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On the Foundations of Combinatorial Theory V, Eulerian Differential Operators

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1. Introduction

In [18], Rota and Mullin develop a theory of binomial enumeration by making an extensive study of polynomials of binomial type, that is sequences $p_0(X)$, $p_1(X)$, $p_2(X), \ldots$ where $p_n(X)$ is of degree n, and

$$p_n(X + Y) = \sum_{j \geq 0} \binom{n}{j} p_j(X) p_{n-j}(Y). \tag{1.1}$$

As they remark early in their paper, such sequences arise naturally in problems of enumeration. For example, if $p_n(X) = X(X - 1) \ldots (X - n + 1)$, then $p_n(X)$ enumerates the number of one-to-one mappings of a set of n elements into a set of X elements. In this instance, equation (1.1) is an obvious combinatorial assertion. Namely, $p_n(X + Y)$ is now the number of one-to-one mappings of a set of n

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elements into a set of $X + Y$ elements, while $\binom{n}{j} p_j(X) p_{n-j}(Y)$ is the number of such one-to-one mappings with exactly j elements mapped into the set of X elements.

In [12], p. 257, Rota and Goldman suggest the importance of a similar study for polynomials related to enumeration problems in finite vector spaces. Namely they suggest consideration of sequences of polynomials satisfying

$$b_n(X, Z) = \sum_{j \geq 0} \binom{n}{j}_q b_j(X, Y) b_{n-j}(Y, Z), \tag{1.2}$$

where $\binom{n}{j}_q$ is the Gaussian polynomial (see Section 2 for definition). They note, however, that such systems of polynomials are seemingly rare. Apparently only one example of such polynomials appears in the literature [12], p. 252, equation (1); however, as we shall see in Section 6, there are infinitely many systems satisfying (1.2) (see Theorem 7).

The object of this paper is to develop a theory for enumeration problems in finite vector spaces that is analogous to the theory Rota and Mullin [18] developed for finite sets.

In Sections 4 and 5, our theory very much parallels the work of Rota and Mullin; however in succeeding sections, the two theories are seen to go their separate ways. As it turns out the theory developed here has application not only to finite vector spaces (Section 11) but also to certain areas of classical analysis, for example, the Rogers–Ramanujan identities (Section 9).

2. Notation

The theory of basic hypergeometric series has always been plagued with a one-to-one correspondence between systems of notation and active researchers. The following table lists the most common notation.

Table 1

Notation for the Product $\prod_{j=0}^{n-1} (1 - aq^j)$

Author	Bailey	Fine	Jackson	Rota, Goldman	Slater	Watson
A work in which notation is used	[1]	[10]	[15]	[12]	[21]	[24]
Notation	$(a)_{q,n}$	$[n; aq^{-1}; q]$	$(1 - q)^n[\log_q a]_n$	$P_n(1, a)$	$(a; q)_n$	$\Pi_n(-a, q)$

Some other works in the subject use a great variety of symbols for special cases of $\prod(1 - aq^j)$ (see for example [2], p. 421).

For reasons that will become apparent as our work progresses, we shall use both the notation of Rota and Goldman as well as that of Slater. Thus the following symbols will be used throughout our work.

$$\begin{aligned} P_n(x, z) &= (x - z)(x - zq)\dots(x - zq^{n-1}) \\ &= x^n\left(1 - \frac{z}{x}\right)\left(1 - \frac{z}{x}q\right)\dots\left(1 - \frac{z}{x}q^{n-1}\right) \\ &= x^n\left(\frac{z}{x};q\right)_n = x^n\left(\frac{z}{x}\right)_n . \end{aligned}$$

For the Gaussian polynomial, we use the notation of Rota and Goldman [12], p. 240:

$$\binom{n}{m}_q = \begin{cases} \dfrac{(q)_n}{(q)_m(q)_{n-m}}, & \text{if } 0 \leq m \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

We also require some conventions concerning finite vector spaces. Upper case script Latin letters $\mathcal{N}, \mathcal{T}, \mathcal{U}, \mathcal{W}, \mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ will denote finite vector spaces; upper case Latin letters N, T, U, W, X, Y, and Z will, in this context, denote the number of elements of such spaces, and lower case Latin letters n, t, u, w, x, y, and z will, in this context, denote the dimensions of these spaces. Thus $\mathcal{X}$ is a vector space of dimension x over $GF(q)$ the finite field of q elements, and there are $X = q^x$ elements of $\mathcal{X}$.

Also every time we refer to a "map" or "mapping" we shall mean a one-to-one linear transformation of one finite vector space into another.

A comment should also be made concerning the names given to various operators, sequences, and series that arise in the course of our work. First we have decided against using "q-operator", "q-basic polynomial" etc., although q-terminology is quite extensive in the literature. Rather we shall use the adjective "Eulerian" paying tribute to the first worker in q-series [3], p. 47. However this requires that most things be given three-word names; this is necessary since terms such as "Eulerian operators", "Eulerian numbers", "Eulerian polynomials", already have been used to describe constructs much different from those in this paper (see [7], [8], and [9]).

3. Transformations of a finite vector space

Goldman and Rota [12], p. 252, have shown that $P_n(X, Z)$ is the number of one-to-one linear transformations f from $\mathcal{N}$ (an n-dimensional vector space over $GF(q)$, the finite field of q elements) into $\mathcal{X}$ such that $f(\mathcal{N}) \cap \mathcal{Z} = \{0\}$ where $\mathcal{Z}$ is a subspace of $\mathcal{X}$. They have also shown that $P_n(X, Z)$ satisfies equation (1.2).

We propose to prove an equivalent form of (1.2) for the one variable polynomials $P_n(X, 1)$. Let us count the number of one-to-one linear transformations of $\mathcal{N}$ into $\mathcal{X} \oplus \mathcal{Y}$. Since $\mathcal{X} \oplus \mathcal{Y}$ has XY elements, there are clearly $P_n(XY, 1)$ such mappings. On the other hand, since every element of $\mathcal{X} \oplus \mathcal{Y}$ is of the form $\alpha + \beta$ where $\alpha \in \mathcal{X}$ and $\beta \in \mathcal{Y}$ (α is called the $\mathcal{X}$-component and β the $\mathcal{Y}$-component), let us look at mappings for which those elements of the image with 0 as $\mathcal{X}$-component form a j-dimensional subspace of $\mathscr{X} \oplus \mathscr{Y}$. To form such maps we may choose a j-dimensional subspace of $\mathscr{N}$ in $\binom{n}{j}_q$ ways (see [20], p. 139, [11], [12], [17]) and then map it into $\mathscr{Y}$ in $P_j(Y, 1)$ ways. Since a linear transformation is completely determined by the action on a basis, we choose a basis of $\mathscr{N}$, say $b_1, b_2, \ldots, b_n$ such that $b_1, \ldots, b_j$ is a basis of the above mentioned j-dimensional subspace. I claim now that $f(b_{j+1}), \ldots, f(b_n)$ need only be chosen so that their $\mathscr{X}$-components are linearly independent in $\mathscr{X}$. This follows from the fact that if $f(b_i) = \alpha_i + \beta_i$ and there exist $C_i \in GF(q)$ not all zero such that

$$\sum_{i=j+1}^{n} C_i\alpha_i = 0,$$

then

$$\sum_{i=j+1}^{n} C_i f(b_i) = \sum_{i=j+1}^{n} C_i(\alpha_i + \beta_i)$$
$$= \sum_{i=j+1}^{n} C_i\beta_i.$$

Thus $\sum_{i=j+1}^{n} C_i f(b_i)$ has 0 as $\mathscr{X}$-component and so is in the space spanned by $f(b_1), \ldots, f(b_j)$. Hence there exist $C_1, \ldots, C_j$ in $GF(q)$ such that

$$-\sum_{i=j+1}^{n} C_i f(b_i) = \sum_{i=1}^{j} C_i f(b_i).$$

Therefore

$$f\left(\sum_{i=1}^{n} C_i b_i\right) = 0,$$

and since f is a one-to-one linear transformation

$$\sum_{i=1}^{n} C_i b_i = 0$$

which is impossible since not all the C_i are zero and the b_i form a basis for $\mathscr{N}$. Conversely if the $\mathscr{X}$-components of $f(b_{j+1}), \ldots, f(b_n)$ are linearly independent, then $f(b_1), \ldots, f(b_n)$ span an n-dimensional subspace of $\mathscr{X} \oplus \mathscr{Y}$ with a j-dimensional subspace having 0 as $\mathscr{X}$-component. Thus there are $P_{n-j}(X, 1)$ ways of choosing the $\mathscr{X}$-components of $f(b_{j+1}), \ldots, f(b_n)$ and Y^{n-j} ways of choosing the $\mathscr{Y}$-components.

