PT} D_\infty$ . Since the order of  $D_j$  is  $k + 2 = r$  for each  $j > 0$ , Theorem 10 is established. ■

The genesis of Theorem 10 lies in the discovery [25] that for  $k = 3$ ,

$$D_\infty \stackrel{PT}{\sim} D_0 = D(1; 2, 3, 4; 3) \stackrel{PT}{\sim} C_d(1, 2; 3) \stackrel{PT}{\sim} C(1, 5; 6),$$

an extension of Schur's theorem.

The proof of Theorem 10, may easily be modified to prove that there exist infinitely many equivalence classes modulo  $\sim^{PT}$  that each

contain an infinite number of partition ideals of order  $r$  provided  $r \geq 7$ . The modification is made by requiring that in  $b_1 + b_2 + \cdots + b_s$ ,  $b_i - b_{i+1} \geq k + 1$  if  $b_i = hk + 2$  where  $h$  is fixed and  $k = r - 2$ . This change produces a new equivalence class for each  $h$  but does not affect the essential relationships involving  $f_m(\infty; q)$  and  $f_m(j; q)$  for  $m \equiv 0, -1 \pmod{k}$ . It remains an open question whether or not there exist any equivalence classes with an infinite number of elements of order  $r$  for  $r = 2, 3, 4$ . Much deeper (and more interesting) is the question of whether there exists an equivalence class with an infinite number of orders.

## 8. CONCLUSION

It must be pointed out here that not all partition functions of interest in the theory of partition identities are  $C$ -partition functions for some partition ideal  $C$ . For example, from Euler's famous pentagonal number theorem

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(3n-1)} = (q)_{\infty}.$$

Legendre deduced that

$$p_e(n) - p_o(n) = \begin{cases} (-1)^{\lambda} & \text{if } n = \frac{1}{2}\lambda(3\lambda \pm 1), \\ 0, & \text{otherwise,} \end{cases}$$

where  $p_e(n)$  (resp.  $p_o(n)$ ) is the number of partitions of  $n$  into an even (resp. odd) number of distinct parts. Thus

$$p_e(n) + p_o(n) = p(D(1; 1; 1); n);$$

however  $p_e(n)$  counts only a certain restricted subset of  $D(1; 1; 1) = D_1$ , namely, those  $\{f_i\} \in D_1$  for which  $\sum_{i=1}^{\infty} f_i$  is even. Similarly  $p_o(n)$  counts those elements of  $D_1$  for which  $\sum_{i=1}^{\infty} f_i$  is odd. Thus the study of equivalence classes of  $\mathcal{S}$  modulo  $\sim^{PT}$  is not all that is of interest in partition identities. The internal structure of certain partition ideals is also of interest in certain cases (see also [34]).

In this vein, we also remark that a sieve method has recently been introduced into the theory of partition identities [38]. This technique allows new proofs of Euler's pentagonal number theorem, the Rogers-Ramanujan identities, and certain new partition identities related to the

$A_{k,a}(n)$  of (1.10) for each  $n$ . The sieve acts on partitions classified according to successive ranks [40], and so the relationship with the partition ideals in  $\mathcal{S}$  is not clear at this time; however, one suspects some canonical mapping between such partitions and partitions of the type enumerated by the  $B_{k,a}(n)$  of (1.10).

As is clear from the theorems presented in this paper, the theory of partition identities is intimately related to basic hypergeometric series (through  $q$ -difference equations) and to techniques for transforming Ferrars graphs. Recent work of Goldman and Rota [57, 58], and Knuth [68] also indicates that the combinatorial theory of finite vector spaces may play an important role in future developments.

