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On the General Rogers-Ramanujan Theorem

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$$\sum_{i=j}^{\lambda-j+1} f_i \leq \min(a-1,\lambda-2j+2) \leq \min(a-1,a-2j+2) \leq a-j.$$

Thus for $a \geq \lambda$, $B_{\lambda,k,a}(n)$ denotes the number of partitions of n of the form $b_1+b_2+\ldots+b_s$, with $b_i \geq b_{i+1}$, no parts $\not\equiv 0 \pmod{\lambda+1}$ are repeated, $b_i-b_{i+k-1} \geq \lambda+1$ with strict inequality if $(\lambda+1)|b_i$, and finally there areat most $a-1$ summands that are $\leq \lambda+1$.

Theorem [4; Th. 2]. If λ, k, and a are positive integers with $\lambda/2 \leq a \leq k$, $k \geq 2\lambda-1$, then

$$A_{\lambda,k,a}(n) = B_{\lambda,k,a}(n)$$

for every positive integer n.

In the conclusion of [4], it is pointed out that while the proof of this theorem heavily relies upon the condition "$k \geq 2\lambda-1$" numerical evidence strongly indicates that the theorem is still true provided only that $k \geq \lambda$. Indeed it was pointed out that Schur's theorem is technically not a special case of the above theorem since it is the case $k = \lambda = a = 2$ which does not satisfy $k \geq 2\lambda-1$.

The object of this paper is to prove that the condition "$k \geq \lambda$" is sufficient for the truth of the result.

Theorem 8.3. If λ, k, and a are positive integers with $\lambda/2 < a \leq k$, $k \geq \lambda$, then

$$A_{\lambda,k,a}(n) = B_{\lambda,k,a}(n)$$

for every positive integer n.

The method of proof may be briefly described as the application of the sieving technique developed in [5] and [9] to the q-difference equation approach of the previous work on this theorem [4]. In Section 2, we shall introduce notation and some interesting peripheral aspects of this theorem. Then in Section 3 we sketch the general lines of the proof. In Section 4, we shall study the q-difference equations related to certain basic hypergeometric series. In Sections 5 and 6 we shall prove Theorem 6.3 which is Theorem 8.3 with the added assumption that $a \geqq \lambda$. In Sections 7 and 8 we shall prove Theorem 8.3 in full generality. In the conclusion, we shall describe further open questions that are not yet amenable to the techniques so far developed. Two conjectures that are supported by substantial numerical evidence will be discussed.

2. General comments. Several conventions are followed throughout this paper. First we assume that whenever the integers k and λ appear, $k \geqq \lambda$. Also if we assert that "$f(x,q)$ generates the partitions that satisfy condition A", we mean that if $\rho(m,n)$ is the number of partitions of n with m parts that satisfy condition A, then

$$f(x,q) = \sum_{m\geqq 0} \sum_{n\geqq 0} \rho(m,n)x^m q^n.$$

Before we begin preparations for the proof of Theorem 6.3 we wish to examine some interesting aspects of this result.

Proposition 2.1. If $|q| < 1$,

$$1+\sum_{n=1}^{\infty} A_{\lambda,k,a}(n)q^n$$
$$=\prod_{j=1}^{\infty}(1+q^j)(1+q^{j(\lambda+1)})^{-1}(1-q^{j(\lambda+1)})^{-1}$$
$$\cdot(1-q^{(2k-\lambda+1)(\lambda+1)j-(a-\frac{1}{2}\lambda)(\lambda+1)})$$
$$\cdot(1-q^{(2k-\lambda+1)(\lambda+1)(j-1)(+a-\frac{\lambda}{2})(\lambda+1)})$$
$$\cdot(1-q^{(2k-\lambda+1)(\lambda+1)j}).$$

Proof. This is merely the infinite product expansion of the generating function for $A_{\lambda,k,a}(n)$ (see [6; Section 13-1] for the standard way that such a product is obtained). □

Definition 3. If λ is an even positive integer, we denote by $\mathcal{A}_{\lambda,k,a}(n)$ the number of partitions of n in which the parts are either odd and not divisible by $\lambda+1$ or else divisible by $\lambda+1$ but $\not\equiv 0,\ \pm(a-\frac{1}{2}\lambda)(\lambda+1)$ $(\mathrm{mod}\,(2k-\lambda+1)(\lambda+1))$. If $\lambda \equiv 1 \pmod 4$, we denote by $\mathcal{A}_{\lambda,k,a}(n)$ the number of partitions of n in which the parts are either odd and not divisible by $\frac{1}{2}(\lambda+1)$ or else divisible by $\frac{1}{2}(\lambda+1)$ but not $\equiv \lambda+1 \pmod{2\lambda+2}$ and not $\equiv 0,\ \pm(2a-\lambda)\frac{1}{2}(\lambda+1)$ $(\mathrm{mod}\,(2k-\lambda+1)(\lambda+1))$.

Proposition 2.2. If $\lambda \equiv 0,1,2, \pmod 4$, then

$$\mathcal{A}_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$$

for each positive integer n.

Proof. Assume that λ is even, then the generating function for $A_{\lambda,k,a}(n)$ may be written in the following form.

$$1+\sum_{n=1}^{\infty} A_{\lambda,k,a}(n)q^n = \prod_{j=1}^{\infty} \frac{(1-q^{(2j-1)(\lambda+1)})}{(1-q^{2j-1})} \cdot (1-q^{j(\lambda+1)})^{-1}$$
$$\cdot (1-q^{(2k-\lambda+1)(\lambda+1)j-(a-\frac{1}{2}\lambda)(\lambda+1)})$$
$$\cdot (1-q^{(2k-\lambda+1)(\lambda+1)(j-1)+(a-\frac{1}{2}\lambda)(\lambda+1)})$$
$$\cdot (1-q^{(2k-\lambda+1)(\lambda+1)j}).$$

That this infinite product is equal to the one appearing in Proposition 2.1 follows from two applications of the simple identity of Euler [12; eq. (19.4.7), p. 277]:

$$\prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}\frac{1}{(1-x^{2n-1})}$$

If $\lambda \equiv 1 \pmod 4$, then the generating function for $A_{\lambda,k,a}(n)$ may be written in the following form.

$$1+\sum_{n=1}^{\infty} A_{\lambda,k,a}(n)q^n = \prod_{j=1}^{\infty} \frac{(1-q^{(2j-1)\frac{1}{2}(\lambda+1)})}{(1-q^{2j-1})} \cdot \frac{(1-q^{(2\lambda+2)j-\lambda-1})}{(1-q^{j\frac{1}{2}(\lambda+1)})}$$
$$\cdot (1-q^{(2k-\lambda+1)(\lambda+1)j-(2a-\lambda)\frac{1}{2}(\lambda+1)})$$
$$\cdot (1-q^{(2k-\lambda+1)(\lambda+1)(j-1)+(2a-\lambda)\frac{1}{2}(\lambda+1)})$$
$$\cdot (1-q^{(2k-\lambda+1)(\lambda+1)j}).$$

To see that this infinite product is equal to the one appearing in Proposition 2.1, we need only utilize the previously quoted identity of Euler which shows that

$$\prod_{j=1}^{\infty} \frac{(1-q^{(2j-1)\frac{1}{2}(\lambda+1)})}{(1-q^{2j-1})} \cdot \frac{(1-q^{(2\lambda+2)j-\lambda-1})}{(1-q^{\frac{1}{2}j(\lambda+1)})}$$

$$= \prod_{j=1}^{\infty} \frac{(1+q^{j})}{(1+q^{\frac{1}{2}j(\lambda+1)})(1+q^{j(\lambda+1)})(1-q^{\frac{1}{2}j(\lambda+1)})}$$

$$= \prod_{j=1}^{\infty} (1+q^{j})(1+q^{j(\lambda+1)})^{-1}(1-q^{j(\lambda+1)})^{-1}.$$

Thus in each case the identity of the generating functions proves that $A_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$ for each n. □

The reason for proving Proposition 2.2 lies in the fact that most of the standard statements of the classical partition theorems are generally in terms of $A_{\lambda,k,a}(n)$ and not $A_{\lambda,k,a}(n)$ (except when $\lambda = 0$ in which case there is no difference in the definition of the two). For example, $A_{2,2,2}(n)$ is just the number of partitions of n into parts that are odd and not divisible by 3, i.e. each part is $\equiv 1$ or 5 (mod 6), and $A_{1,2,2}(n)$ is just the number of partitions of n into parts that are $\not\equiv 2 \pmod 4$ and $\not\equiv 0$, $\pm 3 \pmod 8$, i.e. each part is $\equiv 1, 4,$ or $7 \pmod 8$. Thus $A_{2,2,2}(n) = B_{2,2,2}(n)$ is the standard formulation of Schur's theorem [14], and $A_{1,2,2}(n) = B_{1,2,2}(n)$ is the standard formulation of the first Göllnitz-Gordon identity [10; p. 162].

The definition of $A_{\lambda,k,a}(n)$ is appealing in that the only restriction on parts at all is that they lie in certain specified arithmetic progressions. Unfortunately no partition function of this simple character exists when $\lambda \equiv 3 \pmod 4$. This can be proved utilizing the results in [7; Sec. 3].

3. Outline of proof of Theorem 6.3. The approach of this paper resembles that of [4]. The first step is to define a set of functions $J_{\lambda,k,i}(x)$ and $H_{\lambda,k,i}(x)$ (see equations (4.1) and (4.2) below) that turn out to be the unique solution of the following sets of q-difference equations subject to certain simple boundary conditions (Theorems 4.1 and 4.2):

$$H_{\lambda,k,i}(x) - H_{\lambda,k,i-1}(x) = (xq^{\lambda+1})^{i-1} J_{\lambda,k,k-i+1}(xq^{\lambda+1}) \tag{3.1}$$

$$J_{\lambda,k,i}(x) = \sum_{j=0}^{\lambda} x^j q^{\frac{1}{2}j(j+1)} \begin{bmatrix}\lambda\\ j\end{bmatrix} H_{\lambda,k,i-j}(x); \tag{3.2}$$

$$H_{\lambda,k,-i}(x) = -(xq^{\lambda+1})^{-i} H_{\lambda,k,i}(x), \tag{3.3}$$

where

$$\begin{bmatrix}\lambda\\ j\end{bmatrix} = \begin{cases} \dfrac{(1-q^{\lambda})\dots(1-q^{\lambda-j+1})}{(1-q^{j})\dots(1-q)} & 0 \le j \le \lambda \\ 0, & \text{otherwise.}\end{cases} \tag{3.4}$$

From the definition of these functions we easily deduce that (see the proof of Theorem 6.3):

$$(3.5)\qquad J_{\lambda,k,i}(1) = \prod_{j=1}^{\infty} (1+q^j)(1+q^{j(\lambda+1)})^{-1}(1-q^{j(\lambda+1)})^{-1}$$

$$\cdot (1-q^{(2k-\lambda+1)(\lambda+1)j-(i-\frac{1}{2}\lambda)(\lambda+1)})$$

$$\cdot (1-q^{(2k-\lambda+1)(\lambda+1)(j-1)+(i-\frac{1}{2}\lambda)(\lambda+1)})$$

$$\cdot (1-q^{(2k-\lambda+1)(\lambda+1)j})$$

$$= \sum_{n=0}^{\infty} A_{\lambda,k,i}(n)q^n.$$

In Sections 5 and 6, we focus our attention on generating functions related to $B_{\lambda,k,i}(n)$. Let $P_{\lambda,k,i}(m,n)$ denote the number of partitions of n into m parts of the form $n = b_1+b_2+\dots+b_m$, with $b_j \geq b_{j+1}$, no parts $\not\equiv 0 \pmod{\lambda+1}$ are repeated, $b_j-b_{j+k-1} \geq \lambda+1$ with strict inequality if $(\lambda+1)|b_j$, and finally there are at most $i-1$ parts that are $\leq \lambda+1$. Then we define

$$(3.6)\qquad J^{\dagger}_{\lambda,k,i}(x) = \sum_{m\geq 0}\sum_{n\geq 0} P_{\lambda,k,i}(m,n)x^m q^n,$$

and letting $Q_{\lambda,k,i}(m,n)$ denote the number of those partitions enumerated by $P_{\lambda,k,i}(m,n)$ that have no parts smaller than $\lambda+1$, we define

$$(3.7)\qquad H^{\dagger}_{\lambda,k,i}(x) = \sum_{m\geq 0}\sum_{n\geq 0} Q_{\lambda,k,i}(m,n)x^m q^n.$$

These functions turn out to satisfy a rather complex set of q-difference equations:

(3.8) $$H^{\dagger}_{\lambda,k,i}(x) - H^{\dagger}_{\lambda,k,i-1}(x) = (xq^{\lambda+1})^{i-1} J_{\lambda,k,k-i+1}(xq^{\lambda+1}),\ 1 \le i \le k;$$

(3.9) $$J^{\dagger}_{\lambda,k,i}(x) = \sum_{j=0}^{i} x^j q^{\frac{1}{2}j(j+1)} \begin{bmatrix}\lambda \\ j\end{bmatrix} H^{\dagger}_{\lambda,k,i-j}(x)$$

$$+ \sum_{\ell \ge 2} (-1)^{\ell-1} \sum_{r=0}^{k} g_1(\ell;k,\lambda,r,i;x;q) H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)}),$$

where $g_1(\ell;k,\lambda,r,i;x;q)$ is a polynomial in q and x depending upon the 5 parameters listed.

Everything prior to (3.9) is precisely like what occurred in [4]; however, the assumption "$k \ge 2\lambda-1$" allows the replacement of (3.9) by a much simpler q-difference equation since then the only non-zero terms of $\sum_{\ell \ge 2}$ occur for $\ell=2$.

The final step in the proof of the theorem is to relate the $H^{\dagger}_{\lambda,k,i}(x)$ and $H_{\lambda,k,i}(x)$. The relation is shown to be

(3.10) $$H_{\lambda,k,i}(x) = H^{\dagger}_{\lambda,k,i}(x)$$

$$+ \sum_{\ell \ge 2} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)}) \gamma(\ell;k,\lambda,r,i;x;q).$$

From here we are able to deduce that

(3.11) $$H_{\lambda,k,i}(x) = H^{\dagger}_{\lambda,k,i}(x), \quad 1 \le i \le k - \lambda + 1$$

Since $\gamma(\ell;k,\lambda,r,i;x;q) \equiv 0$ for $1 \leqq i \leqq k - \lambda+1$. Thus for $\lambda \leqq i \leqq k$,

$$(3.12)\quad J_{\lambda,k,i}(x) = x^{-k+i}(H_{\lambda,k,k-i+1}(xq^{-\lambda-1})-H_{\lambda,k,k-i}(xq^{-\lambda-1}))\quad \text{(by (3.1))}$$

$$= x^{-k+i}(H^{\dagger}_{\lambda,k,k-i+1}(xq^{-\lambda-1})-H^{\dagger}_{\lambda,k,k-i}(xq^{-\lambda-1}))\quad \text{(by (3.10))}$$

$$= J^{\dagger}_{\lambda,k,i}(x)\quad \text{(by (3.8))}.$$

Hence for $\lambda \leqq i \leqq k$

$$\sum_{n\geqq 0} B_{\lambda,k,i}(n)q^n = J^{\dagger}_{\lambda,k,i}(1)$$

$$= J_{\lambda,k,i}(1)\quad \text{(by (3.12))}$$

$$= \sum_{n\geqq 0} A_{\lambda,k,i}(n)q^n\quad \text{(by (3.5))}.$$

Thus Theorem 6.3 follows from the identity of the related generating functions. The full proof of Theorem 8.3 relies on the results established in Theorem 6.3, and the technique of proof is much the same.

4. The q-difference equations. In [4], the relevant q-difference equations were quoted from [3] where they are proved in the full generality of general basic hypergeometric series. By restricting attention to only the special case required, we may deduce the relevant q-difference equations in a slightly more straightforward manner. We are concerned with the following q-series:

(4.1)

$$H_{\lambda,k,i}(x) = \frac{(-xq)_\infty}{(x^2q^{2\lambda+2};q^{2\lambda+2})_\infty} \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(\lambda+1)((2k-\lambda+1)n^2+(2k-2i+1)n)}$$

$$\cdot \frac{(1-x^i q^{(2n+1)i(\lambda+1)})(x^2q^{2\lambda+2};q^{2\lambda+2})_n(-q)_{\lambda n+n}}{(q^{2\lambda+2};q^{2\lambda+2})_n(-xq)_{\lambda n+\lambda+n}} ;$$

(4.2)

$$J_{\lambda,k,i}(x) = \frac{(-xq)_\infty}{(x^2q^{2\lambda+2};q^{2\lambda+2})_\infty} \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(\lambda+1)((2k-\lambda+1)n^2+(2k-2i+1)n)}$$

$$\cdot \frac{(x^2q^{2\lambda+2};q^{2\lambda+2})_n(-q)_{\lambda n+n}}{(q^{2\lambda+2};q^{2\lambda+2})_n(-xq)_{\lambda n+n}} \left(1 - \frac{x^i q^{\frac{1}{2}(\lambda+1)(2n+1)(2i-\lambda)}(-q^{\lambda n+n+1})_\lambda}{(-xq^{n\lambda+n+1})_\lambda}\right)$$

where $(a;q)_n = (a)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$, and $(a;q)_\infty = (a)_\infty = \lim_{n\to\infty} (a;q)_n$.

In order to prove the relevant q-difference equations (i.e. (3.1)-(3.3)) for these functions, we shall need the following identity due to Cauchy [12; Th. 348, p. 280]:

$$(A;q)_\lambda = \sum_{j=0}^{\infty} (-1)^j q^{\frac{1}{2}j(j-1)} \left[{\lambda \atop j}\right] A^j ,$$

Where $\left[{\lambda \atop j}\right]$ is defined in (3.4).

Theorem 4.1. Equations (3.1), (3.2), and (3.3) hold for the functions $H_{\lambda,k,i}(x)$ and $J_{\lambda,k,i}(x)$ defined by equations (4.1) and (4.2) respectively.