Consequently the total number of one-to-one linear transformations of $\mathscr{N}$ into $\mathscr{X} \oplus \mathscr{Y}$ with j-dimensional image having 0 as $\mathscr{X}$-component is

$$\binom{n}{j}_q P_j(Y, 1)Y^{n-j}P_{n-j}(X, 1).$$

Hence summing over all j, we see that

$$P_n(XY, 1) = \sum_{j=0}^{n} \binom{n}{j}_q P_j(Y, 1)Y^{n-j}P_{n-j}(X, 1),$$

and replacing j by $n - j$, we obtain

$$P_n(XY, 1) = \sum_{j \geq 0} \binom{n}{j}_q P_j(X, 1) Y^j P_{n-j}(Y, 1). \tag{3.1}$$

We remark that (3.1) is equivalent to the q-binomial theorem of Goldman and Rota [12], p. 252, equation (1), by the substitutions $x \to XY$, $y \to Y$, $z \to 1$ (to reverse $X \to x/y$, $Y \to y/z$, then multiply (3.1) by z^n). However, our derivation here closely parallels the derivation of (1.1) in the special case $p_n(x) = x(x-1)\ldots(x-n+1)$. Thus we are led to the following definition:

DEFINITION 1. We say that $p_0(X)$, $p_1(X)$, $p_2(X), \ldots$ is an *Eulerian family* of polynomials if

(i) $p_0(X) = 1$,
(ii) $p_n(X)$ is of degree n,
(iii) for each n,

$$p_n(XY) = \sum_{j \geq 0} \binom{n}{j}_q p_j(X) Y^j p_{n-j}(Y).$$

This definition is analogous to the definition of Rota and Mullin for polynomials of binomial type [18], p. 169.

The next section is devoted to constructing a theory of operators analogous to the delta operators of Rota and Mullin [18], p. 180, and the differential operators of Berge [4], p. 73.

4. Eulerian differential operators

The role of the shift operators of [18], p. 179, used in binomial enumeration is now played by the *Eulerian shift operator*:

$$\eta^a p(X) = p(Xq^a) = p(XA),$$

where $A = q^a$.

To be consistent with our notation for finite vector spaces, we shall write $X = q^x$, $Y = q^y$, $A = q^a$, $B = q^b$, and so on; this notational convention allows us to exhibit symmetries that might otherwise be hidden. Our polynomials will all lie in algebra (over the real numbers $\mathbf{R}$) of all polynomials of one variable $X = q^x$, to be denoted by $\mathbf{P}$.

Convergence questions here are trivial and will largely be ignored. Generally we shall treat series as formal power series, and q will denote a prime power; however, the results obtained in Section 9 may be treated as analytic results valid for $|q| < 1$, or as q-adic results valid for q prime.

DEFINITION 2. An *Eulerian differential operator* τ is a linear operator on P that satisfies the following conditions:

$$q^{-a} \tau \eta^a = \eta^a \tau, \tag{4.1}$$

and

$$\tau X^n \not\equiv 0 \quad \text{for each} \quad n > 0. \tag{4.2}$$

The most well-known Eulerian differential operator is the q-differentiation[1] operator D_q:

$$D_q = \frac{1}{X}(1 - \eta).$$

Note that

$$\begin{aligned} D_q P_n(X, 1) &= X^{-1}\{(X - 1)(X - q)\dots(X - q^{n-1}) - (Xq - 1)(Xq - q)\dots \\ &\quad \times (Xq - q^{n-1})\} \\ &= X^{-1}P_{n-1}(X, 1)\{X - q^{n-1} - q^{n-1}(Xq - 1)\} \\ &= (1 - q^n)P_{n-1}(X, 1). \end{aligned}$$

LEMMA 1. *If τ is an Eulerian differential operator, then $\tau C = 0$ for each constant C.*

Proof: Since $q^{-a}\tau\eta^a = \eta^a\tau$, we see that $q^{-a}\tau C = \eta^a\tau C$. Let $\tau C = r(X) \in \mathbf{P}$. Then we have

$$q^{-a}r(X) = r(Xq^a).$$

If $r(X) \equiv 0$, this identity is obvious. If $r(X) \not\equiv 0$, let d be the leading coefficient of r and n the degree. Hence comparing coefficients of X^n in the above identity, we find that

$$q^{-a}d = q^{an}d.$$

But since $d \neq 0$ and $n \geq 0$, this equation is impossible. Hence $\tau C = 0$.

LEMMA 2. *If τ is an Eulerian differential operator, and $p(X)$ is any polynomial of degree n, then $\tau p(X)$ is of degree $n - 1$.*

Proof: By (4.1), for each n

$$q^{-a}\tau\eta^a X^n = \eta^a\tau X^n.$$

Hence

$$q^{(n-1)a}\tau X^n = \eta^a\tau X^n.$$

Suppose $\tau X^n = r(X) = eX^j + \dots$; then

$$q^{(n-1)a}r(X) = r(Xq^a).$$

Comparing coefficients of X^j in this equation, we see that

$$q^{(n-1)a}e = q^{aj}e.$$

Since $e \neq 0$ by (4.2), we see that $j = n - 1$. By linearity $\tau p(X)$ is of degree $n - 1$.

DEFINITION 3. Let τ be an Eulerian differential operator. A sequence of polynomials $p_0(X), p_1(X), p_2(X), \dots$ is called the sequence of *Eulerian basic polynomials* for τ if:

(i) $p_0(X) = 1$,
(ii) $p_n(1) = 0$, for each $n > 0$,
(iii) $\tau p_n(X) = (1 - q^n)p_{n-1}(X)$.

[1] F. H. Jackson [14] who introduced q-differentiation actually used $(1 - q)^{-1}D_q$; the operator δ of L. J. Rogers [19] is D_q.

It is clear by mathematical induction that each $p_n(X)$ is of degree n. By Lemma 2 it is clear that we can construct a unique sequence of Eulerian basic polynomials for each τ.

THEOREM 1

(a) *If $p_n(X)$ is an Eulerian basic sequence for some Eulerian differential operator, then it is an Eulerian family of polynomials.*

(b) *If $p_n(X)$ is an Eulerian family of polynomials, then it is an Eulerian basic sequence for some Eulerian differential operator.*

Proof: (a) Iterating property (iii) of Eulerian basic polynomials, we see that

$$\tau^k p_n(X) = (q^{n-k+1})_k p_{n-k}(X),$$

and hence by property (ii)

$$[\tau^k p_n(X)]_{X=1} = \begin{cases} (q)_n, & \text{if } k = n, \\ 0, & \text{if } k < n. \end{cases}$$

Thus

$$p_n(X) = \sum_{k \geq 0} \frac{p_k(X)}{(q)_k} [\tau^k p_n(X)]_{X=1}.$$

By linearity, we see that for each $p(X) \in \mathbf{P}$,

$$p(X) = \sum_{k \geq 0} \frac{p_k(X)}{(q)_k} [\tau^k p(X)]_{X=1}.$$

Now suppose $p(X)$ is the polynomial $p_n(XY)$. Thus

$$p_n(XY) = \sum_{k \geq 0} \frac{p_k(X)}{(q)_k} [\tau^k p_n(XY)]_{X=1}$$

But since $Y = q^y$,

$$\begin{aligned} [\tau^k p_n(XY)]_{X=1} &= [\tau^k \eta^y p_n(X)]_{X=1} \\ &= [q^{ky} \eta^y \tau^k p_n(X)]_{X=1} \\ &= [q^{ky} \eta^y (q^{n-k+1})_k p_{n-k}(X)]_{X=1} \\ &= [Y^k (q^{n-k+1})_k p_{n-k}(XY)]_{X=1} \\ &= Y^k (q^{n-k+1})_k p_{n-k}(Y). \end{aligned}$$

Hence

$$\begin{aligned} p_n(XY) &= \sum_{k \geq 0} \frac{(q^{n-k-1})_k}{(q)_k} p_k(X) Y^k p_{n-k}(Y) \\ &= \sum_{k \geq 0} \binom{n}{k}_q p_k(X) Y^k p_{n-k}(Y). \end{aligned}$$

(b) Conversely suppose $p_n(X)$ is an Eulerian family of polynomials. Putting $Y = 1$ in equation (iii) of Definition 1, we see that

$$p_n(X) = \sum_{j \geq 0} \binom{n}{j}_q p_j(X) p_{n-j}(1).$$

Since this identity is valid for each $n \geq 0$ and since each $p_n(X)$ is of degree n, we see that $p_0(1) = 1$ and that $p_n(1) = 0$ for each $n > 0$. Since $p_0(X)$ is a constant, $p_0(X) = 1$. Thus properties (i) and (ii) of Definition 3 are fulfilled.

Let us now define a linear operator τ on $\mathbf{P}$ by

$$\tau p_0(X) = 0,$$
$$\tau p_n(X) = (1 - q^n)p_{n-1}(X), \qquad \text{for each } n \geq 1.$$

We need only verify that $q^{-y}\tau\eta^y = \eta^y\tau$, where $Y = q^y$.

Clearly if we replace j by $n - k$ in property (iii) of Definition 1, we see that

$$p_n(XY) = \sum_{k \geq 0} \frac{p_k(Y)}{(q)_k} Y^{n-k}\tau^k p_n(X). \tag{4.3}$$

Operating on both sides of (4.3) with τ, we first find that

$$\tau(p_n(XY)) = \tau\eta^y p_n(X),$$

while on the right hand side we have

$$\begin{aligned}
&\sum_{k \geq 0} \frac{p_k(Y)}{(q)_k} Y^{n-k}\tau^{k+1} p_n(X) \\
&\quad = \sum_{k \geq 0} \frac{p_k(Y)}{(q)_k}(1 - q^n)\ldots(1 - q^{n-k})Y^{n-k}p_{n-k-1}(X) \\
&\quad = Y(1 - q^n) \sum_{k \geq 0} \begin{pmatrix} n-1 \\ k \end{pmatrix}_q p_k(Y)Y^{n-1-k}p_{n-1-k}(X) \\
&\quad = Y(1 - q^n)p_{n-1}(XY) \\
&\quad = q^y\eta^y\tau p_n(X).
\end{aligned}$$

Since the $p_n(X)$ form a basis for $\mathbf{P}$, we see by linearity that

$$\tau\eta^y = q^y\eta^y\tau$$

which is equivalent to (4.1)

5. Expansion theorems

DEFINITION 4. If σ is a linear operator on P, we shall say that σ is an *Eulerian shift-invariant operator* if:

$$\sigma\eta^y = \eta^y\sigma$$

for all Y (recall $Y = q^y$).

