ON THE q-ANALOG OF KUMMER'S THEOREM AND APPLICATIONS

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1. Introduction. The q-analogs for Gauss's summation of ${}_2F_1[a, b; c; 1]$ and Saalschutz's summation of ${}_3F_2[a, b, -n; c, a + b - c - n + 1; 1]$ are well known, namely, E. Heine [8; p. 107, Equation (6)] showed that

$$ {}_2\phi_1\left[\begin{matrix} a,\ b;\ q,\ c/ab \\ c \end{matrix}\right] = \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty} \tag{1.1} $$

where

$$ {}_m\phi_n\left[\begin{matrix} a_1\ ,\ \cdots\ ,\ a_m\ ;\ q,\ z \\ b_1\ ,\ \cdots\ ,\ b_n \end{matrix}\right] = \sum_{j=0}^{\infty} \frac{(a_1)_j\ \cdots\ (a_m)_j z^j}{(q)_j (b_1)_j\ \cdots\ (b_n)_j}\ , $$

and $(a)_n = (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$, $(a)_\infty = (a; q)_\infty = \lim_{n\to\infty} (a)_n$. (See also [12; p. 97, Equation (3.3.2.2)].) F. H. Jackson [9; p. 145] showed that

$$ {}_3\phi_2\left[\begin{matrix} a,\ b,\ q^{-n};\ q,\ q \\ c,\ abq/cq^n \end{matrix}\right] = \frac{(c/a)_n (c/b)_n}{(c)_n (c/ab)_n}\ . \tag{1.2} $$

The q-analog of Dixon's summation of ${}_3F_2[a, b, c; 1 + a - b, 1 + a - c; 1]$ was more difficult to find, and indeed only a partial analog is true; namely, W. N. Bailey [5] and F. H. Jackson [10; p. 167, Equation (2)] proved that if $a = q^{-2n}$ where n is a positive integer, then

$$ {}_3\phi_2\left[\begin{matrix} a,\ b,\ c;\ q,\ \dfrac{q^2a^{\frac{1}{2}}}{bc} \\ \dfrac{aq}{b}\ ,\ \dfrac{aq}{c} \end{matrix}\right] = \frac{(b/a)_\infty (c/a)_\infty (qa^{\frac{1}{2}})_\infty (bca^{-\frac{1}{2}})_\infty}{(ba^{-\frac{1}{2}})_\infty (ca^{-\frac{1}{2}})_\infty (a^{-1}qa)_\infty (bca^{-1})_\infty}\ . \tag{1.3} $$

There are three other well-known summations for the ${}_2F_1$ series, namely, Kummer's theorem [12; p. 243, Equation (III. 5)]

$$ {}_2F_1[a, b; 1 + a - b; -1] = \frac{\Gamma(1 + a - b)\Gamma\left(1 + \dfrac{a}{2}\right)}{\Gamma\left(\dfrac{1}{2} + \dfrac{a}{2}\right)\Gamma\left(1 + \dfrac{a}{2} - b\right)}\ , \tag{1.4} $$

Gauss's second theorem [12; p. 243, Equation III. 6)]

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$$(1.5)\qquad {}_2F_1\left[a, b; \frac{1}{2}+\frac{a}{2}+\frac{b}{2}; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}+\frac{b}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{b}{2}\right)},$$

and Bailey's theorem [12; p. 243, Equation (III. 7)]

$$(1.6)\qquad {}_2F_1\left[a, 1-a; c; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{1}{2}+\frac{c}{2}\right)}{\Gamma\left(\frac{c}{2}+\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{c}{2}-\frac{a}{2}\right)}.$$

Of these three, only a q-analog of Kummer's theorem is known, namely, [6; p. 711] (see also [5; p. 173])

$$(1.7)\qquad {}_2\phi_1\begin{bmatrix} a, b; q, -\frac{q}{b} \\ \\ \frac{qa}{b} \end{bmatrix} = \frac{(aq; q^2)_\infty(-q)_\infty(q^2a/b^2; q^2)_\infty}{(qa/b)_\infty(-q/b)_\infty}.$$

The only known proof of (1.7) consists of a specialization of parameters in Jackson's summation of the well-poised ${}_6\phi_5$.