Proof. We begin by proving (3.1).

$$H_{\lambda,k,i}(x)-H_{\lambda,k,i-1}(x) =$$

$$\frac{(-xq)_\infty}{(x^2q^{2\lambda+2};q^{2\lambda+2})_\infty} \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(\lambda+1)((2k-\lambda+1)n^2+(2k+1)n)}$$

$$\cdot \frac{(x^2q^{2\lambda+2};q^{2\lambda+2})_n(-q)_{\lambda n+n}}{(q^{2\lambda+2};q^{2\lambda+2})_n(-xq)_{\lambda n+\lambda+n}} (q^{-in(\lambda+1)}-x^iq^{i(n+1)(\lambda+1)}-q^{-(i-1)n(\lambda+1)}$$

$$+ x^{i-1}q^{(i-1)(n+1)(\lambda+1)}).$$

Now

$$q^{-in(\lambda+1)}-xq^{i(n+1)(\lambda+1)}-q^{-(i-1)n(\lambda+1)}+x^{i-1}q^{(i-1)(n+1)(\lambda+1)}$$

$$= \{q^{-in(\lambda+1)}(1-q^{n(\lambda+1)})\} + \{x^{i-1}q^{(i-1)(n+1)(\lambda+1)}(1-xq^{(n+1)(\lambda+1)})\}.$$

$$= E_1(n) + E_2(n).$$

Therefore we may write $H_{\lambda,k,i}(x)-H_{\lambda,i,i-1}(x)$ as the sum of two series, the first containing $E_1(n)$ as a factor of the n^{th} term and the second containing $E_2(n)$. Hence after some algebraic simplification we find that

$$H_{\lambda,k,i}(x)-H_{\lambda,k,i-1}(x) = \frac{(-xq)_\infty}{(x^2q^{2\lambda+2};q^{2\lambda+2})}$$

$$\sum_{n=1}^{\infty} \frac{(-1^n x^{kn} q^{\frac{1}{2}(\lambda+1)((2k-\lambda+1)n^2+(2k-2i+1)n)} (x^2q^{2\lambda+2};q^{2\lambda+2})_n(-q)_{\lambda n+n-1}}{(q^{2\lambda+2};q^{2\lambda+2})_{n-1}(-xq)_{\lambda n+\lambda+n}}$$

$$+ \frac{(xq^{\lambda+1})^{i-1}(-xq)_\infty}{(x^2q^{2\lambda+2};q^{2\lambda+2})_\infty}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{kn} q^{\frac{1}{2}(\lambda+1)((2k-\lambda+1)n^2+(2k+2i-1)n)} (x^2q^{2\lambda+2};q^{2\lambda+2})_{n+1}(-q)_{\lambda n+n}}{(q^{2\lambda+2};q^{2\lambda+2})_n(-xq)_{\lambda n+\lambda+n+1}}$$

We now replace n by $n+1$ in the first of these series and then combine the series term by term. Therefore after some further algebraic simplification, we obtain that

$$H_{\lambda,k,i}(x)-H_{\lambda,k,i-1}(x)$$

$$= \frac{(xq^{\lambda+1})^{i-1}(-xq^{\lambda+2})_\infty}{(x^2q^{4\lambda+4};q^{2\lambda+2})_\infty} \sum_{n=0}^{\infty} (-1)^n (xq^{\lambda+1})^{kn} q^{\frac{1}{2}(\lambda+1)((2k-\lambda+1)n^2+(2i-1)n)}$$

$$\cdot \frac{(x^2q^{4\lambda+4};q^{2\lambda+2})_n(-q)_{\lambda n+n}}{(q^{2\lambda+2};q^{2\lambda+2})_n(-xq^{\lambda+2})_{\lambda n+n}} \left(1- \frac{(xq^{\lambda+1})^{k-i+1} q^{\frac{1}{2}(\lambda+1)(2n+1)(2k-2i+2-\lambda)} (-q^{\lambda n+n+1})_\lambda}{(-xq^{\lambda n+\lambda+n+2})_\lambda}\right)$$

$$= (xq^{\lambda+1})^{i-1} J_{\lambda,k,k-i+1}(xq^{\lambda+1}),$$

and thus (3.1) is proved for these functions.

Equation (3.2) is somewhat less difficult.

$$J_{\lambda,k,i}(x) = \frac{(-xq)_\infty}{(x^2q^{2\lambda+2};q^{2\lambda+2})} \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(\lambda+1)((2k-\lambda+1)n^2+(2k-2i+1)n)}$$

$$\cdot\frac{(x^2q^{2\lambda+2};q^{2\lambda+2})_n(-q)_{\lambda n+n}}{(q^{2\lambda+2};q^{2\lambda+2})_n(-xq)_{\lambda n+n+\lambda}}((-xq^{n\lambda+n+1})_\lambda - x^i q^{\frac{1}{2}(\lambda+1)(2n+1)(2i-\lambda)}(-q^{\lambda n+n+1})_\lambda)$$

$$= \frac{(-xq)_\infty}{(x^2q^{2\lambda+2};q^{2\lambda+2})_\infty} \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(\lambda+1)((2k-\lambda+1)n^2+(2k-2i+1)n)}$$

$$\cdot\frac{(x^2q^{2\lambda+2};q^{2\lambda+2})_n(-q)_{\lambda n+n}}{(q^{2\lambda+2};q^{2\lambda+2})_n(-xq)_{\lambda n+n+\lambda}}$$

$$\cdot(\sum_{j=0}^{\lambda} x^j q^{\frac{1}{2}j(j+1)+j(n\lambda+n)}\left[{\lambda \atop j}\right] - \sum_{j=0}^{\lambda} x^i q^{\frac{1}{2}(\lambda-j)(\lambda-j+1)+(\lambda-j)(\lambda n+n)+\frac{1}{2}(\lambda+1)(2n+1)(2i-\lambda)}\left[{\lambda \atop j}\right])$$

(by Cauchy's identity)

$$= \sum_{j=0}^{\lambda} x^j q^{\frac{1}{2}j(j+1)}\left[{\lambda \atop j}\right]\frac{(-xq)_\infty}{(x^2q^{2\lambda+2};q^{2\lambda+2})} \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(\lambda+1)((2k-\lambda+1)n^2+(2k-2(i-j)+1)n)}$$

$$\frac{(x^2q^{2\lambda+2};q^{2\lambda+2})_n(-q)_{\lambda n+n}}{(q^{2\lambda+2};q^{2\lambda+2})_n(-xq)_{\lambda n+n+\lambda}}(1 - x^{i-j}q^{(2n+1)(i-j)(\lambda+1)})$$

$$= \sum_{j=0}^{\lambda} x^j q^{\frac{1}{2}j(j+1)}\left[{\lambda \atop j}\right]H_{\lambda,k,i-j}(x).$$

Hence (3.2) is established.

Finally we remark that (3.3) follows directly from the simple identity

$$q^{in(\lambda+1)}-x^{-i}q^{-i(n+1(\lambda+1)} = -x^{-i}q^{-i(\lambda+1)}(q^{-in(\lambda+1)}-x^{i}q^{i(n+1)(\lambda+1)}).$$

Therefore Theorem 4.1 is proved. □

Next we shall prove that (subject to simple boundary conditions) the functions satisfying (3.1), (3.2), and (3.3) are unique. This result is equivalent to Lemma 3.1 of [4] but is included here for completeness.

Theorem 4.2. Let $|q| < 1$ set $\sigma_j(\lambda) = q^{\frac{1}{2}j(j+1)}\left[{\lambda \atop j}\right]$ and let $H^{*}_{\lambda,k,i}(x)$ $(0 \leq i \leq k)$ and $J^{*}_{\lambda,k,i}(x)$ $(1 \leq i \leq k)$ be any $2k+1$ functions analytic in x in the neighborhood of 0 that satisfy

$$(4.3)\qquad H^{*}_{\lambda,k,i}(x)-H^{*}_{\lambda,k,i-1}(x) = (xq^{\lambda+1})^{i-1}J^{*}_{\lambda,k,k-i+1}(xq^{\lambda+1}),\quad 1 \leq i \leq k,$$

$$(4.4)\qquad H^{*}_{\lambda,k,0}(x) = 0,$$

$$(4.5)\qquad J^{*}_{\lambda,k,i}(x) = \sum_{j=0}^{i} x^{j}\sigma_j(\lambda)H^{*}_{\lambda,k,i-j}(x)$$

$$- x^{i}\sum_{j=i+1}^{\lambda} q^{(\lambda+1)(i-j)}\sigma_j(\lambda)H^{*}_{\lambda,k,j-i}(x),$$

where the second sum is zero if $i \geq \lambda$,

$$(4.6)\qquad H^{*}_{\lambda,k,i}(0) = J^{*}_{\lambda,k,i}(0) = 1,\qquad 1 \leq i \leq k.$$

Then $H^*_{\lambda,k,i}(x) = H_{\lambda,k,i}(x)$ and $J^*_{\lambda,k,i}(x) = J_{\lambda,k,i}(x)$, where $H_{\lambda,k,i}(x)$ and $J_{\lambda,k,i}(x)$ are defined by (4.1) and (4.2) respectively.

Proof. We first establish that the $H_{\lambda,k,i}(x)$ and $J_{\lambda,k,i}(x)$ do in fact fulfill the conditions of the theorem. The analyticity condition follows directly from the defining equations (4.1) and (4.2) as does (4.6). Equation (4.3) is the same as (3.1), and equation (4.4) follows from (3.3) by setting $i = 0$. Finally by (3.2) and (3.3), we see that

$$\begin{aligned} J_{\lambda,k,i}(x) &= \sum_{j=0}^{\lambda} x^j \sigma_j(\lambda) H_{\lambda,k,i-j}(x) \\ &= \sum_{j=0}^{i} x^j \sigma_j(\lambda) H_{\lambda,k,i-j}(x) \\ &\quad + \sum_{j=i+1}^{\lambda} x^j \sigma_j(\lambda) H_{\lambda,k,i-j}(x) \\ &= \sum_{j=0}^{i} x^j \sigma_j(\lambda) H_{\lambda,k,i-j}(x) \\ &\quad - x^i \sum_{j=i+1}^{\lambda} q^{(\lambda+1)(i-j)} \sigma_j(\lambda) H_{\lambda,k,j-i}(x) \end{aligned}$$

Thus to complete the proof of this theorem we need only show that the solution set of these equations is unique. From (4.3) we see that we need only show that the $H^*_{\lambda,k,i}(x)$ are unique, since this equation defines the $J^*_{\lambda,k,i}(x)$ in terms of the $H^*_{\lambda,k,i}(x)$.

Combining (4.3) and (4.5), we see that

$$(4.7)\qquad H^*_{\lambda,k,i}(x) - H^*_{\lambda,k,i-1}(x) =$$

$$(xq^{\lambda+1})^{i-1}\sum_{j=0}^{k-i+1} x^j q^{j(\lambda+1)}\sigma_j(\lambda)H^*_{\lambda,k,k-i+1-j}(xq^{\lambda+1})$$

$$-\ (xq^{\lambda+1})^k \sum_{j=k-i+2}^{\lambda} q^{(\lambda+1)(k-i+1-j)}\sigma_j(\lambda)H^*_{\lambda,k,j-k+i-1}(xq^{\lambda+1}).$$

By our analyticity assumption, we may write

$$H^*_{\lambda,k,i}(x) = \sum_{n=0}^{\infty} \eta_n(i)x^n,$$

where by (4.6) $\eta_0(i) = 1$ for $1 \le i \le k$, $\eta_0(0) = 0$. By (4.7), we obtain that

$$(4.8)\qquad \eta_n(i) - \eta_n(i-1) = \sum_{j=0}^{k-i+1} q^{n(\lambda+1)}\sigma_j(\lambda)\eta_{n-j-i+1}(k-i+1-j)$$

$$-\sum_{j=k-i+2}^{\lambda} q^{(\lambda+1)(k-i+1-j)}\sigma_j(\lambda)\eta_{n-k}(j-k+i-1),$$

for $1 \le i \le k$.

We now observe that the subscripts of the $\eta_A(B)$ on the right-hand side of (4.8) are all $< n$ except for the case $j = 0$, $i = 1$. Hence we may rewrite the system (4.8) in the following abbreviated manner:

$$\eta_n(1)-q^{n(\lambda+1)}\eta_n(k) = \text{linear combination of } \eta_j(B) \text{ with } j < n$$

$$\eta_n(2)-\eta_n(1) = \text{linear combination of } \eta_j(B) \text{ with } j < n$$

$$\vdots$$

$$\eta_n(k)-\eta_n(k-1) = \text{linear combination of } \eta_j(B) \text{ with } j < n.$$

Thus we may solve for each $\eta_n(i)$ in terms of $\eta_A(B)$ with $A < n$ since the determinant of this system is $1-q^{n(\lambda+1)} \neq 0$ (because $|q| < 1$). Therefore the $\eta_n(i)$ are seen to be unique by mathematical induction. Hence the $H^*_{\lambda,k,i}(x)$ and therefore the $J^*_{\lambda,k,i}(x)$ must be unique. Therefore $H^*_{\lambda,k,i}(x) = H_{\lambda,k,i}(x)$, and $J^*_{\lambda,k,i}(x) = J_{\lambda,k,i}(x)$. □

5. **The auxiliary partition functions.** We must now consider auxiliary partition functions similar to those considered in Section 2 of [4]. Throughout this section, however, we shall only assume $k \geq \lambda$ and this will greatly increase the complexity of our considerations.

Definition 4. Let $\pi(A_0,A_1,\ldots,A_{\ell-1};D_1,\ldots,D_{\ell-1};k;\lambda;n) = \pi(\{A\}_\ell;\{D\}_\ell;k;\lambda;n)$ denote the number of partitions of n of the form $f_1\cdot 1+f_2\cdot 2+\cdots+f_{\ell\lambda+\ell-1}\cdot(\ell\lambda+\ell-1)$ (hence f_i is the number of times the summand i appears) where

(5.1) $$f_{c\lambda+c+1}+\cdots+f_{c\lambda+c+\lambda} = A_c, \quad 0 \leq c \leq \ell-1;$$

(5.2) $$k \geq f_{c\lambda+c} = k-D_c, \quad 1 \leq c \leq \ell-1,$$

(5.3) $f_m + \ldots + f_{m+\lambda} \geq k$ for some m in each of the $\ell-1$ closed intervals

$[1,\lambda+1]$, $[\lambda+2, 2\lambda+2], \ldots, [(\ell-2)(\lambda+1)+1, (\ell-1)(\lambda+1)]$;

(5.4) $f_a > 1$ implies $(\lambda+1) \mid a$.

Definition 5.

$$g(\{A\}_\ell; \{D\}_\ell; k; \lambda; q) = \sum_{n\geq 0} \pi(\{A\}_\ell; \{D\}_\ell; k; \lambda; n) q^n.$$

Theorem 5.1. Let ℓ be an integer ≥ 1. If $A_0, A_1, \ldots, A_{\ell-1}$, $D_1, D_2, \ldots D_{\ell-1}$ are integers satisfying $A_j \leq D_j + D_{j+1} - \lambda$, $1 \leq j \leq \ell-2$, $A_0 \leq D_1$, and if $\ell > 1$ $A_{\ell-1} \leq D_{\ell-1}$, then

$$g(\{A\}_\ell; \{D\}_\ell; k; \lambda; q) = G(\{A\}_\ell; \{D\}_\ell; k; \lambda; q),$$

where

(5.5) $$G(\{A\}_\ell; \{D\}_\ell; k; \lambda; q)$$

$$= q^{\sum_{b=1}^{\ell-1} (k-D_b+A_b) b(\lambda+1)} \sigma_{D_1}(\lambda) \sigma_{D_2}(\lambda) \ldots \sigma_{D_{-1}}(\lambda) \sigma_{A_0+A_1+\ldots+A_{\ell-1}-D_1-\ldots D_{\ell-1}}(\lambda).$$

Proof. We may assume throughout the proof that each $D_b \leq \lambda$, for if $D_b > \lambda$ then $G(\{A\}_\ell; \{D\}_\ell; k; \lambda; q)$ is trivially zero since $\sigma_{D_b}(\lambda) = 0$, and $g(\{A\}_\ell; \{D\}_\ell; k; \lambda; q) = 0$, since for $m \in [(b-1)(\lambda+1)+1, b(\lambda+1)]$ by (5.2) and (5.4)

$$f_m+\ldots+f_{m+\lambda} \leq \lambda+k-D_b < k$$

and so no partitions exist fulfilling both (5.3) and (5.4) in this case.

We proceed by a double mathematical induction on ℓ and λ.

First we consider $\lambda = 0$. In this case, we see that there is only one possible partition to be enumerated by $\pi(\{A\}_\ell;\{D\}_\ell;k;\lambda;n)$ and it is $\sum_{j=1}^{\ell-1} f_i\cdot j$, where $f_c = k-D_c$ for each c, and in order that this partition satisfy (5.1)-(5.4) we must have $A_0 = 0$ and $A_c = D_c = 0$ for each $c \geq 1$. Thus the only possible partition is $k\cdot 1+k\cdot 2+\ldots+k(\ell-1) = \frac{1}{2}k\ell(\ell-1)$. Therefore

$$g(\{A\}_\ell;\{D\}_\ell;k;0;q) = \begin{cases} q^{\frac{1}{2}k\ell(\ell-1)} & \text{if } A_0=A_1=\ldots=A_{\ell-1}=D_1=D_2=\ldots=D_{\ell-1}=0 \\ 0 & \text{in all other admissible cases.} \end{cases}$$

Now let us examine $G(\{A\}_\ell;\{D\}_\ell;k;0;q)$. In order that $\sigma_j(0) \neq 0$ we must have $j = 0$; therefore $G(\{A\}_\ell;\{D\}_\ell;k;0;q)$ is zero unless $D_1 = D_2\ldots=D_{\ell-1}=0$ and $A_0+A_1+\ldots+A_{\ell-1} = 0$. But since this implies each $A_i \leq 0$ for $0 \leq i \leq \ell-1$ (by conditions of our theorem) we must have $A_0=A_1=\ldots=A_{\ell-1}=0$ also. Hence

$$G(\{A\}_\ell;\{D\}_\ell;k;\lambda;q) = \begin{cases} q^{\frac{1}{2}k\ell(\ell-1)} & \text{if } A_0=A_1=\ldots=A_{\ell-1}=D_1=\ldots=D_{\ell-1}= 0 \\ 0 & \text{in all other admissable cases.} \end{cases}$$

Hence the theorem is true for $\lambda = 0$.

Next we consider the other initial case $\ell=1$. From Definition 4 we see that since (5.2), (5.3), and (5.4) are vacuous, $\pi(A_0;;k;\lambda;n)$ is just the number of partitions of n into A_0 distinct parts each $\leq \lambda$. Therefore

$$g(A_0;;k;\lambda;q) = \sigma_{A_0}(\lambda)$$

$$= G(A_0;;k;\lambda;q),$$

since $\sigma_{A_0}(\lambda) = q^{\frac{1}{2}A_0(A_0+1)}\left[{\lambda \atop A_0}\right]$ is the generating function for partitions into A_0 distinct parts each $\leq \lambda$. [12; Th. 348, p. 280]. Thus the theorem is true for $\ell = 1$.