THEOREM 2. (Eulerian expansion theorem). *Let σ be an Eulerian shift-invariant operator, and let τ be an Eulerian differential operator with associated Eulerian family $p_n(X)$. Then*

$$\sigma = \sum_{k \geq 0} \frac{a_k}{(q)_k} X^k\tau^k,$$

where $a_k = [\sigma p_k(X)]_{X=1}$.

Proof: Since the $p_n(X)$ form the Eulerian family associated with τ, we rewrite property (iii) of Definition 1 as

$$p_n(XY) = \sum_{k \geq 0} \frac{p_k(Y)}{(q)_k} X^k \tau^k p_n(X).$$

We now apply σ as an operator on polynomials in Y to the above equation. Thus

$$\sigma \eta^x p_n(Y) = \sum_{k \geq 0} \frac{\sigma p_k(Y)}{(q)_k} X^k \tau^k p_n(X).$$

By linearity, we can extend this identity to all elements of **P**. Thus we see that

$$\sigma \eta^x p(Y) = \sum_{k \geq 0} \frac{\sigma p_k(Y)}{(q)_k} X^k \tau^k p(X).$$

Since $\sigma \eta^x = \eta^x \sigma$, we see that

$$\eta^x \sigma p(Y) = \sum_{k \geq 0} \frac{\sigma p_k(Y)}{(q)_k} X^k \tau^k p(X).$$

Consequently

$$(\sigma p)(XY) = \sum_{k \geq 0} \frac{\sigma p_k(Y)}{(q)_k} X^k \tau^k p(X).$$

Setting $Y = 1$, we obtain Theorem 2.

So far our theory is a perfect q-analog of the results of Rota and Mullin [18]. However, as is well-known in classical analysis, the q-analogs of ordinary hypergeometric series are not a mirror image of the ordinary theory. The following result exhibits the beginning of the divergence of these theories as the ring structure Rota and Mullin obtained for their operators is replaced in the analog by an additive group structure.

THEOREM 3. *Let τ be an Eulerian differential operator, and let E be the* additive group *of formal Eulerian series over* **R**. *Then there exists an isomorphism from E onto the* additive group *T of Eulerian shift invariant operators which carries*

$$f(t) = \sum_{k \geq 0} \frac{a_k t^k}{(q)_k} \quad \text{into} \quad \sum_{k \geq 0} \frac{a_k X^k \tau^k}{(q)_k}.$$

Proof: The mapping is already linear, and by Theorem 2 it is onto.

We remark that the factor X^k is what prohibits our obtaining the ring structure of formal Eulerian series. At this point we find that the corollaries that Rota and Mullin [18; p. 189, Cor. 1 and 2] easily derived from their strong Theorem 3 are not corollaries of our Theorem 3.

We shall now prove a strengthened form of Lemma 2 that will be important in future developments.

THEOREM 4. *Let τ be an Eulerian differential operator, then there exist constants $e_0 = 0, e_1, e_2, \ldots$ where $e_n \neq 0$ for each $n > 0$ such that*

$$\tau X^n = e_n X^{n-1}.$$

Conversely for any sequence of constants $e_0 = 0, e_1, e_2, \ldots$ where $e_n \neq 0$ for each $n > 0$, the linear operator τ on **P** defined by $\tau X^n = e_n X^{n-1}$ is an Eulerian differential operator.

Proof: First we assume τ is an Eulerian differential operator. Let $\tau X^n = s_n(X)$. Now

$Ys_n(XY) = Y\eta^y s_n(X) = Y\eta^y \tau X^n = \tau\eta^y X^n = Y^n \tau X^n = Y^n s_n(X)$. Setting $X = 1$, we see that

$$s_n(Y) = s_n(1)Y^{n-1}.$$

Since $\tau 1 \equiv 0$, and $\tau X^n \not\equiv 0$ for each $n > 0$, we see that $s_0(1) = 0$ and $s_n(1) \neq 0$ for each $n > 0$. The first half of the theorem now follows with $e_n = s_n(1)$.

Conversely we consider τ defined by $\tau X^n = e_n X^{n-1}$. By linearity, we see that τ is well-defined on **P**. Furthermore

$$Y\eta^y \tau X^n = Y\eta^y e_n X^{n-1} = Y e_n Y^{n-1} X^{n-1} = e_n Y^n X^{n-1} = \tau\eta^y X^n,$$

and since the X^n form a basis for **P**, we see that in general $\eta^y \tau = Y^{-1}\tau\eta^y$. Thus since also $\tau 1 \equiv 0$ and $\tau X^n \not\equiv 0$ for each $n > 0$, we see that τ is an Eulerian differential operator.

COROLLARY. *Let τ be an Eulerian differential operator with related Eulerian family of polynomials $p_n(X)$. Let C_n be the leading coefficient of $p_n(X)$, and let e_n be defined by $\tau X^n = e_n X^{n-1}$. Then*

$$e_n = \frac{(1 - q^n)C_{n-1}}{C_n}, \qquad e_0 = 0.$$

Proof: By Theorem 4 we know that $e_0 = 0$. Now

$$\begin{aligned} C_n e_n X^{n-1} + \cdots &= \tau p_n(X) \\ &= (1 - q^n)p_{n-1}(X) \\ &= (1 - q^n)C_{n-1}X^{n-1} + \cdots. \end{aligned}$$

Comparing coefficients of X^{n-1}, we obtain the desired result.

Theorem 4 and its corollary give us much information about Eulerian differential operators and Eulerian families. To obtain further information (especially an analog to Corollary 2 of Theorem 3 in [18]), we move to a full-fledged study of the relevant generating functions.

6. Generating functions

DEFINITION 5. We say that $p_0(X, Z)$, $p_1(X, Z)$, $p_2(X, Z), \ldots$ is a *homogeneous Eulerian family* of polynomials if each $p_n(X, Z)$ is a homogeneous polynomial of degree n in X and Z such that

(i) $$p_0(X, Z) = 1,$$

(ii) $$p_n(X, 0) \not\equiv 0,$$

(iii) $$p_n(X, Z) = \sum_{j \geq 0} \binom{n}{j}_q p_j(X, Y)p_{n-j}(Y, Z).$$

The relationship between homogeneous Eulerian families and ordinary Eulerian families is explicitly described in the following theorem.

THEOREM 5. *There is a one-to-one correspondence ϕ between homogeneous Eulerian families and ordinary Eulerian families given by*

$$\phi : p_n(X, Z) \to p_n(X, 1),$$
$$\phi^{-1} : p_n(X) \to Z^n p_n(X/Z) = p_n(X, Z).$$

Proof: Suppose the $p_n(X, Z)$ form a homogeneous Eulerian family. Then $p_n(X, 0) = CX^n \not\equiv 0$ by property (ii) of Definition 5. Hence $p_n(X, 1)$ is of degree n. By property (i) of Definition 5, $p_0(X, 1) = 1$. Next by homogeneity and property (iii) of Definition 5, we see that

$$\begin{aligned} p_n(XY, 1) &= \sum_{j \geq 0} \binom{n}{j}_q p_j(XY, Y) p_{n-j}(Y, 1) \\ &= \sum_{j \geq 0} \binom{n}{j}_q p_j(X, 1) Y^j p_{n-j}(Y, 1), \end{aligned}$$

which shows that the $p_n(X, 1)$ form an Eulerian family.

Conversely suppose that the $p_n(X)$ form an Eulerian family. Let

$$p_n(X) = \sum_{j \geq 0}^{n} C_{nj} X^j,$$

where $C_{nn} \neq 0$; then

$$\begin{aligned} p_n(X, Z) &= Z^n p_n(X/Z) \\ &= \sum_{j=0}^{n} C_{nj} X^j Z^{n-j}. \end{aligned}$$

Clearly $p_0(X, Z) = 1$, and

$$p_n(X, 0) = C_{nn} X^n \not\equiv 0.$$

Finally

$$\begin{aligned} p_n(X, Z) = Z^n p_n(X/Z) &= Z^n p_n\left(\frac{X}{Y} \cdot \frac{Y}{Z}\right) \\ &= Z^n \sum_{j \geq 0} \binom{n}{j}_q p_j\left(\frac{X}{Y}\right) \left(\frac{Y}{Z}\right)^j p_{n-j}\left(\frac{Y}{Z}\right) \\ &= \sum_{j \geq 0} \binom{n}{j}_q Y^j p_j\left(\frac{X}{Y}\right) Z^{n-j} p_{n-j}\left(\frac{Y}{Z}\right) \\ &= \sum_{j \geq 0} \binom{n}{j}_q p_j(X, Y) p_{n-j}(Y, Z). \end{aligned}$$

Noting that $\phi^{-1}\phi p_n(X, Z) = \phi^{-1} p_n(X, 1) = Z^n p_n(X/Z, 1) = p_n(X, Z)$, and $\phi\phi^{-1} p_n(X) = \phi Z^n p_n(X/Z) = p_n(X)$, we see that Theorem 5 is established.

The value of Theorem 5 lies in the fact that we may easily determine the form of the generating functions for the homogeneous Eulerian families (see Theorem 6).

We then set $Z = 1$ to determine the generating function for ordinary Eulerian families.