With regard to the lattice-theoretic side of partition identities, we note that if  $L$  is a lattice and  $\sigma$  is a positive, integer-valued valuation on  $L$  such that the sets

$$\{x \in L \mid \sigma(x) = n\}$$

are all finite, then we may study “generalized partition identities” in  $L$  by defining an equivalence on the semi-ideals of  $L$  just as  $\sim^{\text{PT}}$  was defined on  $\mathcal{S}$ . For example, if  $L$  is the lattice of subspaces of an  $n$ -dimensional vector space over the finite field  $GF(q)$ , then the dimension function  $d(S)$  is a positive, integer-valued valuation on  $L$ . Furthermore, if  $p_L(r)$  denotes the cardinality of  $\{S \in L \mid d(S) = r\}$ , then it is known that

$$p_L(r) \equiv \begin{bmatrix} n \\ r \end{bmatrix} = (q)_n / (q)_r (q)_{n-r} \quad [57].$$

Thus the generating function for  $p_L(r)$  is

$$\sum_{r=0}^n p_L(r) x^r = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} x^r = H_n(x),$$

the  $q$ -Hermite polynomial [48]. More interesting questions arise when the lattice  $L$  is infinite.

At the conclusion of his survey article [4], Alder asks about the discovery of families of partition identities. The identities given in (1.9)–(1.12) as well as those discussed in [8 and 9] are all examples of families of identities. If  $B_{\lambda,k,a}$  denotes the partition ideal of which  $B_{\lambda,k,a}(n)$  is the associated partition function, then we see that

$$B_{\lambda,k,a} \subseteq B_{\lambda',k',a'},$$

whenever  $\lambda = \lambda'$ ,  $k' \geq k$ ,  $a' \geq a$ . Thus  $B_{\lambda, k, a}$  may be viewed as an increasing function of  $k$  and  $a$ . General results related to chains of partition ideals would be of interest.

We close with some open problems for which even partial answers would be important.

1. For a given partition ideal  $C$  of order  $k$  determine an asymptotic formula for  $p(C; n)$  that depends only on some parameters related to  $C$ , for example the numbers

$$d_{r,s} = \sup_{\{f_i\} \in C} \{f_r + f_{r+1} + \cdots + f_{r+s}\},$$

where  $0 \leq s \leq k-1$ ,  $\infty > r \geq 1$ .

2. Suppose  $C$  is a partition ideal of order  $k$ . Suppose also there exist integers  $r$  and  $m$  such that for  $\{f_i\} \in C$  with  $f_i = 0$  for  $i < r$  then  $\{f_{i-m}\}$  and  $\{f_{i+m}\}$  are both in  $C$ . Does

$$f(x) = \sum_{\{f_i\} \in C} x^{\sum f_i} q^{\sum f_i^2}$$

satisfy a linear  $q$ -difference equation of finite order whose coefficients are polynomials in  $x$  and  $q$ ?

3. Can  $p(C_1, n) \neq p(C_2, n)$  for only finitely many  $n$ ?

Subbarao has posed the latter question for partition functions involving Euler pairs of various orders.

#### REFERENCES

The following references are intended to supplement the bibliographies of [2, 22, 28, and 89]. The reader is referred to [51, Chap. 3] for older references. Reference [82] is included for reference to work on higher dimensional partitions where a fairly complete bibliography of that subject is given; Ref. [67] provides a good introduction to the analytic number theory aspects of partitions.

1. R. P. AGARWAL, Certain basic hypergeometric identities associated with mock theta functions, *Quart. J. Math.* **20** (1969), 121–128.
2. H. L. ALDER, The nonexistence of certain identities in the theory of partitions and compositions, *Bull. Amer. Math. Soc.* **54** (1948), 712–722.
3. H. L. ALDER, Research problems, *Bull. Amer. Math. Soc.* **62** (1956), 76.
4. H. L. ALDER, Partition identities—from Euler to the present, *Amer. Math. Monthly* **76** (1969), 733–746.
5. H. L. ALDER, Proof of Andrews' conjecture on partition identities, *Proc. Amer. Math. Soc.* **22** (1969), 688–689.