For the induction step we shall use the following notation: $\{A-\varepsilon\}_\ell$ denote $\{A_0-\varepsilon_0, A_1-\varepsilon_1, \ldots, A_{\ell-1}-\varepsilon_{\ell-1}\}$, and $\{D-\varepsilon\}_\ell$ denotes $\{D_1-\varepsilon_1, D_2-\varepsilon_2, \ldots, D_{\ell-1}-\varepsilon_{\ell-1}\}$.

We now prove a recurrence satisfied by both $G(\{A\}_\ell;\{D\}_\ell;k;\lambda;q)$ and $g(\{A\}_\ell;\{D\}_\ell;k;\lambda;q)$ that will provide the passage from $\lambda-1$ to λ, namely:

$$(5.6)\qquad G(\{A\}_\ell;\{D\}_\ell;k;\lambda;q) = q^{\sum_{j=1}^{\ell-1}(k-D_j+A_{j-1})j+\ell A_{\ell-1}}$$

$$\cdot \sum_{\varepsilon_0=0}^{1}\sum_{\varepsilon_1=0}^{1}\cdots\sum_{\varepsilon_{\ell-1}=0}^{1} G(\{A-\varepsilon\}_\ell;\{D-\varepsilon\}_\ell;k;\lambda-1;q).$$

First we prove that G satisfies (5.6). We recall that [13; p. 85]

$$(5.7)\qquad \left[{\lambda \atop j}\right] = \left[{\lambda-1 \atop j-1}\right] + q^j\left[{\lambda-1 \atop j}\right];$$

therefore

$$(5.8) \qquad \sigma_j(\lambda) = q^j(\sigma_{j-1}(\lambda-1) + \sigma_j(\lambda-1)).$$

$$= q^j \sum_{\varepsilon=0}^{1} \sigma_{j-\varepsilon}(\lambda-1).$$

We now apply (5.8) to each of the ℓ σ-functions appearing as factors in (5.5). Hence

$$G(\{A\}_\ell;\{D\}_\ell;k;\lambda;q)$$

$$= \sum_{\varepsilon_0=0}^{1} \sum_{\varepsilon_1=0}^{1} \cdots \sum_{\varepsilon_{\ell-1}=0}^{1} q^{\sum_{b=1}^{\ell-1}(k-D_b+A_b)b(\lambda+1)} \sigma_{D_1-\varepsilon_1}(\lambda-1)\sigma_{D_2-\varepsilon_2}(\lambda-1)\cdots\sigma_{D_{\ell-1}-\varepsilon_{\ell-1}}(\lambda-1)$$

$$\cdot\ \sigma_{A_0-\varepsilon_0+A_1+\cdots+A_{\ell-1}-D_1-D_2-\cdots-D_{\ell-1}}(\lambda-1)$$

$$\cdot\ q^{A_0+A_1+\cdots+A_{\ell-1}}$$

$$= q^{\sum_{b=1}^{\ell-1}(k-D_b+A_b)b + \sum_{b=1}^{\ell} A_{b-1}}$$

$$\cdot \sum_{\varepsilon_0=0}^{1} \sum_{\varepsilon_1=0}^{1} \cdots \sum_{\varepsilon_{\ell-1}=0}^{1} q^{\sum_{b=1}^{\ell-1}(k-(D_b-\varepsilon_b)+A_b-\varepsilon_b)b\lambda} \sigma_{D_1-\varepsilon_1}(\lambda-1)\ {}_{D_2-\varepsilon_2}(\lambda-1)\cdots\ {}_{D_{\ell-1}-\varepsilon_{\ell-1}}(\lambda-1)$$

$$\cdot\ {}_{A_0-\varepsilon_0+A_1-\varepsilon_1+A_2-\varepsilon_2+\cdots+A_{\ell-1}-\varepsilon_{\ell-1}-(D_1-\varepsilon_1)-\cdots-(D_{\ell-1}-\varepsilon_{\ell-1})}(\lambda-1)$$

$$= q^{\sum_{b=1}^{\ell-1}(k-D_b+A_{b-1})b+\ell A_{\ell-1}}$$

$$\sum_{\varepsilon_0=0}^{1}\sum_{\varepsilon_1=0}^{1}\cdots\sum_{\varepsilon_{\ell-1}=0}^{1} G(\{A-\varepsilon\}_\ell;\{D-\varepsilon\}_\ell;k;\lambda-1;q).$$

Thus we see that $G(\{A\}_\ell;\{D\}_\ell;k;\lambda;q)$ satisfies (5.6).

Next we wish to show that $g(\{A\}_\ell;\{D\}_\ell;k;\lambda;q)$ satisfies the same functional equation. We shall prove the equivalent relation for the coefficients involved; namely

(5.9) $$\pi(\{A\}_\ell;\{D\}_\ell;k;\lambda;n)$$

$$= \sum_{\varepsilon_0=0}^{1}\sum_{\varepsilon_1=0}^{1}\cdots\sum_{\varepsilon_{\ell-1}=0}^{1} \pi(\{A-\varepsilon\}_\ell;\{D-\varepsilon\}_\ell;k;\lambda-1;n-\sum_{j=1}^{\ell-1}(k-D_j+A_{j-1})j-\ell A_{\ell-1}).$$

Tp prove (5.9) we transform the partitions enumerated by $\pi(\{A\}_\ell;\{D\}_\ell;k;\lambda;n)$ as follows: we subtract 1 from each summand lying in $[1,\lambda+1]$; 2 from each summand lying in $[\lambda+2,2\lambda+2]$, and in general, j from each summand lying in $[(j-1)(\lambda+1)+1,j\lambda+j]$. Referring to equations (5.1) and (5.2), we see that the number being partitioned is now reduced to $n-\sum_{j=1}^{\ell-1}(k-D_j+A_{j-1})j-\ell A_{\ell-1}$. We now consider 2^ℓ different disjoint classes of partitions depending upon the relations

(5.10) $$\varepsilon_0 = f_1;\varepsilon_1 = f_{\lambda+2};\varepsilon_2 = f_{2\lambda+3};\ldots;\varepsilon_{\ell-1} = f_{(\ell-1)(\lambda+1)+1},$$

where $\varepsilon_j = 0,1$ for $0 \le j \le \ell-1$.

In a particular class as given by (5.10), we see that for the transformed partitions each A_j is replaced by $A_j-\varepsilon_j$; furthermore the frequency of appearance of $j\lambda$ is now $k-D_j+\varepsilon_j$, i.e. D_j is replaced by $D_j-\varepsilon_j$, and finally we remark that the role played by λ in the original partitions is now played by $\lambda-1$.

Since the above transformation is obviously reversible, it therefore establishes a bijection between the partitions enumerated by $\pi(\{A\}_\ell;\{D\}_\ell;k;\lambda;n)$ and the partitions enumerated by the totality of the 2^ℓ partition functions

$$\pi(\{A-\varepsilon\}_\ell;\{D-\varepsilon\}_\ell;k;\lambda-1;n-\sum_{j=1}^{\ell-1}(k-D_j+A_{j-1})j-\ell A_{\ell-1}).$$

This is equivalent to the assertion that (5.9) is valid. Therefore (5.6) holds for $g(\{A\}_\ell;\{D\}_\ell;k;\lambda;q)$.

The following recurrence which holds for both $G(\{A\}_\ell;\{D\}_\ell;k;\lambda;q)$ and $g(\{A\}_\ell;\{D\}_\ell;k;\lambda;q)$ supplies the necessary passage from $\ell-1$ to ℓ. If the conditions $A_j \le D_j+D_{j+1}-\lambda$, $1 \le j \le \ell-2$, $A_0 = D_1, A_{\ell-1} \le D_{\ell-1}$, $\ell > 1$, hold, then

(5.11)
$$g(D_1,A_1,\ldots,A_{\ell-1},D_1,\ldots,D_{\ell-1};k;\lambda;q)$$

$$=\begin{cases}\sigma_{D_1}(\lambda)q^{\sum_{b=1}^{\ell-1}(k-D_b+A_b)(\lambda+1)} g(\{A'\}_{\ell-1};\{D'\}_{\ell-1};k;\lambda;q) & \text{if } 0 \le D_1 = \lambda \\ 0 \quad \text{if } D_1 > \lambda \quad \text{or} \quad D_1 < 0, \end{cases}$$

where

$$\{A'\}_{\ell-1} = \{A'_0, A'_1, \ldots, A'_{\ell-2}\}$$
$$= \{A_1, \ldots, A_{\ell-1}\}$$

and

$$\{D'\}_{\ell-1} = \{D'_1, D'_2, \ldots D'_{\ell-1}\}$$
$$= \{D_2, \ldots, D_{\ell-1}\}.$$

First we prove (5.11) for g. By (5.3) with $m \in [1,\lambda+1]$

$$k \leq f_m+\ldots+f_{m+\lambda} \leq k-D_1+\lambda,$$

or

$$D_1 \leq \lambda,$$

and by (5.2) $D_1 \geq 0$.

Thus the lower line of (5.11) must obtain if $D_1 > \lambda$ or $D_1 < 0$.

Since $D_1 = A_0$, then

$$f_1+\ldots+f_{\lambda+1} = A_0+k-D_1 = k$$

and so (5.3) is automatic when $m \in [1,\lambda+1]$. We now transform the partitions under consideration by deleting the A_0 $(=D_1)$ parts belonging to $[1,\lambda]$, deleting the $k-D_1$ appearances of $\lambda+1$, and subtracting $\lambda+1$ from each of the remaining summands. This transformation produces a bijection between partitions enumerated by $\pi(D_1,A_1,\ldots,A_{\ell-1};\{D\}_\ell;k;\lambda;n)$ and those enumerated by

$$\sum_{j\geq 0} \pi(A_0;;k;\lambda;j) \cdot \pi(A_1,\ldots,A_{\ell-1};D_2,\ldots,D_{\ell-1};k;\lambda;n-j-\sum_{b=1}^{\ell-1}(k-D_b+A_b)(\lambda+1))$$

Hence for $0 \leq D_1 \leq \lambda$

$$g(D_1,A_1,\ldots,A_{\ell-1};D_1,\ldots,D_{\ell-1};k;\lambda;q) = \sigma_{D_1}(\lambda)q^{\sum_{b=1}^{\ell-1}(k-D_b+A_b)(\lambda+1)} g(A_1,\ldots,A_{\ell-1};D_2,\ldots,D_{\ell-1};k;\lambda;q),$$

which is the top line of (5.11).

Now we prove (5.11) for G. We note immediately from (5.5) that

$$G(\{A\}_\ell;\{D\}_\ell;k;\lambda;q) = 0$$

if either $D_1 < 0$ or $D_1 > \lambda$ since in these cases $\sigma_{D_1}(\lambda) = 0$. If $0 \leq D_1 \leq \lambda$, then

$$G(D_1,A_1,\ldots,A_{\ell-1};\{D\}_\ell;k;\lambda;q) = \sigma_{D_1}(\lambda)q^{\sum_{b=1}^{\ell-1}(k-D_b+A_b)b(\lambda+1)} \sigma_{D_2}(\lambda)\ldots\sigma_{D_{\ell-1}}(\lambda)\sigma_{A_1+\ldots+A_{\ell-1}-D_2-\ldots-D_{\ell-1}}(\lambda)$$

$$= \sigma_{D_1}(\lambda) q^{\sum_{b=1}^{\ell-1} (k-D_b+A_b)(\lambda+1)} G(\{A'\}_{\ell-1};\{D'\}_{\ell-1};k;\lambda;q),$$

as desired. Hence (5.11) holds for both g and G.

It is now an easy matter to finish the double induction on ℓ and λ. We have already established the theorem for $\lambda = 0$ and also for $\ell = 1$. If we assume the truth of the theorem for all (ℓ',λ') such that either $\lambda' < \lambda$ or $\lambda' = \lambda$, $\ell' < \ell$, then we see that the right-hand side of (5.6) is unaltered when "G" is replaced by "g", provided $A_0 < D_1$. To see this we must check that the $\{A-\varepsilon\}_\ell$ and $D-\varepsilon\}_\ell$ all satisfy the conditions of the theorem: first $A_j \leq D_j+D_{j+1}-\lambda$ implies $A_j-\varepsilon_j \leq D_j-\varepsilon_j+D_{j+1}-\varepsilon_{j+1}-(\lambda-1)$ since $\varepsilon_{j+1} = 0$ or 1, and if originally $A_0 < D_1$, then $A_0-\varepsilon_0 \leq D_1-\varepsilon_1$ since $\varepsilon_1-\varepsilon_0 \leq 1$, also $A_{\ell-1}-\varepsilon_{\ell-1} \leq D_{\ell-1}-\varepsilon_{\ell-1}$. Note that the assumption $A_0 < D_1$ is clearly critical to our use of (5.6). Thus to complete the induction we must establish the theorem for ℓ and λ when $A_0 = D_1$. In this case we turn to (5.11) and we see that the right hand sides are identical by the induction hypothesis for $\ell-1$ and the fact that the conditions on the $\{A'\}_{\ell-1}$ and $\{D'\}_{\ell-1}$ are exactly as they were on the $\{A\}_\ell$ and $\{D\}_\ell$ except for $A_0' \leq D_1'$ which is easily established since $A_0' = A_1 \leq D_1+D_2-\lambda \leq D_2$ for $D_1 \leq \lambda$ may be assumed by the first paragraph of this proof. Hence by (5.11) (and the induction hypothesis with $\lambda' = \lambda$, $\ell' = \ell-1$)

$$g(\{A\}_\ell;\{D\}_\ell;k;\lambda;q) = G(\{A\}_\ell;\{D\}_\ell;k;\lambda;q)$$

when $A_0 = D_1$. Thus in all cases the truth of the theorem for (ℓ',λ') with either $\lambda' < \lambda$ or $\lambda' = \lambda$, $\ell < \ell'$ implies the truth of the theorem for (ℓ,λ). Thus by a double mathematical induction first on λ and then on ℓ

we see that Theorem 5.1 is valid.

6. The general theorem for $a \geqq \lambda$. The arguments in this section will follow the outline presented in Section 3.

Theorem 6.1. Let $|q| < 1$, $|x| < |q|^{-1}$, then the functions $H^{\dagger}_{\lambda,k,i}(x)$ and $J^{\dagger}_{\lambda,k,i}(x)$ defined by (3.6) and (3.7) are analytic in x in this region; they satisfy (3.8) and (3.9), with

$$(6.1) \qquad g_1(\ell;k,\lambda,r,i;x;q) = \sum_{C} x^{k(\ell-1)+A_0+A_1+\dots+A_{\ell-1}-D_1-D_2-\dots-D_{\ell-1}} g(\{A\}_\ell;\{D\}_\ell;k;\lambda;q),$$

where C is the set of those $(2\ell-1)$-tuples $(A_0,A_1,\dots,A_{\ell-1},D_1,D_2,\dots,D_{\ell-1})$ that satisfy

$$(6.2)_1 \qquad A_0+k-D_1 \leq i-1,$$

$$(6.2)_2 \qquad k-D_{\ell-1}+A_{\ell-1} = k-r$$

$$(6.2)_3 \qquad 0 \leq A_0 \leq \lambda,$$

$$(6.2)_4 \qquad 1 \leq D_j \leq k, \quad (1 \leq j \leq \ell-1),$$

$$(6.2)_5 \qquad 0 \leq A_j \leq D_j, \quad (1 \leq j \leq \ell-2),$$

$(6.2)_6$ $$k-D_j+A_j+k-D_{j+1} \leq k-1, \quad (1 \leq j \leq \ell-2);$$

furthermore

(6.3) $$H^{\dagger}_{\lambda,k,0}(x) = 0,$$

(6.4) $$H^{\dagger}_{\lambda,k,i}(0) = J^{\dagger}_{\lambda,k,i}(0) = 1, \quad 1 \leq i \leq k.$$

Proof. We note that (6.3) is immediate because no partition can have $\lambda+1$ occur ≤ -1 times. As for (6.4), we observe that the only partition with zero parts is the empty partition of zero, a partition which is enumerated by both $Q_{\lambda,k,i}(0,0)$ and $P_{\lambda,k,i}(0,0)$ for $1 \leq i \leq n$. Consequently (6.4) holds.

Comparing coefficients on each side of (3.8), we see that we may equivalently prove that

(6.5) $$Q_{\lambda,k,i}(m,n)-Q_{\lambda,k,i-1}(m,n) = P_{\lambda,k,k-i+1}(m-i+1,n-m(\lambda+1)).$$

Now $Q_{\lambda,k,i}(m,n)-Q_{\lambda,k,i-1}(m,n)$ enumerates the partition of n into m parts of the type enumerated by $Q_{\lambda,k,i}(m,n)$ with the added restrictions that all parts are $\geq \lambda+1$, and that $\lambda+1$ appears as a part exactly $i-1$ times. We now delete the $i-1$ copies of $\lambda+1$ from each partition under consideration, and we subtract $\lambda+1$ from the remaining parts. The number being partitioned is reduced to $n-m(\lambda+1)$, and the number of parts is reduced to $m-i+1$. Since originally

$$f_{\lambda+2}+\ldots+f_{2\lambda+2} \leqq k-1-f_{\lambda+1} = k-i = (k-i+1)-1,$$

we find that after the above trasformation we have

$$f_1+f_2+\ldots+f_{\lambda+1} \leqq (k-i+1)-1.$$

Otherwise the conditions on the summands are just shifted by unaltered; consequently the transformed partitions are of the type enumerated by

$$P_{\lambda,k,k-i+1}(m-i+1,n-m(\lambda+1)).$$

Since the above is clearly reversible, we see that we have established a bijection between the partitions enumerated by $Q_{\lambda,k,i}(m,n)-Q_{\lambda,k,i-1}(m,n)$ and those enumerated by $P_{\lambda,k,k-i+1}(m-i+1,n-m(\lambda+1))$. This establishes (6.5), and consequently (3.8) is established.

We now treat equation (3.9). First we consider

$$(6.6) \quad \left\{\sum_{j=0}^{i} x^j q^{\frac{1}{2}j(j+1)} \left[{\lambda \atop j}\right] H^{\dagger}_{\lambda,k,i-j}(x)\right\} J^{\dagger}_{\lambda,k,i}(x)$$
$$= \left\{\sum_{j=0}^{i} x^j \sigma_j(\lambda) H^{\dagger}_{\lambda,k,i-j}(x)\right\} - J^{\dagger}_{\lambda,k,i}(x)$$
$$= S_1(x).$$

Now since $\sigma_j(\lambda)$ is the generating function for partitions with j distinct parts each $\leqq \lambda$, we see that $x^j\sigma_j(\lambda)H^{\dagger}_{\lambda,k,i-j}(x)$ is the generating function for partitions of the type enumerated by $P_{\lambda,k,i}(m,n)$ (m and n arbitrary) with the conditions that (i) there are exactly j parts $\leq \lambda$, and (ii) the inequality $f_m+f_{m+1}+\ldots+f_{m+\lambda} \geq k$ might occur for some m with $1 \leq m \leq \lambda$. Condition (ii) is of course a violation of one of the conditions originally imposed on $B_{\lambda,k,i}(n)$ wherein always $f_m+f_{m+1}+\ldots+f_{m+\lambda} \leq k-1$. This possible violation arises from the fact that we are multiplying together the two generating functions $x^j\sigma_j(\lambda)$ and $H^{\dagger}_{\lambda,k,i-j}(x)$.