THEOREM 6. *Let $p_n(X, Z)$ be a homogeneous Eulerian family with $C_n = p_n(1, 0) \neq 0$, and let*

$$f(t) = \sum_{k \geq 0} \frac{C_k t^k}{(q)_k}.$$

Then

$$\sum_{n \geq 0} \frac{p_n(X, Z) t^n}{(q)_n} = \frac{f(Xt)}{f(Zt)}.$$

Proof: Let

$$F(X, Z; t) = \sum_{n \geq 0} \frac{p_n(X, Z)}{(q)_n} t^n$$

Then

$$\begin{aligned} F(X, Z; t) &= \sum_{n=0}^{\infty} \frac{t^n}{(q)_n} \sum_{\substack{j+k=n \\ j \geq 0, k \geq 0}} \frac{(q)_n}{(q)_j (q)_k} p_j(X, Y) p_k(Y, Z) \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{p_j(X, Y) t^j}{(q)_j} \cdot \frac{p_k(X, y) t^k}{(q)_k} \\ &= F(X, Y; t) F(Y, Z; t). \end{aligned}$$

Replace Z by 0 and then replace Y by Z; this yields

$$F(X, 0; t) = F(X, Z; t) F(Z, 0; t).$$

Hence

$$\begin{aligned} \sum_{n \geq 0} \frac{p_n(X, Z) t^n}{(q)_n} &= F(X, Z; t) \\ &= \frac{F(X, 0; t)}{F(Z, 0; t)} \\ &= \frac{\sum_{n \geq 0} \frac{C_n X^n t^n}{(q)_n}}{\sum_{n \geq 0} \frac{C_n Z^n t^n}{(q)_n}} \\ &= \frac{f(Xt)}{f(Zt)}. \end{aligned}$$

COROLLARY. *If $p_n(X)$ is an Eulerian family of polynomials, and if C_n is the leading coefficient of $p_n(X)$, then*

$$\sum_{n \geq 0} \frac{p_n(X) t^n}{(q)_n} = \frac{f(Xt)}{f(t)},$$

where

$$f(t) = \sum_{n \geq 0} \frac{C_n t^n}{(q)_n}.$$

Proof: In Theorem 6, let $p_n(X, Z) = Z^n p_n(X/Z)$. Then set $Z = 1$.

THEOREM 7. *If $C_0 = 1, C_1, C_2, \ldots$ is any sequence of non-zero real numbers, and if*

$$f(t) = \sum_{k \geq 0} \frac{C_k t^k}{(q)_k},$$

then the expressions $p_n(X, Z)$ defined by

$$\sum_{n \geq 0} \frac{p_n(X, Z) t^n}{(q)_n} = \frac{f(Xt)}{f(Zt)}$$

form a homogeneous Eulerian family of polynomials.

Proof: The argument required here merely reverses the steps in Theorem 6 so we omit it.

Finally we derive a further result for the generating functions which greatly resembles Corollary 2 of Theorem 3 in [18].

THEOREM 8. *Let the $p_n(X)$ form an Eulerian family of polynomials with C_n as the leading coefficient and let*

$$f(t) = \sum_{n \geq 0} \frac{C_n t^n}{(q)_n}.$$

Then

$$f(t) = \exp\left\{ \sum_{n \geq 1} \frac{p_n'(1) t^n}{(q)_n n} \right\}.$$

Proof: By Theorem 6, if

$$F(X, Z; t) = \frac{f(Xt)}{f(Zt)},$$

then

$$F(X, Z; t) = F(X, Y; t) F(Y, Z; t)$$

Therefore replacing X by XY, then Y by X and Z by 1, we see that

$$\begin{aligned} F(XY, 1; t) &= F(XY, X; t) F(X, 1; t) \\ &= F(Y, 1; Xt) F(X, 1; t). \end{aligned}$$

Hence

$$\begin{aligned} &\frac{F(XY, 1; t) - F(X, 1; t)}{Y - 1} \\ &\quad = F(X, 1; t) \left\{ \frac{F(Y, 1; Xt) - 1}{Y - 1} \right\} \\ &\quad = F(X, 1; t) \sum_{n \geq 1} \frac{p_n(Y)}{Y - 1} \frac{X^n t^n}{(q)_n}. \end{aligned}$$

Letting $Y \to 1$ (or $y \to \infty$ where $Y = q^y$), we obtain

$$X\frac{d}{dX}F(X, 1; t) = F(X, 1; t)\sum_{n\geq 1}\frac{p_n'(1)X^n t^n}{(q)_n};$$

this follows from the fact that, by Theorem 1. $p_n(1) = 0$ for each $n > 0$.

Thus $F(X, 1; t)$ satisfies a first order differential equation in X, namely

$$Xy' - y\sum_{n\geq 1}\frac{p_n'(1)X^n t^n}{(q)_n} = 0.$$

The solutions of this equation are of the form

$$y = K(t)\exp\left\{\sum_{n\geq 1}\frac{p_n'(1)X^n t^n}{(q)_n n}\right\}$$

Since $F(0, 1; t) = (f(t))^{-1} = K(t)$, we see that

$$\frac{f(Xt)}{f(t)} = F(X, 1; t) = (f(t))^{-1}\exp\left\{\sum_{n\geq 1}\frac{p_n'(1)X^n t^n}{(q)_n n}\right\}.$$

Hence

$$f(Xt) = \exp\left\{\sum_{n\geq 1}\frac{p_n'(1)X^n t^n}{(q)_n n}\right\},$$

and this formula is clearly equivalent to the result stated in Theorem 8.

7. Further expansion theorems

In this section we shall be primarily interested in the relationship between expansions of Eulerian differential operators τ and the generating function obtained from the Eulerian family related to τ.

First we observe that any Eulerian differential operator has an expansion in terms of the q-derivative $D_q = (1/X)(1 - \eta)$.

THEOREM 9. *Let τ be any Eulerian differential operator, then*

$$\tau = \frac{1}{X}\sum_{n\geq 0}\frac{a_n X^n D_q^n}{(q)_n},$$

where

$$a_n = [\tau(X - 1)(X - q)\ldots(X - q^{n-1})]_{X=1}.$$

Proof: First we note that $X\tau$ is Eulerian shift-invariant. This follows from the fact that

$$\eta^y(X\tau) = XY\eta^y\tau = XYY^{-1}\tau\eta^y = (X\tau)\eta^y.$$

Hence if $\sigma = X\tau$, then by Theorem 2

$$\sigma = \sum_{n\geq 0}\frac{a_n X^n D_q^n}{(q)_n}$$

where $a_n = [\sigma p_n(X)]_{X=1}$ and where $p_n(X)$ is the Eulerian family associated with

D_q. As we observed just after Definition 2, $p_n(X) = (X - 1)\ldots(X - q^{n-1})$. Thus Theorem 9 follows from the fact that $\sigma = X\tau$.

Most of the Eulerian differential operators we shall meet are expressed in terms of η rather than D_q. The following theorem relates such operators to their respective generating functions.

THEOREM 10. *Suppose τ is an Eulerian differential operator that has a Laurent series expansion in η of the form*

$$\frac{1}{X}\sum_{n=-B}^{\infty} b_n\eta^n = \frac{1}{X}L(\eta).$$

Let $p_n(X)$ be the associated Eulerian family of polynomials with C_n the leading coefficient of $p_n(X)$. Then

$$C_n = \frac{(q)_n}{\prod_{j=1}^{n} L(q^j)}.$$

Proof: We observe that

$$\tau X^m = \frac{1}{X}L(\eta)X^m = \frac{1}{X}\sum_{n=-B}^{\infty} b_n q^{mn}X^m = X^{m-1}L(q^m).$$

Thus in the notation of Theorem 4,

$$e_n = L(q^n),$$

and by the Corollary of Theorem 4

$$C_n = \frac{(1 - q^n)}{L(q^n)}C_{n-1}.$$

Iterating this equation and recalling that $C_0 = 1$, we see that

$$C_n = \frac{(q)_m}{\prod_{j=1}^{m} L(q^j)}.$$

We shall now examine some further results that are related to the symbolic method utilized by Goldman and Rota in [11].

THEOREM 11. *Let τ be an Eulerian differential operator with associated Eulerian family $p_n(X)$. Let*

$$f(t) = \sum_{n\geq 0}\frac{C_n t^n}{(q)_n}$$

where C_n is the leading coefficient of $p_n(X)$. Suppose that

$$g(X, t) = \sum_{n\geq 0}\frac{\pi_n(X)t^n}{(q)_n},$$

where the $\pi_n(X)$ are polynomials in X, $\pi_0(X) = 1$, $\pi_n(1) = 0$ for each $n > 0$, and

$$\tau g(X, t) = tg(X, t).$$

Then

$$g(X, t) = f(Xt)/f(t).$$

Proof: We observe that

$$\tau g(X, t) = \sum_{n \geq 0} \frac{[\tau \pi_n(X)]t^n}{(q)_n}.$$

By by hypothesis

$$\begin{aligned} \tau g(X, t) &= tg(X, t) \\ &= \sum_{n \geq 0} \frac{\pi_n(X) t^{n+1}}{(q)_n} \\ &= \sum_{n \geq 0} \frac{(1 - q^n)\pi_{n-1}(X) t^n}{(q)_n}. \end{aligned}$$

By comparing coefficients of $t^n/(q)_n$ in our two series for $\tau g(X, t)$, we see that

$$\tau \pi_n(X) = (1 - q^n)\pi_{n-1}(X), \pi_n(1) = 0 \qquad \text{for each } n > 0.$$

and

$\pi_0(X) = 1$.
However the only family of polynomials satisfying these conditions is $p_n(X)$. Thus

$$p_n(X) = \pi_n(X).$$

Therefore

$$g(X, t) = \sum_{n=0} \frac{p_n(X)t^n}{(q)_n} = f(Xt)/f(t)$$

as asserted.