6. G. E. ANDREWS, A simple proof of Jacobi's triple product identity, *Proc. Amer. Math. Soc.* **16** (1965), 333–334.
7. G. E. ANDREWS, A generalization of the Göllnitz–Gordon partition theorems, *Proc. Amer. Math. Soc.* **18** (1967), 945–952.
8. G. E. ANDREWS, Partition theorems related to the Rogers–Ramanujan identities, *J. Combinatorial Theory* **2** (1967), 422–430.
9. G. E. ANDREWS, Some new partition theorems, *J. Combinatorial Theory* **2** (1967), 431–436.
10. G. E. ANDREWS, A generalization of a partition theorem of MacMahon, *J. Combinatorial Theory* **2** (1967), 100–101.
11. G. E. ANDREWS, On Schur's second partition theorem, *Glasgow Math. J.* **8** (1967), 127–132.
12. G. E. ANDREWS, On basic hypergeometric series, mock-theta functions, and partitions I, *Quart. J. Math.* **17** (1966), 64–80.
13. G. E. ANDREWS, On basic hypergeometric series, mock-theta functions, and partitions II, *Quart. J. Math.* **17** (1966), 132–143.
14. G. E. ANDREWS,  $q$ -Identities of Auluck, Carlitz, and Rogers, *Duke Math. J.* **33** (1966), 575–582.
15. G. E. ANDREWS, On partition functions related to Schur's second partition theorem, *Proc. Amer. Math. Soc.* **19** (1968), 441–444.
16. G. E. ANDREWS, Some formulae for the Fibonacci sequence with generalizations, *Fibonacci Quart.* **7** (1969), 113–130.
17. G. E. ANDREWS, A general theorem on partitions with difference conditions, *Amer. J. Math.* **91** (1969), 18–24.
18. G. E. ANDREWS, A new generalization of Schur's second partition theorem, *Acta Arithmetica* **14** (1968), 429–434.
19. G. E. ANDREWS, On a partition theorem of Göllnitz and related formulae, *J. Reine Angew. Math.* **236** (1969), 37–42.
20. G. E. ANDREWS, A generalization of the classical partition theorems, *Trans. Amer. Math. Soc.* **145** (1969), 205–221.
21. G. E. ANDREWS, On  $q$ -difference equations for certain well-poised basic hypergeometric series, *Quart. J. Math.* **19** (1968), 433–447.
22. G. E. ANDREWS, A polynomial identity which implies the Rogers–Ramanujan identities, *Scripta Math.* **23** (1970), 297–305.
23. G. E. ANDREWS, Two theorems of Euler and a general partition theorem, *Proc. Amer. Math. Soc.* **20** (1969), 499–502.
24. G. E. ANDREWS, Some new partition theorems II, *J. Combinatorial Theory* **7** (1969), 262–263.
25. G. E. ANDREWS, An extension of Schur's second partition theorem, *Notices Amer. Math. Soc.* **17** (1970), 300.
26. G. E. ANDREWS, A new property of partitions with applications to the Rogers–Ramanujan identities, *J. Combinatorial Theory*, **10** (1971), 266–270.
27. G. E. ANDREWS, Note on a partition theorem, *Glasgow Math. J.* **11** (1970), 108–109.
28. G. E. ANDREWS, On a calculus of partition functions, *Pacific J. Math.* **31** (1969), 555–562.
29. G. E. ANDREWS, On a partition problem of J. J. Sylvester, *J. London Math. Soc.* **2** (1970), 571–576.