If we now sum $x^j\sigma_j(\lambda)H^{\dagger}_{\lambda,k,i-j}(x)$ for $0 \leq j \leq i$, we see that the result is the generating function for partitions of the type generated by $J^{\dagger}_{\lambda,k,i}(x)$ with the added condition on the partitions that possibly $f_m+f_{m+1}+\ldots+f_{m+\lambda} \geq k$ for some m with $1 \leq m \leq \lambda$.

Hence $S_1(x)$ as defined by (6.6) generates the partitions which satisfy $f_m+f_{m+1}+\ldots+f_{m+\lambda} \geq k$ for some m with $1 \leq m \leq \lambda$, but otherwise fulfill the conditions imposed upon the partitions generated by $J_{\lambda,k,i}(x)$.

We now define $g_1(2;k,\lambda,r,i;x;q)$ to be the generating function for partitions of the form $f_1\cdot 1+f_2\cdot 2+\ldots+f_{2\lambda+1}\cdot(2\lambda+1)$ where

$(6.7)_1$ $$f_{\lambda+2}+\ldots+f_{2\lambda+1} = k-r$$

$(6.7)_2$ $$f_1+\ldots+f_{\lambda+1} \leq i-1$$

$(6.7)_3$ $$f_m+f_{m+1}+\ldots f_{m+\lambda} \geq k \text{ for some } m \in [1,\lambda]$$

$(6.7)_4$ $$f_a > 1 \text{ implies } (\lambda+1)|a.$$

Therefore

$$g_1(2;k,\lambda,r,i;x;q)H^{\dagger}_{\lambda,k,r}(xq^{\lambda+1})$$

is the generating function for partitions of the type enumerated by $S_1(x)$ that have exactly k-r parts in the interval $[\lambda+2,2\lambda+1]$ but that might also be such that $f_m+f_{m+1}+\ldots+f_{m+\lambda} \geqq k$ for some m now in $[\lambda+2,2\lambda+1]$. Consequently

$$\sum_{r=0}^{k} g_1(2;k,\lambda,r,i;x;q)H^{\dagger}_{\lambda,k,r}(xq^{\lambda+1}) - S_1(x)$$
$$= S_2(x)$$

is the generating function for partitions $\sum_{j=1}^{\infty} f_j j$ satisfying

$(6.8)_1$ $\qquad f_1+\ldots+f_{\lambda+1} \leqq i-1,$

$(6.8)_2$ $\qquad f_{c(\lambda+1)}+\ldots+f_{(c+1)(\lambda+1)} \leqq k-1$ for all $c \geqq 1,$

$(6.8)_3$ $\qquad f_m+\ldots+f_{m+\lambda} \leqq k-1$ for all $m \notin [2,\lambda], [\lambda+2,2\lambda+1],$

$(6.8)_4$ $\qquad f_m+\ldots+f_{m+\lambda} \geqq k$ for some m in each of $[2,\lambda], [\lambda+2,2\lambda+1],$

$(6.8)_5$ $\qquad f_a > 1$ implies $(\lambda+1)|a.$

The recursive process utilized here is now clear. For each $\ell \geq 2$, we define $g_1(\ell;k,\lambda,r,i;x;q)$ to be the generating function for partitions of the form $\sum_{j=1}^{\ell(\lambda+1)-1} f_j j$ where

$$(6.9)_1 \qquad f_{\ell(\lambda+1)-\lambda}+\dots+f_{\ell(\lambda+1)-1} = k-r$$

$$(6.9)_2 \qquad f_1+\dots+f_\lambda \qquad -1$$

$$(6.9)_3 \qquad f_{c(\lambda+1)}+\dots+f_{(c+1)(\lambda+1)} \leq k-1 \quad \text{for} \quad 1 \leq c \leq \ell-2$$

$$(6.9)_4 \qquad f_m+f_{m+1}+\dots+f_{m+\lambda} \geq k \quad \text{for some } m \text{ in each of}$$

$$[1,\lambda],\ [\lambda+2,2\lambda+1],\dots,[(\lambda-2)(\lambda+1)+1,(\ell-1)(\lambda+1)-1],$$

$$(6.9)_5 \qquad f_a > 1 \quad \text{implies} \quad (\lambda+1) \mid a.$$

Therefore

$$g_1(\ell;k,\lambda,r,i;x;q)H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)})$$

is the generating function for partitions of the type enumerated by $S_{\ell-1}(x)$ that have exactly k-r parts in the interval $[\ell(\lambda+1)-\lambda,\ell(\lambda+1)-1]$ but might also be such that $f_m+f_{m+1}+\dots+f_{m+\lambda} \geq k$ for some m now in $[\ell(\lambda+1)-\lambda,\ \ell(\lambda+1)-1]$. Therefore

$$\sum_{r=0}^{k} g_1(\ell;k,\lambda,r,i;x;q)H_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)}) - S_{\ell-1}(x)$$

$$= S_\ell(x)$$

is the generating function for partitions that satisfy

$(6.10)_1$ $\quad f_1+\dots+f_{\lambda+1} \leqq i-1,$

$(6.10)_2$ $\quad f_{c(\lambda+1)}+\dots+f_{(c+1)(\lambda+1)} \leqq k-1,$ for all $c \geqq 1.$

$(6.10)_3$ $\quad f_m+\dots+f_{m+\lambda} \leqq k-1,$ for all $m \notin [2,\lambda],\ [\lambda+2,2\lambda+1],$

$[2\lambda+3,3\lambda+2],\dots,[\ell(\lambda+1)-\lambda,\ell(\lambda+1)-1],$

$(6.10)_4$ $\quad f_m+\dots+f_{m+\lambda} \geqq k$ for some m in each of $[2,\lambda],$

$[\lambda+2,2\lambda+1],\ [2\lambda+3,3\lambda+2],\dots,[\ell(\lambda+1)-\lambda,\ell(\lambda+1)-1],$

$(6.10)_5$ $\quad f_a > 1$ implies $(\lambda+1)|a.$

Before proceeding we note that the recursive definition of $S_\ell(x)$ implies that

$$S_\ell(x) = \sum_{j=0}^{\ell-2} (-1)^j \sum_{r=0}^{k} g_1(\ell-j;k,\lambda,r,i;x;q)H^\dagger_{\lambda,k,r}(xq^{(\ell-1-j)(\lambda+1)})+(-1)^\ell J_{\lambda,k,i}(x)$$

$$- (-1)^\ell \sum_{j=0}^{i} x^j\sigma_j(\lambda)H_{\lambda,k,i-j}(x).$$

or

$$(6.11)\qquad J_{\lambda,k,i}(x) = \sum_{j=0}^{i} x^j \sigma_j(\lambda) H^{\dagger}_{\lambda,k,i-j}(x)$$

$$- \sum_{j=2}^{\ell} (-1)^j \sum_{r=0}^{k} g_1(j;k,\lambda,r,i;x;q) H^{\dagger}_{\lambda,k,r}(xq^{(j-1)(\lambda+1)})$$

$$+ (-1)^{\ell} S_{\ell}(x).$$

There are now two steps left to the completion of the proof of Theorem 6.1. First we must show that the functions $g_1(\ell;k,\lambda,r,i;x;q)$ as defined above are, in fact, just the polynomials given by equation (6.1), and second we must show that $S_{\ell}(x) = 0$ for ℓ sufficiently large (actually for $\ell \geq \lambda+2$).

Consider the partitions satisfying $(6.9)_1$–$(6.9)_5$; these are the partitions generated by $g_1(\ell;k,\lambda,r,i;x;q)$. We split these partitions into subclasses where

$(6.12)_1$ $\quad f_1+f_2+\ldots+f_{\lambda+1} = A_0 \leq i-1,$

$(6.12)_2$ $\quad f_{c(\lambda+1)+1}+\ldots+f_{c(\lambda+1)+\lambda} = A_c, \quad 0 \leq c \leq \ell-1,$

$(6.12)_3$ $\quad f_{c\lambda+c} = k-D_c, \quad 1 \leq c \leq \ell-1.$

$(6.12)_4$ $\quad f_m+\ldots+f_{m+\lambda} \geq k$ for some m in each of

$[1,\lambda+1], [\lambda+2,2\lambda+2],\ldots,[(\ell-2)(\lambda+1)+1,(\ell-1)(\lambda+1)],$

$(6.12)_4$ $\qquad f_a > 1$ implies $(\lambda+1)|a$,

where we must require just those $(2\ell-1)$-tuples $(A_0, A_1, \ldots, A_{\ell-1}, D_1, \ldots, D_{\ell-1})$ satisfying the conditions $(6.2)_1$-$(6.2)_6$ (I remark that $A_j \le D_j$ in $(6.2)_5$ is redundant as it is implied by $(6.2)_4$ and $(6.2)_6$). But the partitions in the above mentioned subclass are precisely those generated by $g(\{A\}_\ell; \{D\}_\ell; k; \lambda; q)$. Consequently

$$g_1(\ell;k,\lambda,r,i;x;q)$$
$$= \sum_C x^{k(\ell-1)+A_0+A_1+\ldots+A_{\ell-1}-D_1-D_2-\ldots-D_{\ell-1}} g(\{A\}_\ell; \{D\}_\ell; k; \lambda; q),$$

where C is the set of those $(2\ell-1)$-tuples satisfying $(6.2)_1$-$(6.2)_6$.

The fact that $g_1(\ell;k,\lambda,r,i;x;q)$ is a polynomial in x and q is immediate from the definition since the partitions generated are among those in which each part is $\le \ell(\lambda+1)-1$, parts not divisible by $(\lambda+1)$ are not repeated, and parts divisible by $\lambda+1$ are repeated at most $k-1$ times. Therefore in $g_1(\ell;k,\lambda,r,i;x;q)$ the coefficient of $x^M q^N$ is zero if either

$$M > \lambda+(k-1) + \lambda + (k-1) + \ldots + (k-1) + \lambda$$
$$= \ell\lambda + (\ell-1)(k-1)$$

or

$$N > 1+2+\ldots+(k-1)(\lambda+1)+(\lambda+2)+\ldots+(k-1)(2\lambda+2)+\ldots+(\ell(\lambda+1)-1)$$
$$= \binom{\ell(\lambda+1)}{2} + (k-2)(\lambda+1)\binom{\ell}{2}.$$

Now we shall show (in preparation for proving $S_\ell(x) = 0$ for $\ell \geq \lambda+2$) that

$$(6.13) \qquad g(\{A\}_\ell;\{D\}_\ell;k,\lambda;q) = 0$$

for $\ell > \lambda+2$

In order that (6.13) be false we must have by Theorem 5.1

$$(6.14) \qquad 0 \leq D_c \leq \lambda, \qquad 1 \leq c \leq \ell-1$$

$$(6.15) \qquad 0 \leq A_0+A_1+\ldots+A_{\ell-1}-D_1-D_2-\ldots-D_{\ell-1} \leq \lambda.$$

We now combine (6.15) and $(6.2)_6$ and we assume $\ell > \lambda+2$.

$$0 \leq \sum_{j=0}^{\ell-1} A_j - \sum_{j=1}^{\ell-1} D_j \qquad \text{(by (6.15))}$$

$$\leq \lambda + \sum_{j=1}^{\ell-2} (D_j+D_{j+1}-k-1)+D_{\ell-1}-r- \sum_{j=1}^{\ell-1} D_j \quad \text{(by}(6.2)_2,\ (6.2)_3, \text{ and } (6.2_6))$$

$$= \lambda + \sum_{j=2}^{\ell-1} D_j - (k+1)(\ell-2)-r$$

$$\leq \lambda +(\ell-2)\lambda-(k+1)(\ell-2) \quad \text{(by (6.14) and } r \geq 0)$$

$$= \lambda -(k-\lambda+1)(\ell-2)$$

$$\leq \lambda-\ell+2 \qquad \text{(since } k \geq \lambda)$$

$$< 0 \qquad \text{(since } \ell > \lambda+2),$$

which is impossible.

We immediately deduce from (6.13) and (6.1) that

$$g_1(\ell;k,\lambda,r,i;x;q) = 0$$

for $\ell > \lambda+2$.

Finally we observe that $S_\ell(x)$ always has nonnegative coefficients since the coefficient of $x^M q^N$ in $S_\ell(x)$ is the number of partitions of N into M parts satisfying certain conditions. Hence for $\ell > \lambda+2$,

$$\begin{aligned} S_\ell(x) &= \sum_{r=0}^{k} g_1(\ell;k,\lambda,r,i;x;q)H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)}) \\ &\quad - S_{\ell-1}(x) \\ &= -S_{\ell-1}(x), \end{aligned}$$

and the only way the functions here can have nonnegative coefficients is if each is identically zero. Therefore

$$S_\ell(x) = 0 \quad \text{for} \quad \ell \geq \lambda+2.$$

Thus (3.9) now follows from (6.11) when we take $\ell \geq \lambda+2$ in (6.11). We observe the infinite series $\sum_{\ell \geq 2}$ in (3.9) actually terminates and may be replaced by $\sum_{2 \leq \ell \leq \lambda+2}$.

This completes the proof of Theorem 6.1. □

Theorem 6.2. Let $|q| < 1$, $|x| < |q|^{-1}$, then

$$J_{\lambda,k,i}(x) = J^{\dagger}_{\lambda,k,i}(x), \quad \text{for} \quad \lambda \le i \le k,$$

where $J_{\lambda,k,i}(x)$ is defined by (4.2) and $J^{\dagger}_{\lambda,k,i}(x)$ is defined by (3.6).

Proof. We define for $0 \le i \le k$

$$H^{*}_{\lambda,k,i}(x) = H^{\dagger}_{\lambda,k,i}(x) + \sum_{\ell \ge 2} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)})\gamma(\ell;k,\lambda,r,i;x;q), \tag{6.16}$$

where

$$\gamma(\ell;k,\lambda,r,i;x;q) = x^{(\ell-1)k} \sum_{\mathcal{D}(i)} \sigma_{D_1}(\lambda)\dots\sigma_{D_{\ell-1}}(\lambda) q^{\sum_{b=1}^{\ell-2}(k-D_b+A_b)b(\lambda+1)+(k-r)(\ell-1)(\lambda+1)} \tag{6.17}$$

with $\mathcal{D}(i)$ the set of those $(2\ell-3)$-tuples $(A_1,A_2,\dots,A_{\ell-2},D_1,D_2,\dots,D_{\ell-1})$ that satisfy

$$D_2+\dots+D_{\ell-2}-A_1-\dots-A_{\ell-2} \le i-k-r-1 \tag{6.18$_1$}$$

$$1 \le D_j \le k, \quad 1 \le j \le \ell-1, \tag{6.18$_2$}$$

$(6.18)_3$ $$0 \le A_j \le D_j, \quad (1 \le j \le \ell-2)$$

$(6.18)_4$ $$0 \le A_c \le D_c+D_{c+1}-k-1, \quad 1 \le c \le \ell-2.$$

(we remark that when $\ell = 2$, is defined instead by $-D_1 \le i-k-r-1$.

Next we define

(6.19) $$J^*_{\lambda,k,i}(x) = x^{-k+i}(H^*_{\lambda,k,k-i+1}(xq^{-\lambda-1})-H^*_{\lambda,k,k-1}(xq^{-\lambda-1})).$$

The remainder of the proof is divided into two parts. First we shall show that the $H^*_{k,\lambda,i}(x)$ and $J^*_{\lambda,k,i}(x)$ fulfill all the conditions of Theorem 4.2. Second we shall show that $H^*_{\lambda,k,i}(x) = H^*_{\lambda,k,i}(x)$ for $0 \le i \le k-\lambda+1$. Once these two facts are established, Theorem 6.2 follows directly as we shall see.

We first consider the analyticity conditions of Theorem 4.2. We begin by showing that

(6.20) $$\gamma(\ell;k,\lambda,r,i;x;q) = 0 \quad \text{for} \quad r > 0 \quad \text{whenever} \quad \ell \ge \frac{i}{k-\lambda+1} + 1.$$

To see (6.20), we combine the conditions $(6.18)_1$ and $(6.18)_3$ with the observation that for $\gamma(\ell;k,\lambda,r,i;x;q)$ to be nonzero we must have $0 \le D_j \le \lambda$ for $1 \le j \le \ell -1$. Hence for $r > 0$

$i-k-1 > i-k-r-1$

$$\geqq D_2+\dots+D_{\ell-2}-A_1-\dots-A_{\ell-2}$$

$$\geqq \sum_{j=2}^{\ell-2} D_j - \sum_{j=1}^{\ell-2} (D_j+D_{j+1}-k-1)$$

$$= (\ell-2)(k+1) - \sum_{j=1}^{\ell-1} D_j$$

$$\geqq (\ell-2)(k+1) - (\ell-1)\lambda$$

$$= (k-\lambda+1)(\ell-1)-k-1.$$

Therefore, we must have

$$\ell < \frac{i}{k-\lambda+1} +1$$

in order that $\gamma(\ell;k,\lambda,r,i;x;q) \neq 0$; thus, (6.20) is established.