COROLLARY. *Let τ, $p_n(X)$, and $f(t)$ be defined as in Theorem* 11. *Suppose that*

$$h(t) = \sum_{n \geq 0} \frac{d_n t^n}{(q)_n}, \qquad (d_0 = 1),$$

and

$$\tau h(Xt) = th(Xt).$$

Then

$$h(t) = f(t).$$

Proof: Define

$$g(X, t) = h(Xt)/h(t).$$

Then $g(X, t)$ fulfills the conditions of Theorem 11. Therefore

$$\frac{h(Xt)}{h(t)} = g(X, t) = \frac{f(Xt)}{f(t)}.$$

Setting $X = 0$, we see that $h(t) = f(t)$.

8. Eulerian Sheffer polynomials

In [15], Rota and Kahaner extend the work in [18] to Sheffer polynomials. Let us recall that a Sheffer set relative to the delta operator Q is a sequence of polynomials $s_0(x), s_1(x), s_2(x), \ldots$ such that

$$Qs_n(x) = ns_{n-1}(x),$$

and

$$s_0(x) = 1.$$

It is then possible to prove that

$$\sum_{n \geq 0} \frac{s_n(x)t^n}{n!} = h(t) \exp\left\{ x \sum_{n \geq 0} \frac{p_n'(0)t^n}{n!} \right\}$$

where $p_n(x)$ is the basic polynomial set associated with Q and $h(t)$ is a formal power series in t with $h(0) = 1$. Conversely one can show that any family of polynomials $s_n(x)$ defined by a function of the above form is a Sheffer set relative to Q. The Eulerian analogs of these facts will be important in Section 10, and so we develop them now.

DEFINITION 6. Let τ be an Eulerian differential operator. A sequence of polynomials $s_0(X), s_1(X), s_2(X), \ldots$ is called an *Eulerian Sheffer family* relative to τ if:

(i) $$s_0(X) = 1$$

(ii) $$\tau s_n(X) = (1 - q^n)s_{n-1}(X).$$

THEOREM 12. *Let τ be an Eulerian differential operator with $p_n(X)$ the associated Eulerian family. If $s_n(X)$ is an Eulerian Sheffer family relative to τ, then*

$$s_n(XY) = \sum_{j \geq 0} \binom{n}{q}_q s_j(X)X^{n-j}p_{n-j}(Y) \tag{8.1}$$

for each n. Conversely any family of polynomials satisfying (8.1) *with* $s_0(X) = 1$ *is an Eulerian Sheffer family relative to τ.*

Proof: Suppose first that the $s_n(X)$ form an Eulerian Sheffer family relative to τ. Thus if

$$S(X;t) = \sum_{n \geq 0} \frac{s_n(X)t^n}{(q)_n},$$

we see that

$$\tau \frac{S(X;t)}{S(1;t)} = \sum_{n \geq 0} \frac{(1 - q^n)s_{n-1}(X)t^n}{(q)_n}(S(1,t))^{-1}$$
$$= tS(X;t)/S(1.t).$$

Hence by Theorem 11,

$$\frac{S(X,t)}{S(1,t)} = \sum_{n \geq 0} \frac{p_n(X)t^n}{(q)_n}.$$

Therefore

$$\sum_{n\geq 0}\frac{s_n(X)t^n}{(q)_n}=\sum_{n\geq 0}\frac{s_n(1)t^n}{(q)_n}\sum_{m\geq 0}\frac{p_m(X)t^m}{(q)_m}.$$

Comparing coefficients of t^n on each side of this equation, we see that

$$s_n(X)=\sum_{j\geq 0}\binom{n}{j}_q p_j(X)s_{n-j}(1).$$

Therefore

$$\begin{aligned}
s_n(XY) &= \sum_{j=0}^{n}\binom{n}{j}_q\sum_{r=0}^{j}\binom{j}{r}p_r(X)Y^r p_{j-r}(Y)s_{n-j}(1)\\
&= \sum_{r=0}^{n}\frac{(q)_n}{(q)_r}p_r(X)Y^r\sum_{j=r}^{n}\frac{1}{(q)_{n-j}(q)_{j-r}}p_{j-r}(Y)s_{n-j}(1)\\
&= \sum_{r=0}^{n}\binom{n}{r}_q p_r(X)Y^r\sum_{j=0}^{n-r}\binom{n-r}{j}_q p_j(Y)s_{n-r-j}(1)\\
&= \sum_{r\geq 0}\binom{n}{r}_q p_r(X)Y^r s_{n-r}(Y).
\end{aligned}$$

Conversely suppose that $s_0(X)=1$ and the $s_n(X)$ satisfy (8.1). Then by setting $X=1$ in (8.1) and then replacing Y by X, we see that

$$\begin{aligned}
\tau s_n(X) &= \sum_{j\geq 0}\binom{n}{j}_q \tau p_j(X)s_{n-j}(1)\\
&= \sum_{j\geq 0}\binom{n}{j}_q (1-q^j)p_{j-1}(X)s_{n-j}(1)\\
&= (1-q^n)\sum_{j\geq 1}\binom{n-1}{j-1}_q p_{j-1}(X)s_{n-j}(1)\\
&= (1-q^n)\sum_{j\geq 0}\binom{n-1}{j}_q p_j(X)s_{n-1-j}(1)\\
&= (1-q^n)s_{n-1}(X).
\end{aligned}$$

COROLLARY. *If $s_n(X)$ is an Eulerian Sheffer family relative to τ, then*

$$\sum_{n\geq 0}\frac{s_n(X)t^n}{(q)_n}=h(t)\sum_{n\geq 0}\frac{p_n(X)t^n}{(q)_n},$$

where $h(t)$ is a formal Eulerian series with $h(0)=1$. Conversely any family of polynomials defined by the above type of function is an Eulerian Sheffer family relative to τ.

Proof: The first part follows directly from the first part of the proof of Theorem 12. On the other hand, suppose that $s_n(X)$ is defined by the above equation. Then

$$s_0(X)=h(0)p_0(X)=1,$$

and

$$\sum_{n=0}^{\infty} \frac{\tau s_n(X)t^n}{(q)_n} = h(t) \sum_{n\geq 0} \frac{\tau p_n(X)t^n}{(q)_n}$$
$$= h(t) \sum_{n\geq 0} \frac{(1-q^n)p_{n-1}(X)t^n}{(q)_n}$$
$$= th(t) \sum_{n\geq 0} \frac{p_n(X)t^n}{(q)_n}$$
$$= t \sum_{n\geq 0} \frac{s_n(X)t^n}{(q)_n}$$
$$= \sum_{n\geq 0} \frac{(1-q^n)s_{n-1}(X)t^n}{(q)_n}$$

Comparing coefficients of t^n, we see that

$$\tau s_n(X) = (1-q^n)s_{n-1}(X).$$

Therefore the $s_n(X)$ from an Eulerian Sheffer family of polynomials relative to τ.

There are at least two examples of Eulerian Sheffer polynomials that have been studied extensively. First we consider

$$H_n(X) = \sum_{j\geq 0} \binom{n}{j}_q X^j,$$

the q-Hermite polynomials studied by Carlitz [5], [6] and introduced independently by Szegö [22] and Rogers [19].

$$H_0(X) = 1,$$
$$D_q H_n(X) = \sum_{j\geq 0} \binom{n}{j}_q (1-q^j)X^{j-1}$$
$$= (1-q^n) \sum_{j\geq 0} \binom{n-1}{j}_q X^j$$
$$= (1-q^n)H_{n-1}(X).$$

Thus the $H_n(X)$ form an Eulerian Sheffer family relative to D_q. Carlitz [5] also has considered a related set of polynomials

$$q^{n(n-1)/2}G_n(-X) = q^{n(n-1)/2} \sum_{j\geq 0} \binom{n}{j}_q q^{j(j-n)}(-X)^j.$$

Now

$$q^{0(0-1)/2}G_0(-X) = 1,$$

and with $\Delta q = q/X(1-\eta^{-1})$

$$\begin{aligned}\Delta_q q^{n(n-1)/2}G_n(-X) &= q^{n(n-1)/2} \sum_{j\geq 0}\binom{n}{j}_q q^{j(j-n)}(-1)^{j-1}X^{j-1}q^{-j+1}(1-q^j)\\ &= q^{n(n-1)/2}(1-q^n)\sum_{j\geq 0}\binom{n-1}{j-1}_q q^{j(j-n)-j+1}(-X)^{j-1}\\ &= (1-q^n)q^{(n-1)(n-2)/2}\sum_{j\geq 0}\binom{n-1}{j}_q q^{j(j-n+1)}(-X)^{j-1}\\ &= (1-q^n)q^{(n-1)(n-2)/2}G_n(-X).\end{aligned}$$

Therefore $q^{n(n-1)/2}G_n(-X)$ is an Eulerian Sheffer family relative to Δ_q.