Our object here is to provide a very simple proof of (1.7) and to show that the following q-analogs of Gauss's second theorem and Bailey's theorem hold:

$$(1.8)\qquad \sum_{n=0}^{\infty}\frac{(a)_n(b)_nq^{\frac{1}{2}n(n+1)}}{(q)_n(qab; q^2)_n} = \frac{(-q)_\infty(aq; q^2)_\infty(bq; q^2)_\infty}{(qab; q^2)_\infty},$$

$$(1.9)\qquad \sum_{n=0}^{\infty}\frac{(b)_n(q/b)_nc^nq^{\frac{1}{2}n(n-1)}}{(q^2; q^2)_n(c)_n} = \frac{(qc/b; q^2)_\infty(bc; q^2)_\infty}{(c)_\infty}.$$

2. The q-analog of Kummer's theorem. Here we utilize the summation technique that was successfully employed in [1], [2], [3] and [4]. We shall need the elementary summation [12; p. 92, Equation (3.2.2.11)]

$$(2.1)\qquad \sum_{n=0}^{\infty}\frac{(A)_nT^n}{(q)_n} \equiv {}_1\phi_0[A; q, T] = (AT)_\infty/(T)_\infty.$$

Therefore

$$\begin{aligned} {}_2\phi_1\begin{bmatrix} a, b; q, -q/b \\ qa/b \end{bmatrix} &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n(-q/b)^n}{(q)_n(aq/b)_n} \\ &= \frac{(a)_\infty}{(aq/b)_\infty}\sum_{n=0}^{\infty}\frac{(b)_n(-q/b)^n(aq^{n+1}/b)_\infty}{(q)_n(aq^n)_\infty} \\ &= \frac{(a)_\infty}{(aq/b)_\infty}\sum_{n=0}^{\infty}\frac{(b)_n(-q/b)^n}{(q)_n}\sum_{m=0}^{\infty}\frac{(q/b)_ma^mq^{nm}}{(q)_m} \end{aligned}$$

$$= \frac{(a)_\infty}{(aq/b)_\infty} \sum_{m=0}^{\infty} \frac{(q/b)_m a^m (-q^{m+1})_\infty}{(q)_m (-q^{m+1}/b)_\infty}$$

$$= \frac{(a)_\infty (-q)_\infty}{(aq/b)_\infty (-q/b)_\infty} \sum_{m=0}^{\infty} \frac{(q^2/b^2; q^2)_m a^m}{(q^2; q^2)_m}$$

$$= \frac{(a)_\infty (-q)_\infty (aq^2/b^2; q^2)_\infty}{(aq/b)_\infty (-q/b)(a; q^2)_\infty}$$

$$= \frac{(aq; q^2)_\infty (-q)_\infty (aq^2/b^2; q^2)_\infty}{(aq/b)_\infty (-q/b)_\infty},$$

and so (1.7) is established. Technically what we have done is to set $\alpha = b$, $\beta = a$, $\gamma = qa/b$, and $\tau = -q/b$ in [3; Equation (I 1)]; we then have observed that the resulting ${}_2\phi_1$ with base q reduces to a ${}_1\phi_0$ with base q^2.

3. The q-analogs of Gauss's second theorem and Bailey's theorem. As is well-known [12; p. 32], both (1.5) and (1.6) may be deduced by the application of Kummer's theorem to the following identity:

$$(3.1) \qquad (1-z)^{-a} {}_2F_1 \left[\begin{matrix} a, b; -z/(1-z) \\ c \end{matrix} \right] = {}_2F_1 \left[\begin{matrix} a, c-b; z \\ c \end{matrix} \right].$$

The following lemma is the q-analog of (3.1), and from it we may deduce (1.8) and (1.9) utilizing the q-analog of Kummer's theorem. Actually this lemma was given by F. H. Jackson [9; p. 145, Equation (4)]; his proof entails the development of a q-analog of Euler's transformation of power series. We include a short proof that shows this result to be a limiting case of an identity of N. Hall [7].