Since $0 \leqq i \leqq k$ and $k \geqq \lambda$, we see that $\gamma(\ell;k,\lambda,r,i;x;q) = 0$ for $\ell > k+1$. Now $H^{\dagger}_{\lambda,k,i}(x)$ and $J^{\dagger}_{\lambda,k,i}(x)$ are analytic in x around the origin provided $|q| < 1$ by comparision with the function

$$\prod_{n=1}^{\infty} (1+xq^n+x^2q^{2n}+\dots+x^{k-1}q^{(k-1)n})$$

which generates the partitions in which every part may be repated up to $k-1$ times. Thus since $H^{*}_{\lambda,k,i}(x)$ is a finite linear combination of the $H^{\dagger}_{\lambda,k,j}(x)$ with coefficients that are polynomials in x, we see that $H^{*}_{\lambda,k,i}(x)$ is analytic around the origin. Furthermore

$$J^{*}_{\lambda,k,i}(x) = x^{-k+i}(H^{*}_{\lambda,k,k-i+1}(xq^{-\lambda-1})-H^{*}_{\lambda,k,k-i}(xq^{-\lambda-1})) \quad \text{(by (6.17))}$$

$$= x^{-k+i}(H^{\dagger}_{\lambda,k,k-i+1}(xq^{-\lambda-1})$$

$$+\sum_{\ell\geqq 2}(-1)^{\ell-1}\sum_{\ell\geqq 1}^{k}H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i+1;xq^{-\lambda-1};q)$$

$$-H^{\dagger}_{\lambda,k,k-i}(xq^{-\lambda-1})-\sum_{\ell\geqq 2}(-1)^{\ell-1}\sum_{r=1}^{k}H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i;xq^{-\lambda-1};q))$$

Hence

$$(6.21) \qquad J^{*}_{\lambda,k,i}(x) = J^{\dagger}_{\lambda,k,i}(x)$$

$$+x^{-k+i}\sum_{\ell\geqq 2}(-1)^{\ell-1}\sum_{r=2}^{k}H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i+1;xq^{-\lambda-1};q)$$

$$-x^{-k+1}\sum_{\ell\geqq 2}(-1)^{\ell-1}\sum_{r=1}^{k}H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i;xq^{-\lambda-1};q) \quad \text{(by (3,8))},$$

and since $\ell \geqq 2$, we see by (6.17) that $x^{-k+i}\gamma(\ell;k,\lambda,r,s;xq^{-\lambda-1};q)$ is a polynomial in x (the possible powers of q are immaterial here). Thus $J^{*}_{\lambda,k,i}(x)$ is a finite linear combination of the $H^{\dagger}_{\lambda,k,j}(x)$ and $J^{\dagger}_{\lambda,k,j}(x)$ with coefficients that are polynomials in x; therefore $J^{*}_{\lambda,k,i}(x)$ is analytic around the origin.

Equation (4.3) follows directly from (6.19) when x is replaced by $xq^{\lambda+1}$ and i is replaced by $k-i+1$.

From (6.20), we see that

$$H^*_{\lambda,k,i}(x) = H^\dagger_{\lambda,k,i}(x) \tag{6.22}$$

for $0 \leqq i \leqq k-\lambda+1$, since in this case there are no integers ℓ that satisfy $2 \leqq \ell < \frac{i}{k-\lambda+1} + 1$. Hence (4.4) follows from the fact that $H^\dagger_{\lambda,k,0}(x) = 0$ (by (6.3)). As for (4.6), we see that $\gamma(\ell;k,\lambda,r,i;0;q) = 0$ for all $\ell \geqq 2$, hence (by (6.4))

$$H^*_{\lambda,k,i}(0) = H_{\lambda,k,i}(0) = 1 \quad \text{for} \quad 1 \leqq i \leqq k,$$

and since x^k divides $\gamma(\ell;k,\lambda,r,i;x;q)$ for $\ell \geqq 2$, we see from (6.20) and (6.4) that

$$J^*_{\lambda,k,i}(0) = J^\dagger_{\lambda,k,i}(0) = 1 \quad \text{for} \quad 1 \leqq i \leqq k.$$

There remains the problem of the establishment of (4.5).

$$J^*_{\lambda,k,i}(x) = x^{-k+i}(H^*_{\lambda,k,k-i+1}(xq^{-\lambda-1}) - H_{\lambda,k,k-i}(xq^{-\lambda-1})) \qquad \text{(by (6.19))}$$

$$= x^{-k+i}(H^\dagger_{\lambda,k,k-i+1}(xq^{-\lambda-1})$$

$$+ \sum_{\ell \geqq 2} (-1)^{\ell-1} \sum_{r=1}^{k} H^\dagger_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i+1;xq^{-\lambda-1};q)$$

$$- H^{\dagger}_{\lambda,k,k-i}(xq^{-\lambda-1})$$

$$- \sum_{\ell\geq 2} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i;xq^{-\lambda-1};q)).$$

Now we recall from the remark following $(6.18)_3$ that

$$\gamma(2;k,\lambda,r,i;x;q) = x^k q^{-r(\lambda+1)} \sum_{D_1=k+r+1-i}^{\lambda} \sigma_{D_1}(\lambda);$$

therefore splitting off the terms at $\ell = 2$, we see that

$$J^{*}_{\lambda,k,i}(x) = x^{-k+i}(H^{\dagger}_{\lambda,k,k-i+1}(xq^{-\lambda-1})$$

$$- \sum_{r=1}^{\lambda-i} H^{\dagger}_{\lambda,k,r}(x)x^k q^{-r(\lambda+1)} \sum_{D_1=i+r}^{\lambda} \sigma_{D_1}(\lambda)$$

$$+ \sum_{\ell\geq 3} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i+1;xq^{-\lambda-1};q)$$

$$- H^{\dagger}_{\lambda,k,k-i}(xq^{-\lambda-1})$$

$$+ \sum_{r=1}^{\lambda-i-1} H^{\dagger}_{\lambda,k,r}(x)x^k q^{-r(\lambda+1)} \sum_{D_1=i+r+1}^{\lambda} \sigma_{D_1}(\lambda)$$

$$- \sum_{\ell\geq 3} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i;xq^{-\lambda-1};q)).$$

Combining the second and fifth lines of the above equation, we see that

$$J^{*}_{\lambda,k,i}(x) = x^{-k+i}(H^{\dagger}_{\lambda,k,k-i+1}(xq^{-\lambda-1})-H^{\dagger}_{\lambda,k,k-i}(xq^{-\lambda-1})$$

$$-\sum_{r=1}^{\lambda-i} H^{\dagger}_{\lambda,k,r}(x)x^{k}q^{-r(\lambda+1)}\sigma_{i+r}(\lambda)$$

$$+\sum_{\ell\geqq 3}(-1)^{\ell-1}\sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i+1;xq^{-\lambda-1};q)$$

$$-\sum_{\ell\geqq 3}(-1)^{\ell-1}\sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i;xq^{-\lambda-1};q))$$

$$= J^{\dagger}_{\lambda,k,i}(x) - \sum_{r=1}^{\lambda-i} H^{\dagger}_{\lambda,k,r}(x)x^{i}q^{-r(\lambda+1)}\sigma_{i+r}(\lambda)$$

$$+ x^{-k+i}\sum_{\ell\geqq 3}(-1)^{\ell-1}\sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i+1;xq^{-\lambda-1};q)$$

$$- x^{-k+i}\sum_{\ell\geqq 3}(-1)^{\ell-1}\sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i;xq^{-\lambda-1};q) \quad \text{(by (3.8))}$$

$$= \sum_{j=0}^{i} x^{j}\sigma_{j}(\lambda)H^{\dagger}_{\lambda,k,i-j}(x)$$

$$+ \sum_{\ell\geqq 2}(-1)^{\ell-1}\sum_{r=0}^{k} g_{1}(\ell;k,\lambda,r,i;x;q)H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)})$$

$$- \sum_{\rho=1}^{\lambda-i} H_{\lambda,k,\rho}(x) x^i q^{-\rho(\lambda+1)} \sigma_{i+\rho}(\lambda)$$

$$+ x^{-k+i} \sum_{\ell\geq 3} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)}) \gamma(\ell;k,\lambda,r,k-i+1;xq^{-\lambda-1};q)$$

$$-x^{-k+i} \sum_{\ell\geq 3} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)}) \gamma(\ell;k,\lambda,r,k-i;xq^{-\lambda-1};q) \quad \text{(by (3.9))}.$$

We now apply (6.16) to the $H^{\dagger}_{\lambda,k,j}(x)$ in the first and third sums above. Therefore

$$J^{*}_{\lambda,k,i}(x) = \sum_{j=0}^{i} x^j \sigma_j(\lambda) H^{*}_{\lambda,k,i-j}(x)$$

$$- \sum_{j=0}^{i} x^j \sigma_j(\lambda) \sum_{\ell\geq 2} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)}) \gamma(\ell;k,\lambda,r,i-j;x;q)$$

$$+ \sum_{\ell\geq 2} (-1)^{\ell-1} \sum_{r=1}^{k} g_1(\ell;k,\lambda,r,i,;x;q)\, H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)})$$

$$- \sum_{\rho=1}^{\lambda-i} x^i q^{-\rho(\lambda+1)} \sigma_{\rho+i}(\lambda) H^{*}_{\lambda,k,\rho}(\lambda)$$

$$+ \sum_{\rho=1}^{\lambda-i} x^i q^{-\rho(\lambda+1)} \sigma_{\rho+i}(\lambda) \sum_{\ell\geq 2} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)}) \gamma(\ell;k,\lambda,r,\rho;x;q)$$

$$+ x^{-k+i} \sum_{\ell\geq 3} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i+1;xq^{-\lambda-1};q)$$

$$- x^{-k+i} \sum_{\ell\geq 3} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-2)(\lambda+1)})\gamma(\ell;k,\lambda,r,k-i;xq^{-\lambda-1};q)$$

Hence

$$(6.23) \qquad J^{*}_{\lambda,k,i}(x) =$$

$$\sum_{j=0}^{i} x^{j}\sigma_{j}(\lambda)H^{*}_{\lambda,k,i-j}(x) - \sum_{\rho=1}^{\ell-i} x^{i}q^{-\rho(\lambda+1)}\sigma_{\rho+i}(\lambda)H^{*}_{\lambda,k,\rho}(x)$$

$$+ \sum_{\ell\geq 2} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)}) \cdot \Bigg\{ - \sum_{j=0}^{i} x^{j}\sigma_{j}(\lambda)\gamma(\ell;k,\lambda,r,i-j;x;q)$$

$$+ g_{1}(\ell;k,\lambda,r,i;x;q)$$

$$+ \sum_{\rho=1}^{\lambda-i} x^{i}q^{-\rho(\lambda+1)}\sigma_{\rho+i}(\lambda)\gamma(\ell;k,\lambda,r,\rho;x;q)$$

$$- x^{-k+i}\gamma(\ell+1;k,\lambda,r,k-i+1;xq^{-\lambda-1};q)$$

$$+ x^{-k+i}\gamma(\ell+1;k,\lambda,r,k-i;xq^{-\lambda-1};q)\Bigg\} .$$

Let us denote the expression inside the curly brackets by E. Comparing (6.23 with (4.5), we see that to establish (4.5) it is sufficient to prove that $E = 0$. We see that the assertion "$E = 0$" follows immediately from the following two (somewhat troublesome) identities:

$$(6.24) \qquad g_{1}(\ell;k,\lambda,r,i;x;q) = \sum_{j=0}^{i} x^{j}\sigma_{j}(\lambda)\gamma(\ell;k,\lambda,r,i-j;x;q),$$

and

$$(6.25)\qquad \gamma(\ell+1;k,\lambda,r,k-i+1;xq^{-\lambda-1};q)$$

$$-\gamma(\ell+1;k,\lambda,r,k-i;xq^{-\lambda-1};q)$$

$$= x^k \sum_{\rho=1}^{\lambda-i} q^{-\rho(\lambda+1)}\sigma_{\rho+i}(\lambda)\gamma(\ell;k,\lambda,r,\rho;x;q).$$

We begin by proving (6.24), the simplier identity. If in the conditions for $\mathcal{C}$ (i.e. equations $(6.2)_1$-$(6.2)_6$) we replace A_0 by $j+D_1+D_2+\ldots+D_{\ell-1}-A_1-A_2-\ldots-A_{\ell-1}$, then $(6.2)_1$ and $(6.2)_2$ produce

$$(6.26)\qquad D_2+\ldots+D_{\ell-2}-A_1-\ldots-A_{\ell-2} \leq i-k-j-1-r.$$

Note also by $(6.2)_6$ that

$$D_1+D_2+\ldots+D_{\ell-1}-A_1-A_2-\ldots-A_{\ell-1}$$

$$\geq \sum_{h=1}^{\ell-1} D_h - \sum_{h=1}^{\ell-2} (D_h+D_{h+1}-k-1) - D_{\ell-1} + r$$

$$= (\ell-2)(k+1) + r - \sum_{h=1}^{\ell-2} D_{h+1}$$

$$\geq (\ell-2)(k+1) + r - (\ell-2)k = \ell - 2 + r \geq 0.$$

Consequently the condition $A_0+k-D_1 \leq i-1$, implies

$$i-1 \geq j+D_1+\ldots+D_{\ell-1}-A_1-\ldots-A_{\ell-1}+k-D_1$$

$$\geq j+k-D_1 \geq j.$$

Therefore we may restrict our considerations to those j's for which $0 \leq j \leq i$, and we see that by (6.1)

$$g_1(\ell;k,\lambda,r,i;x;q)$$

$$= \sum_{j=0}^{i} x^j \sigma_j(\lambda) \sum_{\mathcal{D}^\#} x^{k(\ell-1)} g(\{A\}_\ell;\{D\}_\ell;k;\lambda;q)$$

where $\mathcal{D}^\#$ is the set of those $(2\ell-3)$-tuples $(A_1,A_2,\ldots,A_{\ell-2},D_1,\ldots,D_{\ell-1})$ that satisfy (6.26), $(6.2)_4$, $(6.2)_5$ and $(6.2)_6$; condition $(6.2)_2$ has been eliminated by replacing $A_{\ell-1}$ by $D_{\ell-1}-r$; condition $(6.2)_3$ is also superfluous since

$$A_0 = j+D_1+\ldots+D_{\ell-1}-A_1-\ldots-A_{\ell-1} \geq j \geq 0,$$

and (by $(6.26)_1$)

$$A_0 = j+D_1+\ldots+D_{\ell-1}-A_1-\ldots-A_{\ell-1}$$

$$\leq j+D_1+i-k-j-1-r$$

$$\leq D_1+i-k-1-r$$

$$\leq \lambda+k-k-1-r \leq \lambda-1 \leq \lambda.$$

Hence we see that $\mathcal{D}^{\#} = \mathcal{D}(i-j)$ and consequently

$$g(\ell;k,\lambda,r,i;x;q)$$

$$= \sum_{j=0}^{i} x^{i}\sigma_j(\lambda) \quad \gamma(\ell;k,\lambda,r,i-j;x;q),$$

which is (6.24) as desired.

We must now prove (6.25). First we consider the left-hand side of (6.25); we are summing over $(2(\ell+1)-3)$-tuples $(A_1,\ldots,A_{\ell-1},D_1,\ldots,D_\ell)$ lying in $\mathcal{D}(k-i+1) - \mathcal{D}(k-i)$. Making the substitutions $A_2 = \alpha_1$, $A_3 = \alpha_2,\ldots,$ $A_{\ell-1} = \alpha_{\ell-2}$, $D_1 = \mu$, $D_2 = \delta_1$, $D_3 = \delta_2,\ldots,D_\ell = \delta_{\ell-1}$, I claim that the admissable $(2\ell-1)$-tuples $(A_1,\alpha_1,\ldots,\alpha_{\ell-2},\mu, \delta_1,\ldots,\delta_{\ell-1})$ are exactly those that satisfy

$(6.27)_1$ $$\delta_2+\ldots+\delta_{\ell-2}-\alpha_1-\ldots-\alpha_{\ell-2} \leq \mu-i-r-k-1$$

$(6.27)_2$ $$0 \leq \alpha_c \leq \delta_c + \delta_{c+1} -k-1, \quad 1 \leq c \leq \ell-2.$$

(6.28) $$A_1 = \delta_1+\ldots+\delta_{\ell-2}-\alpha_1-\ldots-\alpha_{\ell-2}+i+r.$$

This assertion is not immediately obvious for there are several defining conditions for $\mathcal{D}(k-i+1) - \mathcal{D}(k-i)$ that appear to have been ignored. First of all (6.28) is the condition that guarantees that

$$D_2 +\ldots+ D_{\ell-1} - A_1 -\ldots- A_{\ell-1} \leq -i-r$$

but also

$$D_2 +\ldots+ D_{\ell-1} - A_1 -\ldots- A_{\ell-1} \not\leq -i-r-1,$$

the conditions implied by $(6.18)_1$.

Condition $(6.18)_2$ is completely superfluous because first $k \geqq \lambda$ and if $D_j > \lambda$, then $\sigma_{D_j}(\lambda) = 0$; therefore the summands vanish whenever $D_j \notin [0,\lambda] \subseteq [0,k]$. Secondly if $D_j = 0$ then $(6.18)_4$ cannot be fulfilled in a nontrivial way since $0 \leqq A_c \leqq 0 + D_{c+1} - k-1 \leqq \lambda-k-1 \leqq -1$ would be necessary to produce a nonzero summand and this string of inequalities cannot hold.

As for $(6.18)_3$, we see that by $(6.18)_4$

$$0 \leqq A_c \leqq D_c + D_{c+1} - k - 1 \leqq D_c + \lambda - k - 1 \leqq D_c - 1 < D_c;$$

therefore $(6.18)_3$ is superfluous.

Finally we remark that $(6.27)_2$ is the translation of $(6.18)_4$ except in the case when A_1 appears. I claim now that $(6.27)_1$ is the only nonsuperfluous condition implied by $(6.14)_4$ when $c = 1$, because

$$A_1 \leqq D_1 + D_2 - k - 1$$

is equivalent to

$$\delta_1+\ldots+\delta_{\ell-2} - \alpha_1-\ldots-\alpha_{\ell-2}+i+r \leqq \mu + \delta_1 - k - 1$$

which is equivalent to $(6.27)_1$, and the condition $A_1 \geqq 0$ is superflous since

$$A_1 = \delta_1+\ldots+\delta_{\ell-2} - \alpha_1-\ldots-\alpha_{\ell-2} + i + r$$

$$= D_2+\ldots+D_{\ell-1}-A_2-\ldots-A_{\ell-1}+i+r$$

$$\geqq \sum_{j=2}^{\ell-1} D_j - \sum_{c=2}^{\ell-1} (D_c + D_{c+1} - k - 1) + i + r$$

$$\geq (\ell-2)(k+1) - \sum_{c=2}^{\ell-1} D_{c+1} + i + r$$

$$\geq (\ell-2)(k+1) - (\ell-2)\lambda + i + r$$

$$= (k-\lambda+1)(\ell-2) + i + r \geq 0.$$

Hence if we substitute the right-hand side of (6.28) for A_1 in (6.24) we see that

(6.29)
$$\gamma(\ell+1;k,\lambda,r,k-i+1;xq^{-\lambda-1};q) - \gamma(\ell+1;k,\lambda,r,k-i;xq^{-\lambda-1};q)$$

$$= x^{\ell k}q^{-(\lambda+1)\ell k} \sum_{\mathcal{D}^*} \sigma_{\mu}(\lambda)\sigma_{\delta_1}(\lambda)\ldots\sigma_{\delta_{\ell-1}}(\lambda)$$

$$\cdot\, q^{(\lambda+1)(k-\mu+\delta_1+\ldots+\delta_{\ell-2}-\alpha_1-\ldots-\alpha_{\ell-2}+i+r) + \sum_{b=2}^{\ell-1}(k-\delta_{b-1}+\alpha_{b-1})b(\lambda+1)+(k-r)\ell(\lambda+1)}$$

$$= x^{\ell k} \sum_{\mathcal{D}^*} \sigma_{\mu}(\lambda)\sigma_{\delta_1}(\lambda)\ldots\sigma_{\delta_{\ell-1}}(\lambda)q^{-(\mu-i)(\lambda+1)+\sum_{b=1}^{\ell-2}(k-\delta_b+\alpha_b)b(\lambda+1)}$$
$$\cdot\, q^{(\ell-1)(k-r)(\lambda+1)},$$

where $\mathcal{D}^*$ is the set of $(2\ell-2)$-tuples $(\alpha_1,\ldots,\alpha_{\ell-2},\mu,\delta_1,\ldots,\delta_{\ell-1})$ that satisfy (6.27), and $(6.27)_2$.