9. Applications to basic hypergeometric series

9.1 q-Differentiation. We have already discussed $D_q = 1/X(1-\eta)$ with related Eulerian family $P_n(X,1) = (X-1)\ldots(X-q^{n-1})$. Since the leading coefficient of $P_n(X,1)$ is always 1, we see that by the Corollary to Theorem 6

$$\sum_{n\geq 0}\frac{P_n(X,1)t^n}{(q)_n} = e(Xt)/e(t), \tag{9.1}$$

where

$$e(t) = \sum_{n\geq 0}\frac{t^n}{(q)_n}.$$

Since $[P_n'(X,1)]_{X=1} = \lim_{X\to 1}\dfrac{P_n(X,1)}{(X-1)} = (q)_{n-1}$, we see that by Theorem 8

$$\begin{aligned}\sum_{n\geq 0}\frac{t^n}{(q)_n} &= e(t)\\ &= \exp\left\{\sum_{n\geq 1}\frac{(q)_{n-1}t^n}{(q)_n n}\right\}\\ &= \exp\left\{\sum_{n\geq 1}\frac{t^n}{(1-q^n)n}\right\}\\ &= \exp\left\{-\sum_{n\geq 1}\sum_{m\geq 0}\frac{t^n q^{nm}}{h}\right\}\\ &= \exp\left\{-\sum_{m\geq 0}\log(1-tq^m)\right\}\\ &= \prod_{m\geq 0}(1-tq^m)^{-1} = (t)_\infty^{-1},\end{aligned}$$

a well-known result due to Euler.

Equation (9.1) may now be rewritten as

$$\sum_{n\geq 0}\frac{P_n(X,1)t^n}{(q)_n} = (t)_\infty/(tX)_\infty, \tag{9.2}$$

the well-known summation due to Heine [21], p. 92, equation (3.2.2.12).

9.2 Backwards q-differentiation. Here we consider the Eulerian differential operator

$$\Delta_q = \frac{q}{X}(1 - \eta^{-1}).$$

$$\begin{aligned}\Delta_q P_n(1, X) &= q\frac{P_n(1, X) - P_n(1, Xq^{-1})}{X} \\ &= qP_{n-1}(1, X)\left\{\frac{1 - Xq^{n-1} - 1 + Xq^{-1}}{X}\right\} \\ &= P_{n-1}(1, X)(1 - q^n).\end{aligned}$$

A repetition of the arguments used in Section 9.1 would yield

$$\sum_{n\geq 0} \frac{P_n(1, X)t^n}{(q)_n} = (Xt)_\infty/(t)_\infty, \tag{9.3}$$

a result equivalent to (9.2).

9.3 The Heine–Gauss theorem. We now examine the Eulerian differential operator

$$\gamma = \frac{1}{1 - b\eta} D_q = \frac{1}{X}\frac{1 - \eta}{1 - b\eta q^{-1}}$$

with associated Eulerian family $g_n(X)$ and generating function $G(t)$, that is

$$\sum_{n\geq 0} \frac{g_n(X)t^n}{(q)_n} = G(Xt)/G(t).$$

Now by Theorem 10

$$\begin{aligned}G(t) &= \sum_{n\geq 0} \frac{t^n}{(q)_n} \frac{(q)_n}{\prod_{j=1}^{n} \frac{(1 - q^j)}{(1 - bq^{j-1})}} \\ &= \sum_{n\geq 0} \frac{(b)_n t^n}{(q)_n}.\end{aligned}$$

Hence

$$G(t) = \frac{(bt)_\infty}{(t)_\infty},$$

by (9.3).

Therefore

$$\sum_{n\geq 0} \frac{g_n(X)t^n}{(q)_n} = \frac{(bXt)_\infty (t)_\infty}{(Xt)_\infty (bt)_\infty}. \tag{9.4}$$

Now let us expand (9.4) in the following manner

$$\sum_{n\geq 0} \frac{g_n(X)t^n}{(q)_n} = \sum_{n\geq 0} \frac{C_n(t)P_n(X, 1)}{(q)_n}. \tag{9.5}$$

That such a formal expansion exists is obvious from the fact that $P_n(X, 1)$ forms a basis for **P** over **R**. We wish to determine $C_n(t)$.

Since

$$\gamma g_n(X) = (1 - q^n) g_{n-1}(X),$$

we see that

$$\begin{aligned} D_q g_n(X) &= (1 - b\eta)(1 - q^n) g_{n-1}(X) \\ &= (1 - q^n)(g_{n-1}(X) - b g_{n-1}(Xq)). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n \geq 0} \frac{C_{n+1}(t) P_n(X, 1)}{(q)_n} &= \sum_{n \geq 0} \frac{C_n(t)(1 - q^n) P_{n-1}(X)}{(q)_n} \\ &= \sum_{n \geq 0} \frac{C_n(t) D_q P_n(X)}{(q)_n} \\ &= D_q \sum_{n \geq 0} \frac{g_n(X) t^n}{(q)_n} \\ &= \sum_{n \geq 1} \frac{(g_{n-1}(X) - b g_{n-1}(Xq)) t^n}{(q)_{n-1}} \\ &= t \sum_{n \geq 0} \frac{(g_n(X) - b g_n(Xq)) t^n}{(q)_n} \\ &= t \sum_{n \geq 0} \frac{C_n(t) P_n(X, 1)}{(q)_n} - bt \sum_{n \geq 0} \frac{C_n(t) P_n(Xq, 1)}{(q)_n}. \end{aligned} \tag{9.6}$$

Now

$$\begin{aligned} P_n(Xq, 1) &= q^n \left(X - \frac{1}{q} \right) P_{n-1}(X, 1) \\ &= q^n (X - q^{n-1}) P_{n-1}(X, 1) + (q^{2n-1} - q^{n-1}) P_{n-1}(X, 1) \\ &= q^n P_n(X, 1) - q^{n-1}(1 - q^n) P_{n-1}(X, 1) \end{aligned}$$

Substituting this identity into (9.6), we see that

$$\begin{aligned} &\sum_{n \geq 0} \frac{C_{n+1}(t) P_n(X, 1)}{(q)_n} \\ &\quad = \sum_{n \geq 0} \frac{t C_n(t)(1 - bq^n) P_n(X, 1) + btq^{n-1}(1 - q^n) C_n(t) P_{n-1}(X, 1)}{(q)_n}. \end{aligned}$$

Comparing coefficients of $P_n(X, 1)/(q)_n$ on both sides of this equation we see that

$$C_{n+1}(t) = t(1 - bq^n) C_n(t) + btq^n C_{n+1}(t).$$

Therefore

$$C_{n+1}(t) = \frac{t(1 - bq^n)}{(1 - btq^n)} C_n(t). \tag{9.7}$$

By iterating (9.7) and noting that $C_0(t) = 1$, we find that

$$C_n(t) = \frac{t^n(b)_n}{(bt)_n}. \tag{9.9}$$

Substituting (9.8) into (9.5), we see that

$$\begin{aligned}\frac{(bXt)_\infty(t)_\infty}{(Xt)_\infty(bt)_\infty} &= \sum_{n\geq 0}\frac{g_n(X)t^n}{(q)_n}\\ &= \sum_{n\geq 0}\frac{C_n(t)P_n(X,1)}{(q)_n}\\ &= \sum_{n\geq 0}\frac{P_n(X,1)(b)_nt^n}{(q)_n(bt)_n}\end{aligned} \tag{9.9}$$

Equation (9.9) is the Heine–Gauss theorem [21], p. 97, equation (3.3.2.5).

9.4 The Rogers–Ramanujan Identities. Here we consider the Eulerian differential operator

$$R_q = \frac{1}{X}(\eta^{-2} - \eta^{-1}).$$

Let $r_n(X)$ denote the associated Eulerian family, and let

$$\sum_{n\geq 0}\frac{r_n(X)t^n}{(q)_n} = \frac{\rho(Xt)}{\rho(t)}.$$

By Theorem 10,

$$\begin{aligned}\rho(t) &= \sum_{n\geq 0}\frac{t^n}{(q)_n}\cdot\frac{(q)_n}{\prod_{j=1}^{n}(q^{-2j} - q^{-j})}\\ &= \sum_{n\geq 0}\frac{t^n}{\prod_{j=1}^{n} q^{-2j}(1 - q^j)}\\ &= \sum_{n\geq 0}\frac{q^{n^2+n}t^n}{(q)_n}.\end{aligned}$$

Thus $\rho(t)$ is indeed one of the functions involved in the Rogers–Ramanujan identities (see [21], p. 103).

Now let us consider the following function:

$$F(t) = \sum_{n\geq 0}\frac{(-1)^n t^{2n} q^{n(5n+3)/2}(1 - tq^{2n+1})}{(q)_n(tq^{n+1})_\infty}$$

Then in the notation of Hardy and Wright [13], p. 294, equation (19.14.11)

$$F(t) = H_1(tq, q),$$

and by [13], p. 294, equation (19.14.15)

$$(\eta^{-2} - \eta^{-1})F(Xt) = XtF(Xt).$$

Thus

$$R_q F(Xt) = tF(Xt).$$

Hence by the corollary to Theorem 11,

$$\rho(t) = F(t). \tag{9.10}$$

Setting $t = q^{-1}$ in (9.10), we obtain

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1 + \sum_{n=1}^{\infty} (-1)^n q^{n(5n-1)/2}(1 + q^n)}{(q)_\infty}$$
$$= (q;q^5)_\infty^{-1}(q^4;q^5)_\infty^{-1}, \tag{9.11}$$

by Jacobi's identity [13], p. 282.

Finally setting $t = 1$ in (9.10), we see that

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{\sum_{n=0}^{\infty} (-1)^n q^{n(5n+3)/2}(1 - q^{2n+1})}{(q)_\infty}$$
$$= (q^2;q^5)_\infty^{-1}(q^3;q^5)_\infty^{-1}, \tag{9.12}$$

by Jacobi's identity [13], p. 282.