LEMMA. $$\sum_{n=0}^{\infty} \frac{(\beta)_n (\alpha)_n (-1)^n q^{\binom{n}{2}} (x\gamma/\beta)^n}{(q)_n (\gamma)_n (x\alpha)_n} = \frac{(x)_\infty}{(x\alpha)_\infty} \sum_{n=0}^{\infty} \frac{(\gamma/\beta)_n (\alpha)_n x^n}{(q)_n (\gamma)_n}.$$

Proof. N. Hall [7] (see also [11; p. 174, Equation (10.1)]) has proved the result

$$ {}_3\phi_2 \left[\begin{matrix} a_1, a_2, a_3; q, b_1 b_2/a_1 a_2 a_3 \\ b_1, b_2 \end{matrix} \right] = \frac{(b_2/a_3)_\infty (b_1 b_2/a_1 a_2)_\infty}{(b_2)_\infty (b_1 b_2/a_1 a_2 a_3)_\infty} {}_3\phi_2 \left[\begin{matrix} b_1/a_1, b_1/a_2, a_3; q, b_2/a_3 \\ b_1, b_1 b_2/a_1 a_2 \end{matrix} \right].$$

Our lemma follows directly by the substitutions $a_1 = \beta$, $a_3 = \alpha$, $b_1 = \gamma$, $b_2 = x\alpha$ if we then let $a_2 \to \infty$.

To obtain (1.8), we set $\alpha = a$, $\beta = b$, $\gamma = q^{\frac{1}{2}} a^{\frac{1}{2}} b^{\frac{1}{2}}$ and $x = -q^{\frac{1}{2}} b^{\frac{1}{2}} a^{-\frac{1}{2}}$ in the lemma. Hence

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n q^{\frac{1}{2}n(n+1)}}{(q)_n(qab;\, q^2)_n} = \frac{(-q^{\frac{1}{2}}b^{\frac{1}{2}}a^{-\frac{1}{2}})_\infty}{(-q^{\frac{1}{2}}b^{\frac{1}{2}}a^{\frac{1}{2}})_\infty}\, {}_2\phi_1 \begin{bmatrix} a,\ q^{\frac{1}{2}}a^{\frac{1}{2}}b^{-\frac{1}{2}};\ q,\ -q^{\frac{1}{2}}b^{\frac{1}{2}}a^{-\frac{1}{2}} \\ q^{\frac{1}{2}}a^{\frac{1}{2}}b^{\frac{1}{2}} \end{bmatrix}$$

$$= \frac{(-q^{\frac{1}{2}}b^{\frac{1}{2}}a^{-\frac{1}{2}})_\infty}{(-q^{\frac{1}{2}}b^{\frac{1}{2}}a^{\frac{1}{2}})_\infty} \frac{(aq;\, q^2)_\infty(-q)_\infty(bq;\, q^2)_\infty}{(q^{\frac{1}{2}}b^{\frac{1}{2}}a^{\frac{1}{2}})_\infty(-q^{\frac{1}{2}}b^{\frac{1}{2}}a^{-\frac{1}{2}})_\infty} \qquad \text{(by (1.7))}$$

$$= \frac{(-q)_\infty(aq;\, q^2)_\infty(bq;\, q^2)_\infty}{(abq;\, q^2)_\infty}$$

which is (1.8).

To obtain (1.9), we set $\alpha = q/b$, $\beta = b$, $\gamma = c$ and $x = -b$ in the lemma. Hence

$$\sum_{n=0}^{\infty} \frac{(b)_n(q/b)_n c^n q^{\frac{1}{2}n(n-1)}}{(q)_n(c)_n(-q)_n} = \sum_{n=0}^{\infty} \frac{(b)_n(q/b)_n c^n q^{\frac{1}{2}n(n-1)}}{(q^2;\, q^2)_n(c)_n}$$

$$= \frac{(-b)_\infty}{(-q)_\infty}\, {}_2\phi_1 \begin{bmatrix} c/b,\ q/b;\ q,\ -b \\ c \end{bmatrix}$$

$$= \frac{(-b)_\infty(qc/b;\, q^2)_\infty(-q)_\infty(bc;\, q^2)_\infty}{(-q)_\infty(c)_\infty(-b)_\infty}$$

$$= \frac{(qc/b;\, q^2)_\infty(bc;\, q^2)_\infty}{(c)_\infty}.$$

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