On the other hand, if we replace $\rho+i$ by μ on the right-hand side of (6.25), we see that

(6.30)
$$x^k \sum_{\rho=1}^{\lambda-i} q^{-\rho(\lambda+1)}\sigma_{\rho+i}(\lambda)\ \gamma(\ell;k,\lambda,r,\rho;x;q)$$

$$= x^k \sum_{\rho=-\infty}^{\infty} q^{-\rho(\lambda+1)}\sigma_{\rho+i}(\lambda)\ \gamma(\ell;k,\lambda,r,\rho;x;q)$$

$$= x^{\ell k} \sum_{\mathcal{D}^*} \sigma_{\mu}(\lambda)\sigma_{D_1}(\lambda)\ldots\sigma_{D_{\ell-1}}(\lambda)q^{\sum_{b=1}^{\ell-2}(k-D_b+A_b)b(\lambda+1)-(\mu-i)(\lambda+}$$

$$q^{(\ell-1)(k-r)(\lambda+1)}$$

where $\mathcal{D}^*$ is the set of $(2\ell-2)$-tuples defined by

$$D_2+\dots+D_{\ell-2}-A_1-\dots-A_{\ell-2} \leq \mu-i-k-1-r$$

$$0 \leq A_c \leq D_c+D_{c+1}-k-1, \quad 1 \leq c \leq \ell-2;$$

note that we have eliminated the superfluous condition $(6.18)_2$ and $(6.18)_3$. The replacement of $\sum_{\rho=1}^{\lambda-i}$ by $\sum_{\rho=-\infty}^{\infty}$ in the second line above is valid since if $\rho > \lambda-i$, then $\sigma_{\rho+i}(\lambda) = 0$, and if $\rho \leq 0$, then the region $\mathcal{D}(\rho)$ contains no $(2\ell-3)$-tuples that produce nonzero summands in $\gamma(\ell;k,\lambda,r,\rho;x;q)$ since we must have

$$\begin{aligned}-k-r-1 &\geq \rho-k-r-1\\ &\geq \sum_{j=2}^{\ell-2} D_j - \sum_{j=1}^{\ell-2} (D_j+D_{j+1}-k-1)\\ &= (\ell-2)(k+1)-(\ell-2)\lambda\\ &= (k-\lambda+1)(\ell-2) \geq 0,\end{aligned}$$

which is impossible. Therefore $\gamma(\ell;k,\lambda,r,\rho;x;q) = 0$ for $\rho \leq 0$.

Comparing (6.30) with (6.29) we see that the right-hand sides are identical (just equate D_j with δ_j and A_j and α_j); thus the left-hand sides are identical and therefore (6.25) is established.

The establishment of (6.24) and (6.25) proves (via (6.23)) that the $J^*_{\lambda,k,i}(x)$ and $H^*_{\lambda,k,i}(x)$ satisfy (4.5). Therefore all the conditions of Theorem 4.2 are fulfilled. This theorem tell us then that

(6.31) $$J^*_{\lambda,k,i}(x) = J_{\lambda,k,i}(x), \quad 1 \leq i \leq k,$$

and

$$H^{*}_{\lambda,k,i}(x) = H_{\lambda,k,i}(x), \quad 0 \leqq i \leqq k. \tag{6.32}$$

Finally for $\lambda \leqq i \leqq k$ (and so $1 \leqq k-i+1 \leqq k-\lambda+1$),

$$\begin{aligned} J^{\dagger}_{\lambda,k,i}(x) &= x^{-k+i}(H^{\dagger}_{\lambda,k,k-i+1}(xq^{-\lambda-1}) - H^{\dagger}_{\lambda,k,k-i}(xq^{-\lambda-1})) && \text{(by (3.8))} \\ &= x^{-k+i}(H^{*}_{\lambda,k,k-i+1}(xq^{-\lambda-1}) - H^{*}_{\lambda,k,k-i}(xq^{-\lambda-1})) && \text{(by (6.22))} \\ &= x^{-k+i}(H_{\lambda,k,k-i+1}(xq^{-\lambda-1}) - H_{\lambda,k,k-i}(xq^{-\lambda-1})) && \text{(by (6.32))} \\ &= J_{\lambda,k,i}(x) && \text{(by (3.1)).} \end{aligned}$$

We therefore have proved Theorem 6.2. □

Theorem 6.3. For $\lambda \leqq a \leqq k$,

$$A_{\lambda,k,a}(n) = B_{\lambda,k,a}(n)$$

for each $n \geqq 1$.

Proof.

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} B_{\lambda,k,a}(n)q^{n} &= J^{\dagger}_{\lambda,k,a}(1) \\ &= J_{\lambda,k,a}(1) && \text{(by Theorem 6.2)} \end{aligned}$$

$$= \frac{(-q)_\infty}{(q^{2\lambda+2};q^{2\lambda+2})_\infty} \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}(\lambda+1)((2k-\lambda+1)n^2+(2k-2i+1)n)} (1-q^{\frac{1}{2}(\lambda+1)(2n+1)(2i-\lambda)})$$

(by (4.2))

$$= \frac{(-q)_\infty}{(q^{2\lambda+2};q^{2\lambda+2})_\infty} (q^{(2k-\lambda+1)(\lambda+1)};q^{(2k-\lambda+1)(\lambda+1)})_\infty$$

$$(q^{(a-\frac{\lambda}{2})(\lambda+1)};q^{(2k-\lambda+1)(\lambda+1)})_\infty (q^{(2k-a-\frac{\lambda}{2}+1)(\lambda+1)};q^{(2k-\lambda+1)(\lambda+1)})_\infty$$

(by Jacobi's identity [12;eq. (19.9.1), p. 283])

$$= 1 + \sum_{n=1}^{\infty} A_{\lambda,k,a}(n)q^n \qquad \text{(by Proposition 2.1)}.$$

Comparing coefficients of q^n in the extremes of this string of equations, we derive Theorem 6.3. □

7. Further auxiliary partition functions. In this section, we shall be greatly extending the work of [4; Section 4]. Just as Section 5 of this paper was much more complex than Section 2 of [4], so also are our problems much more difficult here than in Section 4 of [4].

We begin by remarking that Lemma 4.1 of [4], a result required in our present considerations, was given a proof that is somewhat difficult to follow. Since this is the case, we shall present a new and (hopefully) simpler proof.

Definition 6. For $0 \leq \nu \leq i$, $\frac{1}{2}\lambda \leq i \leq \lambda$ we denote by $\varphi(i,\nu;\lambda;n)$ the number of partitions of n into ν distinct parts of the form $n = \sum_{e=1}^{\lambda} f_e \cdot e$ (here $f_e = 0$ or 1), where at least one of the following $[\frac{1}{2}\lambda]+1$ inequalities holds.

$$(7.1)_b \qquad f_b+\ldots+f_{\lambda+1-b} > i-b, \quad 1 \leq b \leq \frac{1}{2}\lambda + 1.$$

(we note that when $b = \frac{1}{2}\lambda + 1$ the assertion is $0 > i - \frac{1}{2}\lambda - 1$ which is true only for $i = \frac{\lambda}{2}$).

Definition 7. For $0 \leq \nu \leq i$, $\frac{1}{2}\lambda \leq i \leq \lambda$ we define

$$\psi(i,\nu;\lambda;q) = \sum_{n\geq 0} \varphi(i,\nu;\lambda;n)q^n.$$

Theorem 7.1. For $0 \leq \nu \leq i \leq \lambda$, $\frac{1}{2}\lambda \leq i \leq \lambda$

$$\psi(i,\nu;\lambda;q) = q^{(\lambda-i)(\lambda+1)}\sigma_{2i-\nu}(\lambda).$$

Remark. This is just Lemma 4.1 of [4] where now j is replaced by $2i-\nu$. Also the proof of Lemma 4.1 is [4] should be reworded so that it becomes a descending induction on i starting at $i=j$ and descending to smaller values of i.

Proof. We begin with four special cases:

Case 1. $\nu=i$. In this case $(7.1)_1$ is automatic. Therefore $\varphi(i,i;\lambda;n)$ is merely the number of partitions of n into i distinct parts each $\leq \lambda$. Hence

$$\psi(i,i;\lambda;q) = \sigma_i(\lambda),$$

since $\sigma_i(\lambda)$ is well-known [12;Th. 348, p. 280] to be generating function for partitions into i distinct parts each $\leq \lambda$.

Case 2. $\nu=0$. Now the only partition with zero parts is the empty partition of zero, and for this partition we see that the inequalities $(7.1)_b$ can only be fulfilled when $i = \frac{\lambda}{2}$ since the left-hand side is always zero while $i-b \geq \frac{\lambda}{2} - (\frac{\lambda}{2} + 1) = -1$. Thus in this case

$$\psi(i,0;\lambda;q) = \begin{cases} 1 & \text{if } i = \frac{\lambda}{2} \\ 0 & \text{otherwise} \end{cases}$$

On the other hand for $\nu = 0$

$$q^{(\lambda-i)(\lambda+1)}\sigma_{2i-\nu}(\lambda) = \begin{cases} 1 & \text{if } i = \frac{1}{2}\lambda \\ 0 & \text{otherwise} \end{cases}$$

since $2i-\nu \geq \lambda$ in this case with equality only for $i = \frac{\lambda}{2}$. Thus the theorem is valid in this case also.

Case 3. $\nu=1$. Since we observed in Case 2 that $i-b \geq -1$, we see that now the only way to fulfill any of the $(7.1)_b$ is with $b = \frac{\lambda+1}{2}$ (which implies λ is odd) and $f_{\frac{\lambda+1}{2}} = 1$, $i = \frac{\lambda+1}{2}$ or with $b = \frac{\lambda}{2}+1$, $i = \frac{\lambda}{2}$. Hence

$$\psi(i,1,\lambda;q) = \begin{cases} q^{\frac{\lambda+1}{2}} & \text{if } \lambda \text{ is odd and } i = \frac{\lambda+1}{2} \\ \sigma_1(\lambda) & \text{if } \lambda \text{ is even and } i = \frac{\lambda}{2} \\ 0 & \text{otherwise} \end{cases}$$

On the other hand, for $\frac{\lambda}{2} \leq i \leq \lambda$, we see that

$$q^{(1-i)(\lambda+1)}\sigma_{2i-1}(\lambda) = \begin{cases} q^{(1-\frac{\lambda+1}{2})(\lambda+1)}\sigma_{\lambda}(\lambda) & \text{if } i = \frac{\lambda+1}{2} \\ q^{(1-\frac{\lambda}{2})(\lambda+1)}\sigma_{\lambda-1}(\lambda) & \text{if } i = \frac{\lambda}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} q^{\frac{1}{2}(\lambda+1)} & \text{if } \lambda \text{ is odd and } i = \frac{\lambda+1}{2} \\ \sigma_1(\lambda) & \text{if } \lambda \text{ is even and } i = \frac{\lambda}{2} \\ 0 & \text{otherwise} \end{cases}$$

Thus the theorem is valid in this third case.

Case 4. i= . Examining $(7.1)_b$ we see that when $\lambda=i$, we have for some b

$$\lambda+1-b-(b-1) \geqq f_b +\ldots+f_{\lambda+1-b} > i-b = \lambda-b;$$

hence $2>b$. This implies that $(7.1)_b$ can be possibly true only for $b=1$, i.e.

$$\nu = f_1 +\ldots+f_\lambda > \lambda-1,$$

which means $\nu=\lambda$. Therefore

$$\psi(\lambda,\nu;\ \lambda;q) = \begin{cases} q^{\frac{1}{2}\lambda(\lambda+1)} & \text{if} \quad \nu=\lambda \\ 0 & \text{otherwise} \end{cases}$$

But $0 \leqq \nu \leqq i = \lambda$ implies $2i-\nu=2\lambda-\nu \geqq \lambda$ with equality only for $\nu=\lambda$. Therefore

$$q^{(\nu-\lambda)(\lambda+1)}\sigma_{2\lambda-\nu}(\lambda) = \begin{cases} q^{\frac{1}{2}\lambda(\lambda+1)} & \text{if} \quad \nu=\lambda \\ 0 & \text{otherwise} \end{cases}$$

Hence the theorem is true in Case 4.

We now proceed to show that each side of the equation in Theorem 7.1 satisfies the following recurrence for $2 \leqq \nu < i$:

(7.2)
$$\psi(i,\nu;\lambda;q) = q^\nu\{\psi(i-1,\nu;\lambda-2;q) + \psi(i-1,\nu-1;\lambda-2;q) + q^{\lambda-1}\psi(i-1,\nu-1;\lambda-2;q) + q^{\lambda-1}\psi(i-1,\nu-2;\lambda-2;q)\}.$$

Equation (7.2) may be written in terms of the coefficients of the functions involved as follows:

$$\text{(7.3)} \qquad \varphi(i,\nu;\lambda;n) = \varphi(i-1,\nu;\ \lambda-2;n-\nu) + \varphi(i-1,\nu-1;\lambda-2;n-\nu) + \varphi(i-1,\nu-1;\lambda-2;n-\nu-\lambda+1) + \varphi(i-1,\nu-2;\lambda-2;n-\nu-\lambda+1).$$

We prove (7.3) as follows: Let us transform each partition $\sum_{e=1}^{\lambda} f_e \cdot e$ enumerated by $\varphi(i,\nu;\lambda;n)$ by deleting 1 if it appears as a summand, deleting λ if it appears and subtracting 1 from each summand in the open interval $(1,\lambda)$. If we denote the transformed partition by $\sum_{e=1}^{\lambda-2} f_e' \cdot e$, then the conditions $(7.1)_b$ have now become

$$f'_{b-1}+\dots+f'_{\lambda-b} > i-b, \quad 2 \le b \le \tfrac{1}{2}(\lambda+1)$$

or

$$f'_{b'}+\dots+f'_{(\lambda-2)-b'+1} > i-1-b', \quad 1 \le b' \le \tfrac{1}{2}(\lambda-2+1).$$

Thus the transformed inequalities are just the original inequalities with i replaced by $i-1$ and λ by $\lambda-2$.

We now distinguish four classes i) $f_1 = f_\lambda = 0$, ii) $f_1 = 1$, $f_\lambda = 0$, iii) $f_1 = 0$, $f_\lambda = 1$, iv) $f_1 = f_\lambda = 1$. The partitions of class i) transformed are just those enumberated by $\varphi(i-1,\nu;\lambda-2;n-\nu)$; those of class ii) transformed are enumerated by $\varphi(i-1,\nu-1;\lambda-2;n-\nu)$; those of class iii) transformed are enumerated by $\varphi(i-1,\nu-1;\lambda-2;n-\nu-\lambda+1)$, and those of class iv) transformed are enumerated by $\varphi(i-1,\nu-2;\lambda-2;n-\nu-\lambda+1)$. Since our transformation is clearly reversible, we see that (7.3) is established.

To prove that (7.2) holds for $q^{(\nu-i)(\lambda+1)}\sigma_{2i-\nu}(\lambda)$, we observe that

$$\sigma_j(\lambda) = q^{\frac{1}{2}j(j+1)}\begin{bmatrix}\lambda\\ j\end{bmatrix} \qquad \text{(see statement of Theorem 4.2)} \tag{7.4}$$

$$= q^{\frac{1}{2}j(j+1)}\left(\begin{bmatrix}\lambda-1\\ j\end{bmatrix} + q^{-j}\begin{bmatrix}\lambda-1\\ j-1\end{bmatrix}\right) \qquad \text{([13; p. 85])}$$

$$= \sigma_j(\lambda-1) + q^{\lambda}\sigma_{j-1}(\lambda-1).$$

Hence

$$q^{(\nu-i)(\lambda+1)}\sigma_{2i-\nu}(\lambda)$$

$$= q^{(\nu-i)(\lambda+1)}(\sigma_{2i-\nu}(\lambda-1) + q^{\lambda}\sigma_{2i-\nu-1}(\lambda-1)) \qquad \text{(by (7.4))}$$

$$= q^{(\nu-i)(\lambda+1)+2i-\nu}(\sigma_{2i-\nu}(\lambda-2) + \sigma_{2i-\nu-1}(\lambda-2))$$

$$+ q^{(\nu-i)(\lambda+1)+\nu+2i-\nu-1}(\sigma_{2i-\nu-1}(\lambda-2) + \sigma_{2i-\nu-2}(\lambda-2)) \qquad \text{(by (5.7))}$$

$$= q^{\nu}\{q^{(\nu-i+1)(\lambda-1)}\sigma_{2(i-1)-\nu}(\lambda-2) + q^{(\nu-i)(\lambda-1)}\sigma_{2(i-1)-(\nu-1)}(\lambda-2)$$

$$+ q^{\lambda-1}q^{(\nu-i)(\lambda-1)}\sigma_{2(i-1)-(\nu-1)}(\lambda-2) + q^{\lambda-1}q^{(\nu-i-1)(\lambda-1)}\sigma_{2(i-1)-(\lambda-2)}(\lambda-2)\},$$

and so we see that the recurrence (7.2) is fulfilled by $q^{(\nu-i)(\lambda+1)}\sigma_{2i-\nu}(\lambda)$.