Equations (9.11) and (9.12) constitute the Rogers–Ramanujan identities.

The results of this section give a small sampling of the relationship of the theory of Eulerian differential operators to the classical theory of basic hypergeometric series.

10. Applications to Eulerian Rodrigues formulae

One of the most useful results in [18] is Theorem 4 which presents several formulae for the iterative calculation of families of basic polynomials. In particular if Q is the delta operator related to the family of basic polynomials $p_n(x)$, then the Rodrigues-type formula [18; p. 194, equation (4)] may be rewritten as

$$(Qx - xQ)x^{-1}p_n(x) = p_{n-1}(x),$$

or equivalently

$$Qx^{-1}p_{n+1}(x) = nx^{-1}p_n(x). \tag{10.1}$$

Thus the Rodrigues-type formula of Rota and Mullin [18], p. 194, equation (4), is equivalent to the assertion that the family $x^{-1}p_1(x), x^{-1}p_2(x), x^{-1}p_3(x), \ldots$ is a Sheffer set (in the notation of [16] which was described in our Section 8) provided each polynomial is multiplied by $p_1'(0)$ so that $[x^{-1}p_1(x)/p_1'(0)]_{x=1} = 1$.

Hence we may prove the Rodrigues-type formula of Rota and Mullin [18], p. 194, equation (4), if we can establish that $\{x^{-1}p_{n+1}(x)/p_1'(0)\}$ is a Sheffer set relative to Q. This is possible in the following manner[2]

$$\begin{aligned}\sum_{n\geq 0}\frac{x^{-1}p_{n+1}(x)t^n}{p_1'(0)n!} &= \frac{1}{xp_1'(0)}\frac{d}{dt}\sum_{n\geq 0}\frac{p_n(x)t^n}{n!}\\ &= \frac{1}{xp_1'(0)}\frac{d}{dt}\exp\left\{x\sum_{n\geq 0}\frac{p_n'(0)t^n}{n!}\right\}\\ &= \frac{1}{p_1'(0)}\sum_{n\geq 0}\frac{p_{n+1}'(0)t^n}{n!}\exp\left\{x\sum_{n\geq 0}\frac{p_n'(0)t^n}{n!}\right\}\\ &= h(t)\exp\left\{x\sum_{n\geq 0}\frac{p_n'(0)t^n}{n!}\right\}\end{aligned}$$

where $h(0) = 1$. Thus by our remarks in the beginning of Section 8, the

$$x^{-1}p_{n+1}(x)/p_1'(0)$$

do indeed form a Sheffer set relative to Q.

Our object now is to follow the q-analog of this procedure. As we shall see a simple formula like (10.1) does not hold in general for Eulerian families; however, more complicated recurrences can be obtained. We shall content ourselves with examining the polynomials $g_n(X)$ introduced in Section 9.3. Define for $n > 0$

$$G_n(X) = \frac{g_{n+1}(X) - bXg_n(X)(1 - q^n)}{X - 1}, \quad G_0(X) = 1 - b. \tag{10.2}$$

Then if $D_{q,t}$ denotes q-differentiation with respect to t,

$$\begin{aligned}\sum_{n\geq 0}\frac{G_n(X)t^n}{(q)_n} &= (X-1)^{-1}(1 - bXt)\sum_{n\geq 0}\frac{g_{n+1}(X)t^n}{(q)_n}\\ &= (X-1)^{-1}(1 - bXt)D_{q,t}\sum_{n\geq 0}\frac{g_n(X)t^n}{(q)_n}\\ &= \frac{(X-1)^{-1}(1 - bXt)(bXtq)_\infty(tq)_\infty}{t(Xt)_\infty(bt)_\infty}\\ &\quad\times\{(1 - bXt)(1 - t) - (1 - Xt)(1 - bt)\}\\ &= \frac{(1-b)}{1-t}\frac{(bXt)_\infty(t)_\infty}{(Xt)_\infty(bt)_\infty}\\ &= \frac{(1-b)}{(1-t)}\sum_{n\geq 0}\frac{g_n(X)t^n}{(q)_n}.\end{aligned}$$

Therefore by the corollary to Theorem 12, the $(1 - b)^{-1}G_n(X)$ form an Eulerian Sheffer family relative to γ. Hence

$$\gamma G_n(X) = (1 - q^n)G_{n-1}(X) \tag{10.3}$$

As we see (10.3) is quite a bit more complicated than (10.1), and, in general, matters are even worse. The reason is that for basic families $p_n(x)$ associated with delta

[2] I wish to thank Gian-Carlo Rota for supplying me with an equivalent form of this argument.

operators (by Corollary 2 of Theorem 3 in [18]),

$$\log \sum_{n\geq 0} \frac{p_n(x)t^n}{n!} = x \sum_{n\geq 0} \frac{p_n'(0)t^n}{n!},$$

which is a linear function of x. For Eulerian families $\pi_n(X)$ associated with Eulerian differential operators, we see by Theorem 8 that

$$\log \sum_{n\geq 0} \frac{\pi_n(X)t^n}{n!} = \sum_{n\geq 1} \frac{\pi_n'(1)X^n t^n}{(q)_n n} - \sum_{n\geq 1} \frac{\pi_n'(1)t^n}{(q)_n n},$$

and in general this is a very complicated function of X. Thus it is not surprising that recurrences among the $\pi_n(X)$ are more complicated.

In actual fact, Theorem 4 and its corollary provide very effective means for recursively defining Eulerian families.

11. Applications to finite vector spaces

Just as the theory of delta operators developed by Rota and Mullin is useful in the combinatorics of finite sets, so our theory is useful in the combinatorics of finite vector spaces.

First we remark that Rota and Goldman [12], Section 5, have studied $P_n(X, Z)$ in detail and have shown that $P_n(X, Z)$ is the number of one-to-one linear transformations f of $\mathcal{N}$ into $\mathcal{X}$ such that $f(\mathcal{N}) \cap \mathcal{Z} = \{0\}$ where $\mathcal{Z}$ is a subspace of $\mathcal{X}$. They also established combinatorially the q-binomial theorem:

$$P_n(X, Z) = \sum_{l\geq 0} \binom{n}{l}_q P_l(X, Y) P_{n-l}(Y, Z), \tag{11.1}$$

a result equivalent to our (3.1). We have already seen that the $P_n(X, 1)$ form an Eulerian family. We conclude by considering a new Eulerian family, and we show how combinatorial studies may lead to analytic identities.

DEFINITION 7. Let $\mathcal{H}_n(X, U, W)$ denote the number of one-to-one linear transformations f of $\mathcal{N}$ into $\mathcal{U} \oplus \mathcal{X}$ where all non-zero $\mathcal{U}$-components of f-images lie outside of $\mathcal{W}$ a subspace of $\mathcal{U}$.

DEFINITION 8. $h_n(X) = \mathcal{H}_n(X, U, UX^{-1})$.

Combinatorially we may think of $h_n(X)$ as being defined exactly as $\mathcal{H}_n(X, U, W)$ is with the added condition that $w = u - x$.

PROPOSITION 1. For each $n \geq 0$,

$$\mathcal{H}_n(X, U, W) = \sum_{j\geq 0} \binom{n}{j}_q P_j(X, 1) P_{n-j}(U, W) X^{n-j}.$$

Proof: Let us look at the maps f counted by $\mathcal{H}_n(X, U, W)$ for which the subspace of $f(\mathcal{N})$ with 0 as $\mathcal{U}$-component is j-dimensional. The number of such maps is obtained as follows: We can choose a j-dimensional subspace of $\mathcal{N}$ in $\binom{n}{j}_q$ ways, We can then map this chosen subspace into $\mathscr{X}$ in $P_j(X, 1)$ ways. If $v_1, \ldots, v_j$ form a basis for this j-dimensional subspace of $\mathscr{N}$, we can extend to $v_1, \ldots, v_n$ a basis for $\mathscr{N}$. By the same argument used in Section 3, we need only choose $f(v_{j+1}), \ldots, f(v_n)$ so that their $\mathscr{U}$-components are linearly independent (and now outside of $\mathscr{W}$ as well). This can clearly be done in $X^{n-j}P_{n-j}(U, W)$ ways. Hence

$$\mathscr{H}_n(X, U, W) = \sum_{j \geq 0} \binom{n}{j}_q P_j(X, 1)P_{n-j}(U, W)X^{n-j}.$$

PROPOSITION 2. For each $n \geq 0$,

$$h_n(X) = \sum_{j \geq 0} \binom{n}{j}_q P_j(X, 1)P_{n-j}(X, 1)U^{n-j}.$$

Proof:

$$\begin{aligned} h_n(X) &= \mathscr{H}_n(X, U, UX^{-1}) \\ &= \sum_{j \geq 0} \binom{n}{j}_q P_j(X, 1)P_{n-j}(U, UX^{-1})X^{n-j} \\ &= \sum_{j \geq 0} \binom{n}{j}_q P_j(X, 1)(UX^{-1})^{n-j}P_{n-j}(X, 1)X^{n-j} \\ &= \sum_{j \geq 0} \binom{n}{j}_q P_j(X, 1)P_{n-j}(X, 1)U^{n-j}. \end{aligned}$$

PROPOSITION 3. The $h_n(X)$ form an Eulerian family of polynomials.

Proof: By Proposition 2 we see that $h_0(X) = 1$ and $h_n(X)$ is a polynomial of degree n in X for each n. Finally $h_n(XY)$ counts the number of one-to-one linear transformations f of $\mathscr{N}$ into $\mathscr{U} \oplus \mathscr{X} \oplus \mathscr{Y}$ where all non-zero $\mathscr{U}$-components of f-images lie outside of $\mathscr{T}$, a subspace of $\mathscr{U}$ with $u - t = x + y$.