30. G. E. ANDREWS, On a transformation of bilateral series with applications, *Proc. Amer. Math. Soc.* **25** (1970), 554–558.
31. G. E. ANDREWS, On the Rogers–Ramanujan identities and partial  $q$ -difference equations, *Illinois J. Math.*, to appear.
32. G. E. ANDREWS, The use of computers in search of identities of the Rogers–Ramanujan type, *Computers in Number Theory*, Edited by A. O. L. Atkin and B. J. Birch, Academic Press (1971), 377–387.
33. G. E. ANDREWS, On a partition problem of H. L. Alder, *Pacific J. Math.* **36** (1971), 279–284.
34. G. E. ANDREWS, Lambert series, false theta functions, and partitions, *Rocky Mountain J. Math.*, **2** (1972), 49–56.
35. G. E. ANDREWS, On a theorem of Schur and Gleissberg, *Archiv Math.* (Basel), **22** (1971), 165–167.
36. G. E. ANDREWS, On identities implying the Rogers–Ramanujan identities, to appear.
37. G. E. ANDREWS, Two theorems of Gauss and allied identities proved arithmetically, *Pacific J. Math.*, to appear.
38. G. E. ANDREWS, Sieves for theorems of Euler, Rogers, and Ramanujan, *Proceedings of the Kalamazoo Conferences on Arithmetic Functions*, Springer-Verlag, Lecture Notes in Math., No. 251, 1–20.
39. G. E. ANDREWS, M. V. SUBBARAO, AND M. VIDYASAGAR, A family of combinatorial identities, *Canad. Math. Bull.* **15** (1972), 11–18.
40. A. O. L. ATKIN, A note on ranks and conjugacy of partitions, *Quart. J. Math.* **17** (1966), 335–338.
41. F. C. AULUCK, On some new types of partitions associated with generalized Ferrers graphs, *Proc. Cambridge Phil. Soc.* **47** (1951), 679–686.
42. D. P. BANERJEE, On some identities in theory of partition, *Proc. Nat. Acad. Sci. India Sect. A* **34** (1964), 68–73.
43. E. A. BENDER AND J. R. GOLDMAN, The enumerative uses of generating functions, *Indiana Math. J.* **20** (1971), 753–765.
44. G. BIRKHOFF, Lattice Theory, Amer. Math. Soc. Colloq. Publ., No. 25, 1948, revised 1967.
45. G. R. BLAKLEY, Combinatorial remarks on partitions of a multipartite number, *Duke Math. J.* **31** (1964), 335–340.
46. G. R. BLAKLEY, Algebra of formal power series, *Duke Math. J.* **31** (1964), 341–346.
47. G. R. BLAKLEY, Formal solution of nonlinear simultaneous equations: reversion of series in several variables, *Duke Math. J.* **31** (1964), 347–358.
48. L. CARLITZ, Some polynomials related to theta functions, *Ann. Math. Pura Appl.* **41** (1956), 359–373.
49. L. CARLITZ, A note on the Rogers–Ramanujan identities, *Duke Math. J.* **35** (1968), 839–842.
50. M. P. CHEN, A generalization of a partition theorem of Andrews, *Studies and Essays Presented to Yu-Why Chen on His Sixtieth Birthday*, Math. Res. Center, National Taiwan University, Taipei, 1970.
51. L. E. DICKSON, “History of the Theory of Numbers,” Vol. 2, Chelsea, New York, 1952.
52. P. DIENES, “The Taylor Series,” Dover, New York, 1957.
53. F. J. DYSON, A new symmetry of partitions, *J. Combinatorial Theory* **7** (1969), 56–61.
54. L. EULER, *Introductio in Analysin Infinitorum*, Lausanne, 1748.