We may now easily prove Theorem 7.1 by mathematical induction on λ. We assume the theorem is true for each nonnegative integer $< \lambda$. Since we have treated $i = 0,1$ (Cases 2 and 3) and since $i \leq \lambda$, we may assume $\lambda \geq 2$. Since $\nu \leq i \leq \lambda$, we may assume $\nu \leq \lambda-2$ since $\nu=\lambda$ implies $i = \lambda$ (Case 1), $\nu = \lambda-1$ implies either $i = \lambda-1$ (Case 1), or $i = \lambda$

(Case 4.) Furthermore we may assume $0 < i < \nu$ since $i = \nu$ is Case 1. These conditions $0 < i < \nu \leqq \lambda-2$ imply that the arguments of the four functions on the righthand side of (7.2) all satisfy the conditions of the theorem. Hence sine (7.2) is valid for both $\psi(i,\nu;\lambda;q)$ and $q^{(\nu-i)(\lambda+1)}\sigma_{2i-\nu}(\lambda)$, the induction hypothesis implies the two righthand sides are identical. Hence the lefthand sides are identical. Thus Theorem 7.1 is proved. □

Definition 8. Let $\pi^*(A_0,A_1,\ldots,A_{\ell-1};\ D_1,\ldots,D_{\ell-1};k;\lambda;i;n)$ $= \pi^*(\{A\}_\ell;\ \{D\}_\ell;k;\lambda;i;n)$ denote the number of partitions of n of the form $f_1 \cdot 1+f_2 \cdot 2+\ldots+f_{\ell\lambda+\lambda-1}(\ell\lambda+\lambda-1)$ (here f_j is the number of times the summand j apears) where

(7.5) $$f_{c\lambda+c+1}+\ldots+f_{c\lambda+c+\lambda} = A_c,\quad 0 \leqq c \leqq \ell-1;$$

(7.6) $$k \geqq f_{c\lambda+c} = k-D_c,\quad 1 \leqq c \leqq \ell-1;$$

(7.7) $f_m+\ldots+f_{m+\lambda} \geqq k$ for some m in each of the $\ell-1$

intervals $[1,\lambda+1]$, $[\lambda+2,2\lambda+2],\ldots,[(\ell-2)(\lambda+1)+1,(\ell-1)(\lambda+1)$;

(7.8) $f_a > 1$ implies $(\lambda+1)|a$;

$(7.9)_b$ $f_b+\ldots+f_{\lambda+1-b} > i-b$ for some b with $1 \leqq b \leqq \frac{1}{2}\lambda + 1$.

Definition 9.

$$\psi(\{A\}_\ell;\ \{D\}_\ell;\ k;\lambda;i;q) = \sum_{n>0} (\{A\}_\ell;\{D\}_\ell;k,\lambda,i;n)q^n.$$

Theorem 7.2. Let ℓ be an integer ≥ 1, and let $A_0, A_1, \ldots, A_{\ell-1}, D_1, D_2, \ldots, D_{\ell-1}$ denote integers that satisfy $A_j \leq D_j + D_{j+1} - \lambda$ for $1 \leq j \leq \ell-2$, and if $\ell > 1$, $A_{\ell-1} \leq D_{\ell-1}$ and $A_0 \leq D_1 + i - \lambda - 1$, while if $\ell = 1$ we only require $A_0 \leq i$. Then for $\lambda \geq i \geq \frac{1}{2}\lambda \geq 0$

$$\psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;i;q) = \Psi(\{A\}_\ell;\{D\}_\ell;k,\lambda,i;q), \tag{7.10}$$

where

$$\Psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;i;q) = \tag{7.11}$$

$$\sigma_{2i-A_0-A_1-\cdots-A_{\ell-1}+D_1+\cdots+D_{\ell-1}}(\lambda)\sigma_{D_1}(\lambda)\sigma_{D_2}(\lambda)\cdots\sigma_{D_{\ell-1}}(\lambda) \cdot q^{\sum_{b=1}^{\ell-1}(k-D_b+A_b)b(\lambda+1)-(i-A_0-A_1-\cdots-A_{\ell-1}+D_1+\cdots+D_{\ell-1})(\lambda+1)}$$

Proof. We may throughout this proof assume that $0 \leq D_j \leq \lambda$ for each j for otherwise obviously $\Psi \equiv 0$ and also $\psi = 0$ since if $D_j > \lambda$ then condition (7.8) implies that for m in $[(j-1)(\lambda+1)+1, j(\lambda+1)]$

$$f_m + \cdots + f_{m+\lambda} \leq k - D_j + \lambda < k,$$

which contradicts (7.7); and if $D_j < 0$, then either $j = \ell-1$ and $A_{\ell-1} \leq D_{\ell-1} < 0$ would make (7.5) impossible or $A_j \leq D_j + D_{j+1} - \lambda < D_{j+1} - \lambda$ and so either $A_j < 0$ which makes (7.5) impossible or $D_{j+1} > \lambda$ which is the case treated above. When $\ell=1$, we note that the theorem asserts that

$$\psi(A_0;;k,\lambda,i;q) = q^{-(i-A_0)(\lambda+1)}\sigma_{2i-A_0}(\lambda),$$

which is just Theorem 7.1.

We begin by rewriting (7.11) in terms of the functions appearing in Theorem 5.1.

Since

$$\sigma_j(\lambda) = q^{\frac{1}{2}j(j+1)}\left[{\lambda \atop j}\right] = q^{\frac{1}{2}j(j+1)}\left[{\lambda \atop \lambda-j}\right] = q^{(\lambda+1)(j-\frac{\lambda}{2})}\sigma_{\lambda-j}(\lambda), \tag{7.12}$$

we see that

$$\Psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;i;q)$$

$$= \sigma_{\lambda-2i+A_0+A_1+\dots+A_{\ell-1}-D_1-\dots-D_{\ell-1}}(\lambda)\sigma_{D_1}(\lambda)\sigma_{D_2}(\lambda)\dots\sigma_{D_{\ell-1}}(\lambda) \cdot q^{\sum_{b=1}^{\ell-1}(k-D_b+A_b)b(\lambda+1) + (\lambda+1)(i-\frac{\lambda}{2})} \tag{7.13}$$

We shall attack Theorem 7.2 in much the same manner that we attacked Theorem 5.1. We again have a double mathematical induction on ℓ and λ.

The case $\ell=1$ is completely treated in Theorem 7.1 as we remarked in the first sentence of the proof.

If $\lambda = 0$, then i must be 0. In order that $\Psi(\{A\}_\ell;\{D\}_\ell;k,0,i;q)$ be nonzero we must also have $D_1 = D_2 = \dots = D_{\ell-1} = 0$, $A_0+\dots+A_{\ell-1} = 0$. However, by the conditions imposed upon the A_j and D_j in the statement of the theorem we see that $A_0 \leq -1$, $A_j \leq 0$, $1 \leq j \leq \ell-1$ if $\ell > 1$. Consequently

$$\Psi(\{A\}_\ell;\{D\}_\ell;k,0,0;q) = \begin{cases} 1 & \text{if } \ell=1,\ A_0 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

As for $\Psi(\{A\}_\ell;\{D\}_\ell;k;0;i;q)$ we see that (7.5) is false for $c = 1$ if $A_0 = -1$; hence in this case also we must have $\ell=1$ and thus by Theorem 7.1

$$\psi(\{A\}_\ell;\ \{D\}_\ell;k;0;0;q) = \begin{cases} 1 & \text{if } \ell=1,\ A_0 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence the theorem is true for $\lambda = 0$.

If $\lambda=1$, then i must be 1 also. In order that $\Psi(\{A\}_\ell;\{D\}_\ell;\ k;1;1;q)$ be nonzero we must have each $D_j = 0$ or 1. Since $A_0 \leq D_1 + i-\lambda-1 \leq 1+1-1-1 = 0$, we must have $A_0 = 0$, $D_1 = 1$ if $\ell > 1$, and $A_j \leq D_j+D_{j+1}-1$ implies $A_j \leq D_j$ for each j with $1 \leq j \leq \ell-1$. Hence

$$\begin{aligned} &\lambda-2i+A_0+\ldots+A_{\ell-1}-D_1-\ldots-D_{\ell-1} \\ &\leq 1-2+0+D_1+\ldots+D_{\ell-1}-D_1-\ldots-D_{\ell-1} = -1. \end{aligned}$$

Thus again if $\ell > 1$ then $\Psi(\{A\}_\ell;\{D\}_\ell;k,1,1;q) = 0$. If $\ell = 1$, we want $0 \leq 2i-A_0 \leq \lambda = 1$, or $A_0 = 1$ or 2; however $A_0 \neq 2$ since A_0 must be $\leq i$ when $\ell = 1$. Hence by Theorem 7.1,

$$\Psi(\{A\}_\ell;\ \{D\}_\ell\ ;k;1;1;q) = \begin{cases} q & \text{if } A_0 = \ell = 1 \\ 0 & \text{in all other admissible cases.} \end{cases}$$

As for $\psi(\{A\}_\ell;\{D\}_\ell;k;1;1;q)$ we know that we may assume that $0 \leq D_j \leq \lambda = 1$ for each j (see the first paragraph of this proof). Hence if $\ell < 1$, $A_0 \leq D_1+i-\lambda-1 \leq 1+1-1-1 = 0$, and so for (7.5) to hold with $c = 0$ we must have $A_c = 0$. Thus $A_0 = f_1 = 0$ when $\ell > 1$ and so $(7.9)_b$ cannot obtain because b must be 1 and

$$0 = f_1 \not\geq i-b = 1-1 = 0.$$

Hence for $\psi(\{A\}_\ell;\{D\}_\ell;k;1;1;q)$ to be nonzero we must have $\ell = 1$. Whence by Theorem 7.1,

$$\psi(\{A\}_\ell;\{D\}_\ell;k;1;1;q) = \begin{cases} q & \text{if } A_0 = \ell = 1, \\ 0 & \text{in all other admissable cases} \end{cases}$$

Hence the theorem is true when $\lambda = 1$.

We have one more special case to treat before proceeding to the recurrences that will establish the induction. The case to be considered is $i = \lambda$. If $\ell = 1$, this is merely a special case of Theorem 7.1. If $\ell > 1$, then for $\Psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;\lambda;q)$ to be nonzero we must have $0 \leq D_j \leq \lambda$ for each j. Hence $A_0 \leq D_1+\lambda-\lambda-1 \leq \lambda-1$. Thus

$$\lambda-2i+A_0+A_1+\ldots+A_{\ell-1}-D_1-\ldots-D_{\ell-1}$$
$$\leq \lambda-2\lambda+\lambda-1 + \sum_{j=1}^{\ell-2} (D_j+D_{j+1}-\lambda) + D_{\ell-1}-D_1-\ldots-D_{\ell-1}$$
$$= -1 - (\ell-2)\lambda + \sum_{j=2}^{\ell-1} D_j$$
$$\leq -1 - (\ell-2)\lambda + (\ell-2)\lambda = -1.$$

Hence $\Psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;\lambda;q) = 0$ in all admissible cases when $\ell > 1$. As for $\psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;\lambda;q)$, we know that $0 \leq D_j \leq \lambda$ must hold for each j in order that $\psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;\lambda;q) \neq 0$. Thus again

$$A_0 \leq D_1+\lambda-\lambda-1 \leq \lambda-1,$$

and this makes $(7.9)_b$ untenable since by (7.8)

$$\lambda+2-2b \geq f_b+\ldots+f_{\lambda+1-b} > \lambda-b$$

implies that b must be 1 (since $b \geq 1$); but

$$\lambda-1 \geq A_0 = f_1+\ldots+f_\lambda > \lambda-1$$

is false. Hence the theorem is true if $i = \lambda$.

We now prove a recurrence satisfied by both $\psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;i;q)$ and $\Psi(\{A\}_\ell;\{D\}_\ell;k,\lambda;i;q)$ that will provide the passage from $\lambda-2$ to λ, namely under conditions of the theorem assuming additionally $\ell > 1$, $\frac{\lambda}{2} \leq i < \lambda$, $A_{\ell-1} < D_{\ell-1}$

(7.14)
$$\Psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;i;q)$$
$$= q^{A_0+2(k-D_1)+3A_1+4(k-D_2)+\ldots+(2\ell-2)(k-D_{\ell-1})+(2\ell-1)A_{\ell-1}}$$
$$\cdot \sum_{\epsilon_0=0}^{1} \cdots \sum_{\epsilon_{\ell-1}=0}^{1} \sum_{\delta_1=0}^{1} \cdots \sum_{\delta_\ell=0}^{1}$$
$$q^{\delta_\ell(\lambda-1)\ell} \Psi(A_0-\epsilon_0-\delta_1,\ A_1-\epsilon_1-\delta_2,\ldots,A_{\ell-1}-\epsilon_{\delta-1}-\delta_\ell;\{D-\epsilon-\delta\}_\ell;k;\lambda-2,i-1;q).$$

We first prove the Ψ satisfies (7.14). From (5.7) and (7.4), we see that

$$\sigma_j(\lambda) = q^i \sum_{\epsilon=0}^{1} \sigma_{j-\epsilon}(\lambda-1)$$

(7.15)
$$= q^j \sum_{\epsilon=0}^{1} \sum_{\delta=0}^{1} \sigma_{j-\epsilon-\delta}(\lambda-2)q^{\delta(\lambda-1)}.$$

Applying (7.15) to each σ-function in (7.13), we deduce that

$$\Psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;\ i;q)$$
$$= \sum_{\epsilon_0=0}^{1} \sum_{\epsilon_1=0}^{1} \cdots \sum_{\epsilon_{\ell-1}=0}^{1} \sum_{\delta_1=0}^{1} \sum_{\delta_2=0}^{1} \sum_{\delta_\ell=0}^{1}$$
$$q^{\lambda-2i+A_0+\sum_{b=1}^{\ell-1} A_b+\sum_{b=1}^{\ell} \delta_b(\lambda-1)+\sum_{b=1}^{\ell-1}(k-D_b+A_b)b(\lambda+1)+(\lambda+1)(i-\frac{\lambda}{2})(\lambda-2)}.$$
$$\sigma_{\lambda-2i+A_0-\epsilon_0-\delta_\ell+\sum_{b=1}^{\ell-1}(A_b-D_b)}(\lambda-2)\sigma_{D_1-\epsilon_1-\delta_1}(\lambda-2)\cdots\sigma_{D_{\ell-1}-\epsilon_{\ell-1}-\delta_{\ell-1}}(\lambda-2)$$

Now

$$\sum_{b=1}^{\ell-1} (k-D_b+A_b)\, b(\lambda+1) = \sum_{b=1}^{\ell-1} (k-(D_b-\epsilon_b-\delta_b) + (A_b-\epsilon_b-\delta_{b+1}))b(\lambda-1)$$

$$+ 2\sum_{b=1}^{\ell-1} (k-D_b+A_b)b - \sum_{b=1}^{\ell-1} \delta_b(\lambda-1) + \delta_\ell(\lambda-1)(\ell-1).$$

Hence

$$\Psi(\{A\}_\ell;\ \{D\}_\ell;k;\lambda;i;q)$$

$$= q^{\sum_{b=1}^{\ell-1} (k-D_b)2b+ \sum_{b=0}^{\ell-1} A_b(2b+1)}$$

$$\sum_{\epsilon_0=0}^{1}\ \sum_{\epsilon_1=0}^{1} \cdots \sum_{\epsilon_{\ell-1}=0}^{1}\ \sum_{\delta_1=0}^{1}\ \sum_{\delta_2=0}^{1} \cdots \sum_{\delta_\ell=0}^{1}$$

$$q^{\delta_\ell \ell(\lambda-1)}\ \Psi(A_0-\epsilon_0-\delta_1, A_1-\epsilon_1-\delta_2, \ldots, A_{\ell-1}-\epsilon_{\ell-1}-\delta_\ell; \{D-\epsilon-\delta\}_\ell;k;\lambda-2;i-1;q).$$

Hence Ψ satisfies (7.14).

To see that $\Psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;i;q)$ satisfies (7.14) we transform the partitions enumerated by $\pi^*(\{A\}_\ell;\{D\}_\ell;k;\lambda;i;q)$ as follows: we delete 1 if it appears as a summand; we subtract 1 from the remaining summands in $[1,\lambda]$; we subtract 2 from each part $\lambda+1$, and in general we subtract $2b+1$ from each part in the interval $[b(\lambda+1)+1, b(\lambda+1)+\lambda]$ and $2b$ from each part $b(\lambda+1)$, and finally we delete $\ell(\lambda+1)-1$ if it appears initially. We now split such partitions $\sum_{j=1}^{\ell(\lambda+1)-1} f_j \cdot j$ into 4^j classes according to the equations:

$$f_{c(\lambda+1)+1} = \epsilon_c, \quad 0 \leq c \leq \ell-1,$$

and

$$f_{b(\lambda+1)-1} = \delta_b, \quad 1 \leq b \leq \ell$$

The above transformation replaces

$$f_{c(\lambda+1)+1}+\dots+f_{c(\lambda+1)+\lambda} = A_c$$

by

$$f'_{c(\lambda-1)+1}+\dots+f'_{c(\lambda-1)+\lambda-1} = A_c - \epsilon_c - \delta_{c+1}$$

and

$$f_{c(\lambda+1)} = k - D_c$$

by

$$f'_{c(\lambda+1)} = k - D_c+\epsilon_c+\delta_c = k - (D_c - \epsilon_c - \delta_c).$$

The question now is whether the transformed partitions are of the type enumerated by $\pi^*(A_0-\epsilon_0-\delta_1,\dots,A_{\ell-1}-\epsilon_{\ell-1}-\delta_\ell;\{D-\epsilon-\delta\}_\ell;k;\lambda-2;i-1;q)$. We see that (7.7) still holds since if

$$f'_m+\dots+f'_{b(\lambda+1)}+\dots+f_{m+\lambda} \geq k,$$

we have in the transformed partition

$$f_{m-2b+1}+\dots+f_{b(\lambda-1)}+\dots+f_{m+\lambda-2b-1} \geq k;$$

(7.8) is valid by construction, and $(7.9)_b$ follows with λ replaced by

$\lambda-2$ and i replaced by $i-1$ exactly as in the proof of Theorem 7.1. The above transformation is clearly reversible and so establishes a bijection between the partitions enumerated by $\pi^*(\{A\}_\ell;\{D\}_\ell;k;\lambda;i;q)$ and the disjoint union of the sets of partitions enumerated by the 4^ℓ partition functions

$$\pi^*(A_0-\epsilon_0-\delta_1,A_1-\epsilon_1-\delta_2,\ldots,A_{\ell-1}-\epsilon_{\ell-1}-\delta_\ell;\{D-\epsilon-\delta\}_\ell;k;\lambda-2;i-1;n-\sum_{b=0}^{b-1}A_b(2b+1)$$
$$-\sum_{b=1}^{\ell-1}(k-D_b)2b-\delta_\ell(\lambda-1)\ell).$$

This is equivalent to (7.14) for the ψ-function.