Let us look at the maps counted by $h_n(XY)$ for which the subspace of $f(\mathscr{N})$ with 0 as $\mathscr{U} \oplus \mathscr{X}$-component is j-dimensional. The number of such maps is obtained as follows: We can choose a j-dimensional subspace of $\mathscr{N}$ in $\binom{n}{j}_q$ ways. We can then map this subspace into $\mathscr{Y}$ in $P_j(Y, 1)$ ways. Extending a basis $v_1, \ldots, v_j$ of this j-dimensional subspace of $\mathscr{N}$ to a basis $v_1, \ldots, v_n$ for $\mathscr{N}$, we see by the same argument used in Section 3 that we need only choose $f(v_{j+1}), \ldots, f(v_n)$ so that their $\mathscr{U} \oplus \mathscr{X}$-components are linearly independent and their non-zero $\mathscr{U}$-components are outside $\mathscr{T}$ This choice can be made in $Y^{n-j}\mathscr{H}_{n-j}(X, U, T)$ ways. Therefore

$$h_n(XY) = \sum_{j \geq 0} \binom{n}{j}_q P_j(Y, 1)Y^{n-j}\mathscr{H}_{n-j}(X, U, T)$$

Now choose $\mathscr{W}$ so that $\mathscr{U} \supset \mathscr{W} \supset \mathscr{T}$ and $u - w = x$ (consequently $w - t = y$).

Utilizing (11.1), we see that

$$\begin{aligned} h_n(XY) &= \sum_{j\ge 0}\binom{n}{j}_q P_j(Y,1)Y^{n-j}\mathscr{H}_{n-j}(X,U,T) \\ &= \sum_{j\ge 0}\binom{n}{j}_q P_j(Y,1)Y^{n-j}\sum_{r\ge 0}\binom{n-j}{r}_q P_r(X,1)P_{n-j-r}(U,T)X^{n-j-r} \\ &= \sum_{j\ge 0}\binom{n}{j}_q P_j(Y,1)Y^{n-j}\sum_{r\ge 0}\binom{n-j}{r}_q P_r(X,1)X^{n-j-r} \\ &\quad\times \sum_{l\ge 0}\binom{n-j-r}{l}_q P_l(U,W)P_{n-j-r-l}(W,T). \end{aligned} \tag{11.2}$$

Now if $p = r + l$, then

$$\begin{aligned} \binom{n}{j}_q\binom{n-j}{r}_q\binom{n-j-r}{l}_q &= \frac{(q)_n}{(q)_j(q)_r(q)_l(q)_{n-j-r-l}} \\ &= \binom{n}{p}_q\binom{p}{r}_q\binom{n-p}{j}_q. \end{aligned}$$

Hence interchanging the summations in (11.2) and replacing l by $p - r$, we see that

$$\begin{aligned} h_n(XY) &= \sum_{p\ge 0}\binom{n}{p}_q Y^p \sum_{r\ge 0}\binom{p}{r}_q P_r(X,1)P_{p-r}(U,W)X^{p-r} \\ &\quad \sum_{j\ge 0}\binom{n-p}{j}_q P_j(Y,1)P_{n-j-p}(W,T)(XY)^{n-j-p} \\ &= \sum_{p\ge 0}\binom{n}{p}_q Y^p \sum_{r\ge 0}\binom{p}{r}_q P_r(X,1)P_{p-r}(X,1)U^{p-r} \\ &\quad \sum_{j\ge 0}\binom{n-p}{j}_q P_j(Y,1)P_{n-j-p}(Y,1)U^{n-j-p} \\ &= \sum_{p\ge 0}\binom{n}{p}_q Y^p h_p(X)h_{n-p}(Y). \end{aligned}$$

Knowing that the $h_n(X)$ form an Eulerian family, we can derive an identity of Carlitz [5], p. 361, equation (2.2). First we see by inspection of Proposition 2 that the leading coefficient of $h_n(X)$ is

$$H_n(U) = \sum_{j\ge 0}\binom{n}{j}_q U^j,$$

the q-Hermite polynomial mentioned in Section 8. Furthermore for $n > 0$

$$\begin{aligned} h_n'(1) &= \lim_{X\to 1}(X-1)^{-1}h_n(X) \\ &= \lim_{X\to 1}(X-1)^{-1}\sum_{j\ge 0}\binom{n}{j}_q P_j(X,1)U^{n-j}P_{n-j}(X,1) \\ &= U^n(q)_{n-1} + (q)_{n-1} \\ &= (q)_{n-1}(1+U^n). \end{aligned}$$

These facts now allow us to establish the following result:

PROPOSITION 4. ([5], p. 361, equation (2.2))

$$\sum_{n\geq 0}\frac{H_n(U)t^n}{(q)_n}=(t)_\infty^{-1}(tU)_\infty^{-1}.$$

Proof: By Theorem 8,

$$\begin{aligned}\sum_{n\geq 0}\frac{H_n(U)t^n}{(q)_n}&=\exp\left\{\sum_{n\geq 1}\frac{h_n'(1)t^n}{(q)_n n}\right\}\\&=\exp\left\{\sum_{n\geq 1}\frac{(1+U^n)t^n}{(1-q^n)n}\right\}\\&=\exp\left\{\sum_{m\geq 0}\sum_{n\geq 1}\frac{q^{nm}(1+U^n)t^n}{n}\right\}\\&=\exp\left\{-\sum_{m\geq 0}\log(1-tq^m)-\sum_{m\geq 0}\log(1-tUq^m)\right\}\\&=\prod_{m=0}^{\infty}(1-tq^m)^{-1}(1-tUq^m)^{-1}=(t)_\infty^{-1}(tU)_\infty^{-1}.\end{aligned}$$

12. Conclusion

In light of the characterization of Eulerian differential operators given in Theorem 4, and since the factor $1-q^n$ does not appear in this characterization, we may reasonably ask what happens if the sequence $0, 1-q, 1-q^2, \ldots$ were replaced by $u_0=0, u_1, u_2, \ldots$ (where $u_n\neq 0$ for each $n>0$) in Definition 1. Actually we can prove all the theorems through Section 8 with

$$\begin{aligned}&1-q^n &&\text{replaced by}\quad u_n,\\&\binom{n}{r}_q &&\text{replaced by}\quad \frac{u_n u_{n-1}\cdots u_{n-r+1}}{u_r u_{r-1}\cdots u_1},\\&(q)_n &&\text{replaced by}\quad u_n u_{n-1}\cdots u_1\end{aligned}$$

Indeed the results would extend the work of Morgan Ward in [23]; however, such a generalization seems of little immediate value in applications (such as in Sections 3, 9, 10, and 11), and we have, therefore, not bothered to write our results in this more general form.

There are many other possible applications of our theory. For example, the series-product identity of F. H. Jackson [21], p. 96, equation (3.3.1.3)

$$1+\sum_{n\geq 1}\frac{(aq)_{n-1}(1-aq^{2n})(b)_n(c)_n(d)_n(aq/bcd)^n}{(q)_n(aq/b)_n(aq/c)_n(aq/d)_n}=\frac{(aq)_\infty(aq/bc)_\infty(aq/bd)_\infty(aq/cd)_\infty}{(aq/b)_\infty(aq/c)_\infty(aq/d)_\infty(aq/bcd)_\infty}$$

can be transformed into a generating function for a family of Eulerian polynomials by the substitutions $a=Xt$, $b=X$. Further aspects of the classical theory of basic hypergeometric series can be included in the theory of Eulerian differential operators.

As for applications to finite vector spaces, we first remark that it should be possible to extend the results of Section 11 to polynomials of the form

$$\sum_{n\geq 0} \frac{\tilde{h}_n(X)t^n}{(q)_n} = \frac{(t)_\infty(U_1t)_\infty(U_2t)_\infty \ldots (U_rt)_\infty}{(Xt)_\infty(U_1Xt)_\infty(U_2Xt)_\infty \ldots (U_rXt)_\infty}.$$

A more tantalizing problem involves a finite vector space interpretation for the $g_n(X)$ defined in (9.4) (obviously $g_n(X) = P_n(X, 1)$ if $b = 0$).

Professor L. Carlitz has drawn my attention to the paper by A. Sharma and A. Chak (The basic analog of a class of polynomials, Revista di Matematica della Universita di Parma, 5 (1954), 325–337) and to the paper by W. A. Al-Salam (q-Appell polynomials, Annali di Matematica, 77 (1967), 31–45). In the present context, the polynomials studied in these papers are essentially Eulerian Sheffer polynomials related to the Eulerian differential operator D_q. The q-differential operators L_p discussed by Al-Salam on page 43 of his paper are not Eulerian differential operators (as defined here) except when $L_p = D_q$ (cf. our Theorem 9).

Also Professor W. A. Al-Salam has drawn my attention to the extensive literature on generalized Sheffer polynomials. In particular, he pointed out the forthcoming paper by A. M. Chak (An Extension of a Class of Polynomials) in which our Eulerian Sheffer polynomials are named "Appell Polynomials to the Base c". Also Professor Al-Salam mentioned the work by Mourad El Houssieny Ismail (Classification of Polynomial Sets, M.S. Thesis, 1969, University of Alberta) in which an extensive account of generalized Sheffer polynomials is given and in which appears a list of 119 references. Presumably our results in Section 8 duplicate those of Chak; however, other than there, our results appear to be new.

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