55. N. J. FINE, Some new results on partitions, *Proc. Nat. Acad. Sci.* **34** (1948), 616–618.
56. N. J. FINE, Some Basic Hypergeometric Series and Applications, to appear.
57. J. GOLDMAN AND G.-C. ROTA, The number of subspaces of a vector space, in “Recent Progress in Combinatorics,” (W. Tutte, Ed.), pp. 75–83, Academic Press, New York, 1969.
58. J. GOLDMAN AND G.-C. ROTA, On the foundations of combinatorial theory IV: finite vector spaces and Eulerian generating functions, *Studies Appl. Math.* **49** (1970), 239–258.
59. H. GÖLLNITZ, Einfache Partitionen, (unpublished), Diplomarbeit W. S. (1960), Göttingen.
60. H. GÖLLNITZ, Partitionen mit Differenzenbedingungen, *J. Reine Angew. Math.* **225** (1965), 154–190.
61. B. GORDON, A combinatorial generalization of the Rogers–Ramanujan identities, *Amer. J. Math.* **83** (1961), 393–399.
62. B. GORDON, Some continued fractions of the Rogers–Ramanujan type, *Duke Math. J.* **31** (1965), 741–748.
63. R. GUY, Two theorems on partitions, *Math. Gaz.* **42** (1958), 84–86.
64. R. R. HALL, Three identities connected with the numbers 5 and 7, *Quart. J. Math.* **18** (1967), 113–117.
65. G. H. HARDY, “Ramanujan,” Cambridge University Press, Cambridge, 1940.
66. G. H. HARDY AND E. M. WRIGHT, “An Introduction to the Theory of Numbers,” 4th ed., Oxford University Press, Oxford, 1960.
67. M. I. KNOPP, “Modular Functions in Analytic Number Theory,” Markham, Chicago, 1970.
68. D. E. KNUTH, Subspaces, subsets, and partitions, *J. Combinatorial Th. (A)* **10** (1971), 178–180.
69. H. R. LATTMAN,  $q$ -Difference equation proofs for some identities of the Rogers–Ramanujan type, Masters paper, Pennsylvania State Univ., 1969.
70. D. H. LEHMER, Two nonexistence theorems on partitions, *Bull. Amer. Math. Soc.* **52** (1946), 538–544.
71. P. A. MACMAHON, “Combinatory Analysis,” Vol. 2, Cambridge University Press, Cambridge, 1916.
72. E. A. PARBERRY, On a partition function related to a theorem of Schur, Ph.D. dissertation, Pennsylvania State Univ., 1969.
73. L. J. ROGERS, Second memoir on the expansion of certain infinite products, *Proc. London Math. Soc.* **25** (1894), 318–343.
74. L. J. ROGERS, On two theorems of combinatory analysis and some allied identities, *Proc. London Math. Soc.* **16** (1917), 315–336.
75. L. J. ROGERS AND S. RAMANUJAN, Proof of certain identities in combinatory analysis, *Proc. Camb. Phil. Soc.* **19** (1919), 211–214.
76. D. E. RUTHERFORD, “Introduction to Lattice Theory,” Hafner, New York, 1965.
77. I. J. SCHUR, Ein Beitrag zur additiven Zahlentheorie, *Sitzungsber. Akad. Wissensch. Berlin Phys.-Math. Klasse* (1917), 302–321.
78. I. J. SCHUR, Zur additiven Zahlentheorie, *Sitzungsber. Akad. Wissensch. Berlin, Phys.-Math. Klasse* (1926), 488–495.
79. L. J. SLATER, A new proof of Rogers’s transformations of infinite series, *Proc. London Math. Soc.* **53** (1951), 460–475.

80. L. J. SLATER, Further identities of the Rogers–Ramanujan type, *Proc. London Math. Soc.* **54** (1952), 147–167.
81. L. J. SLATER, “Generalized Hypergeometric Functions,” Cambridge, Univ. Press, Cambridge, 1966.
82. R. P. STANLEY, Theory and application of plane partitions (I) and (II), *Studies in Appl. Math.* **50** (1971), 167–188, 259–279.
83. K. B. STOLARSKY, Higher partition functions and their relation to finitely generated nilpotent groups, Ph.D. dissertation, Univ. of Wisconsin, 1968.
84. K. B. STOLARSKY, A vanishing finite sum associated with Jacobi’s triple product identity, *J. Combinatorial Theory* **6** (1969), 392–398.
85. M. V. SUBBARAO, Partition theorems for Euler pairs, *Proc. Amer. Math. Soc.* **28** (1971), 330–336.
86. M. V. SUBBARAO, On a partition theorem of MacMahon–Andrews, *Proc. Amer. Math. Soc.* **27** (1971), 449–450.
87. G. N. WATSON, The mock theta functions II, *Proc. London Math. Soc.* **42** (1937), 272–304.
88. G. E. ANDREWS, Generalizations of the Durfee square, *J. London Math. Soc.* **3** (1971), 563–570.
89. H. GUPTA, Partitions—a survey, *J. Res. Nat. Bur. Stand. B* **74** (1970), 1–29.