Finally we must establish the following recurrence for the passage from $\ell-1$ to ℓ: if $A_{\ell-1}=D_{\ell-1}$

$$\Psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;i;q) \tag{7.16}$$
$$=q^{k(\ell-1)(\lambda+1)}\sigma_{D_{\ell-1}}(\lambda)\Psi(\{A\}_{\ell-1};\{D\}_{\ell-1};k;\lambda;i;q).$$

Equation (7.16) is immediately obvious for the Ψ-function if we set $A_{\ell-1}=D_{\ell-1}$ in (7.11). Turning to the ψ-function we see that $A_{\ell-1}=D_{\ell-1}$ implies (7.7) is valid for $m=(\ell-1)(\lambda+1)$ since

$$f_{(\ell-1)(\lambda+1)}+\cdots+f_{(\lambda-1)(\lambda+1)+\lambda}=k-D_{\ell-1}+A_{\ell-1}=k.$$

Thus the partitions under consideration need only be required to fulfill (7.7) in the first $(\ell-2)$ intervals listed since fulfillment is automatic in the $(\ell-1)$-st. Thus the parts in $[(\ell-1)(\ell+1),\ (\ell-1)(\lambda+1)+\lambda]$ are subject only to

$$f_{(\ell-1)(\lambda+1)+1}+\dots+f_{(\ell-1)(\lambda+1)+\lambda} = A_{\ell-1},$$

$$f_{(\ell-1)(\lambda+1)} = k - D_{\ell-1},$$

$$f_a > 1 \quad \text{implies} \quad (\lambda+1)|a.$$

Thus this portion of the partition is generated by

$$q^{k(\ell-1)(\ell+1)}\sigma_{D_{\ell-1}}(\lambda),$$

and it is independent of the remaining portion of the partition, which is generated by

$$\psi(\{A\}_{\ell-1};\{D\}_{\ell-1};k;\lambda;i;q).$$

Hence the product of these two generating functions yields $\psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;i;q)$, which establishes (7.16).

We now proceed to prove our theorem. We note that we have proved the theorem when $\ell = 1$, or when $\lambda = 0, 1$, or when $i = \lambda$. We now assume the theorem is true for any (ℓ',λ') such that either $\lambda' < \lambda$ or $\lambda' = \lambda$ and $\ell' < \ell$. We may also assume $\lambda \geq 2$ and $\ell \geq 2$, $\frac{\lambda}{2} \leq i < \lambda$.

If $A_{\ell-1} < D_{\ell-1}$, then we appeal to (7.14) for both Υ and ψ. To have such an appeal justified we must verify that the arguments of the functions on the right side of (7.14) fulfill the conditions of the theorem. Since $\ell > 1$, $A_j \leq D_j+D_{j+1}-\lambda$ implies $A_j-\epsilon_j-\delta_{j+1} \leq D_j-\epsilon_j-\delta_j+D_{j+1}-\epsilon_{j+1}-\delta_{j+1}-(\lambda-2)$ because $2-\delta_j-\epsilon_{j+1} \geq 0$. Also $A_0 \leq D_1-\lambda+i-1$, implies $A_0-\epsilon_0-\delta_1 \leq (D_1-\epsilon_1-\delta_1)-(\lambda-2)+(i-1)-1$ because $-\epsilon_0 \leq 0 \leq 1-\epsilon_1$. Also since

$\frac{\lambda}{2} \leq i < \lambda$; $\frac{\lambda-2}{2} \leq i-1 \leq \lambda-2$. Finally since $A_{\ell-1} < D_{\ell-1}$, we see that

$$A_{\ell-1} - \epsilon_{\ell-1} - \delta_{\ell} \leq D_{\ell-1} - \epsilon_{\ell-1} - \delta_{\ell-1}$$

for $-\delta_{\ell} \leq 0 \leq 1-\delta_{\ell-1}$.

If, however, $A_{\ell-1} = D_{\ell-1}$, then we instead appeal to (7.16) for Ψ and ψ . Here the condition on the right hand side are unaltered except that originally $A_{\ell-2} \leq D_{\ell-2}+D_{\ell-1}-\lambda \leq D_{\ell-2}$ which is as desired since we may always assume $0 \leq D_j \leq \lambda$. Thus in any case the truth of the theorem for ℓ and λ follows from the induction hypothesis. Hence Theorem 7.2 is established. □

8. The general theorem. We begin by considering $P_{\lambda,k,a}(m,n)$, the number of partitions of the type enumerated by $B_{\lambda,k,a}(n)$ that have exactly m parts. The discussion following Definition 2 implies that $P_{\lambda,k,q}(m,n) = P_{\lambda,k,a}(m,n)$ whenever $a \geq \lambda$. Our interest in this section centers on $\frac{\lambda}{2} < a < \lambda$.

We also define

$$J_{\lambda,k,a}(x) = \sum_{m\geq 0} \sum_{n\geq 0} P_{\lambda,k,a}(m,n)x^m q^n. \tag{8.1}$$

Theorem 8.1. Let $|q| < 1$, $|x| < |q|^{-1}$, then

$$\begin{aligned} J_{\lambda,k,a}(x) = J^{\dagger}_{\lambda,k,a}(x) - \sum_{r=0}^{a-1} \psi(1;k,\lambda,r,a;x;a)H^{\dagger}_{\lambda,k,a-r}(x) \\ + \sum_{\ell=2}^{\infty} (-1)^{\ell} \sum_{r=1}^{k} \psi(\ell;k,\lambda,r,a;x;q)H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)}), \end{aligned} \tag{8.2}$$

(8.3) $$\psi(\ell;k,\lambda,r,a;x;q)$$

$$= \sum_{E} x^{k(\ell-1)+A_0+A_1+\dots+A_{\ell-1}-D_1-D_2-\dots-D_{\ell-1}} \psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;a;q)$$

where E is the set of those $(2\ell-1)$-tuples $(A_0,A_1,\dots,A_{\ell-1},D_1,D_2,\dots,D_{\ell-1})$ that satisfy

$(8.4)_1$ $$A_0 + k-D_1 \leq a-1$$

$(8.4)_2$ $$k-D_{\ell-1} + A_{\ell-1} = k-r$$ (when $\ell=1$, this is replaced by $A_0=r$)

$(8.4)_3$ $$0 \leq A_0 \leq \lambda,$$

$(8.4)_4$ $$1 \leq D_j \leq k, \quad (1 \leq j \leq \ell-1),$$

$(8.4)_5$ $$0 \leq A_j \leq D_j, \quad (1 \leq j \leq \ell-2),$$

$(8.4)_6$ $$k-D_j+A_j+k-D_{j+1} \leq k-1, \quad (1 \leq j \leq \ell-2).$$

Proof. The argument here is exactly parallel the proof of equation (3.9) given in Theorem 6.1. For this reason our presentation will be slightly terser.

We start with

(8.5) $$S_1(x) = \sum_{r=1}^{a-1} \psi(1;k,\lambda,r,a;x;q)\, H^{\dagger}_{\lambda,k,a-r}(x) - (J^{\dagger}_{\lambda,k,a}(x) - J_{\lambda,k,a}(x)).$$

Now $\psi(1;k,\lambda,r,a;x;q)$ is the generating function for partitions with r distinct parts each $\leq \lambda$ for which at least one of the inequalities

(8.6) $$f_b+\dots+f_{\lambda+1-b} > a-b, \quad 1 \leq b \leq \frac{1}{2}(\lambda+1)$$

holds (we note that the case $b = \frac{1}{2}\lambda + 1$ is now not relevant since $(8.6)_b$ for $b = \frac{1}{2}\lambda + 1$ is true only for $a = \frac{1}{2}\lambda$ and we have under consideration only those a for which $\frac{1}{2}\lambda < a < \lambda$). Hence $\psi(1;k,\lambda,r,a;x;q)H^{+}_{\lambda,k,q-r}(x)$ is the generating function for partitions of the type enumerated by $P_{\lambda,k,a}(m,n)$ (m and n arbitrary) with the conditions that (i) there are exactly r parts $\leq \lambda$, (ii) the inequality $(8.6)_b$ occurs for some b, and (iii) the inequality $f_m+\dots+f_{m+\lambda} \geq k$ might occur for some n with $1 \leq m \leq \lambda$. Therefore $S_1(x)$ (defined by (8.5)) generates the partitions which satisfy $f_m+f_{m+1}+\dots+f_{m+\lambda} \geq k$ for some m with $1 \leq m \leq \lambda$, but otherwise fulfill the conditions on the partitions generated by $J_{\lambda,k,a}(x)$.

For each $\ell \geq 2$, we define $\psi(\ell;k,\lambda,r,a;x;q)$ to be the generating function for partitions of the form $\sum_{j=1}^{\ell(\lambda+1)-1} f_j \cdot j$ where

$(8.7)_1$ $$f_{\ell(\lambda+1)-\lambda}+\dots+f_{\ell(\lambda+1)-1} = k-r$$

$(8.7)_2$ $$f_1+\dots+f_{\lambda+1} \leq a-1$$

$(8.7)_3$ $$f_{c(\lambda+1)}+\dots+f_{(c+1)(\lambda+1)} \leq k-1, \quad \text{for } 1 \leq c \leq \ell-2$$

$(8.7)_4$ $$f_m+\dots+f_{m+\lambda} \geq k \quad \text{for some } m \text{ in each of}$$

$$[1,\lambda],\ [\lambda+2,2\lambda+1],\dots,[(\ell-2)(\lambda+1)+1,\ (\ell-1)(\lambda+1)-1],$$

$(8.7)_5$ $$f_h > 1 \quad \text{implies} \quad (\lambda+1)\,|\,h,$$

$(8.7)_6$ $$f_b+\dots+f_{\lambda+1-b} > a-b \quad \text{for some } b \text{ with } 1 \leq b \leq \frac{1}{2}(\lambda+1).$$

We thus may easily establish (by mathematical induction) that if

$$(8.8)\qquad \sum_{r=0}^{k} \psi(\ell;k,\lambda,r,a;x;q)\, H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)}) - S_{\ell-1}(x) = S_{\ell}(x),$$

then $S_{\ell}(x)$ is the generating function for partitions that satisfy

$$(8.9)_1\qquad f_1+\dots+f_{\lambda+1} \leq a-1,$$

$$(8.9)_2\qquad f_{c(\lambda+1)}+\dots+f_{(c+1)(\lambda+1)} \leq k-1, \quad \text{for all } c \geq 1,$$

$$(8.9)_3\qquad f_m+\dots+f_{m+\lambda} \leq k-1, \quad \text{for all } m \notin [2,\lambda],$$

$$[\lambda+2,2\lambda+1],\dots,\ [\ell(\lambda+1)-\lambda,\ \ell(\lambda+1)-1],$$

$$(8.9)_4\qquad f_m+\dots+f_{m+\lambda} \geq k \quad \text{for some } m \text{ in each of}$$

$$[2,\lambda],[\lambda+2,2\lambda+1],[2\lambda+3,3\lambda+2],\dots,[\ell(\lambda+1)-\lambda,\ell(\lambda+1)-1],$$

$$(8.9)_5\qquad f_h > 1 \quad \text{implies} \quad (\lambda+1)\,|\,h,$$

$$(8.9)_6\qquad f_b+\dots+f_{\lambda+1-b} > a-b \quad \text{for some } b \in [1,\tfrac{1}{2}(\lambda+1)].$$

Before proceeding we note that (8.8) and (8.5) imply that

$$(8.10)\qquad S_{\ell}(x) = \sum_{j=0}^{\ell-2} (-1)^j \sum_{r=0}^{k} \psi(\ell-j;k,\lambda,r,a;x;q)H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1-j)(\lambda+1)})$$

$$+ (-1)^{\ell} (J^{\dagger}_{\lambda,k,a}(x)-J_{\lambda,k,a}(x)) - (-1)^{\ell}\sum_{r=0}^{a-1} \psi(1;k,\lambda,r,a;x;q)H^{\dagger}_{\lambda,k,a-r}(x),$$

or

$$(8.11)\qquad J_{\lambda,k,a}(x) = J^{\dagger}_{\lambda,k,a}(x) - \sum_{r=0}^{a-1} \psi(1;k,\lambda,r,a;x;q)H^{\dagger}_{\lambda,k,a-r}(x)$$

$$- \sum_{j=2} (-1)^j \sum_{r=0}^{k} \psi(j;k,\lambda,r,a;x;q) H^{\dagger}_{\lambda,k,r}(xq^{(j-1)(\lambda+1)})$$
$$- (-1)^{\ell} S_{\ell}(x).$$

As in the case of Theorem 6.1, we have two tasks left to complete our proof. First we must show that the functions $\psi(\ell;k,\lambda,r,a;x;q)$ as defined above are precisely the polynomials given by (8.3), and second we must show that $S_{\ell}(x) = 0$ for ℓ sufficiently large (actually for $\ell \geq \lambda+1$).

We consider the partitions that are generated by $\psi(\ell;k,\lambda,r,a;x;q)$ (i.e. those partitions $\sum_{j=1}^{\ell(\lambda+1)-1} f_j \cdot j$ that satisfy $(8.7)_1$-$(8.7)_6$. We split these partitions into subclasses where

$(8.12)_1$ $\qquad f_1+\ldots+f_{\lambda+1} = A_0 \leq a-1,$

$(8.12)_2$ $\qquad f_{c(\lambda+1)+1}+\ldots+f_{c(\lambda+1)+\lambda} = A_c,$ for $0 \leq c \leq \ell-1,$

$(8.12)_3$ $\qquad f_{c\lambda+c} = k-D_c,$ for $1 \leq c \leq \ell-1,$

$(8.12)_4$ $\qquad f_m+\ldots+f_{m+\lambda} \geq k,$ for some m in each of

$\qquad [1,\lambda+1],\ [\lambda+2,2\lambda+2],\ldots,[(\ell-2)(\lambda+1)+1,(\ell-1)(\lambda+1)],$

$(8.12)_5$ $\qquad f_h > 1$ implies $(\lambda+1)|h,$

$(8.12)_6$ $\qquad f_b+\ldots+f_{\lambda+1-b} > a-b$ for some $b \in [1,\frac{1}{2}(\lambda+1)].$

where we must consider exactly those $(2\ell-1)$-tuples $(A_0,A_1,\ldots,A_{\ell-1},D_1,\ldots,D_{\ell-1})$ that satisfy $(8.4)_1$-$(8.4)_6$ (the condition $A_j \leq D_j$ in $(8.4)_5$ is redundant as it is implies by $(8.4)_4$ and $(8.4)_6$). Since $k \geq \lambda$, we see that $(8.4)_6$ implies

$$A_j \le D_j+D_{j+1}-k-1 \le D_j+D_{j+1}-\lambda,$$

and $(8.4)_1$ implies

$$A_0 \le D_1-k+a-1 \le D_1+a-\lambda-1.$$

Consequently we may invoke Theorem 7.2, and we see that since the partitions of the above subclass are generated by $\psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;a;q)$, we have

$$\psi(\ell;k,\lambda,r,a;x;q)$$

$$= \sum_E x^{k(\ell-1)+A_0+A_1+\dots+A_{\ell-1}-D_1-D_2-\dots-D_{\ell-1}} \psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;a;q)$$

$$(8.13) \qquad = \sum_E x^{k(\ell-1)+A_0+A_1+\dots+A_{\ell-1}-D_1-D_2-\dots-D_{\ell-1}} \Psi(\{A\}_\ell;\{D\}_\ell;k;\lambda;a;q).$$

where E is the set of those $(2\ell-1)$-tuples $(A_0,A_1,\dots,A_{\ell-1},D_1,D_2,\dots,D_{\ell-1})$ that satisfy $(8.4)_1-(8.4)_6$.

The fact that $\psi(\ell;k,\lambda,r,a;x;q)$ is a polynomial in x and q is immediate since the partitions generated are among those in which each part is $\le \ell(\lambda+1)-1$, parts not divisible by $\lambda+1$ are not repeated, and parts divisible by $\lambda+1$ are repeated at most $k-1$ times. Hence for $\psi(\ell;k,\lambda,r,a;x;q)$ (just as for $g_1(\ell;k,\lambda,r,i;x;q)$ in Theorem 6.2) the coefficient of $x^M q^N$ is zero if either

$$M > \ell\lambda + (\ell-1)(k-1),$$

or

$$N > \binom{\ell(\lambda+1)}{2} + (k-2)(\lambda+1)\binom{\ell}{2}.$$

To show that $S_\ell(x) = 0$ for $\ell \geq \lambda+2$, we begin by proving that

(8.14) $$\psi(\{A\}_\ell;\{D\}_\ell;k,\lambda,a;q) = 0$$

for $\ell \geq \lambda+2$.

In order that (8.14) be false we must have (by Theorem 7.2)

(8.15) $$0 \leqq D_c \leqq \lambda,\ 1 \leqq c \leqq \ell-1.$$

(8.16) $$0 \leq 2i-A_0-A_1-\dots-A_{\ell-1}+D_1+D_2+\dots+D_{\ell-1} \leq \lambda$$

We now apply $(8.4)_6$ to (8.16) and we assume $i > \frac{\lambda}{2}$, $k \geqq \lambda$, $\ell \geq \lambda + 2$,

$$\lambda \geqq 2i-\sum_{j=0}^{\ell-1} A_j+\sum_{j=1}^{\ell-1} D_j$$

$$\geqq 2i-\lambda-\sum_{j=1}^{\ell-2}(D_j+D_{j+1}-k-1) - D_{\ell-1}+r+\sum_{j=1}^{\ell-1} D_j$$

$$= 2i-\lambda+(\ell-2)(k+1) - \sum_{j=2}^{\ell-1} D_j+r$$

$$\geqq 2i-\lambda+(\ell-2)(k+1)-(\ell-2)\lambda+r$$

$$\geqq 2i-\lambda+\ell-2+r$$

$$> 2\cdot\frac{\lambda}{2} - \lambda+\lambda+r = \lambda+r \geqq \lambda,$$

which is impossible. Consequently (8.14) holds for $\ell \geqq \lambda+2$, and therefore

$$\psi(\ell;k,\lambda,r,a;x;q)$$

$$= \sum_E x^{k(\ell-1)+A_0+A_1+\dots+A_{\ell-1}-D_1-\dots-D_{\ell-1}}\ \psi(\{A\}_\ell;\{D\}_\ell;k,\lambda,a;q)$$

$$= 0, \quad \text{for } \ell \geqq \lambda+2.$$

Finally we observe that $S_\ell(x)$ always has nonnegative coefficients since the coefficient of $x^M q^N$ in $S_\ell(x)$ is the number of partitions of N into M parts that satisfy certain conditions. Now for $\ell \geqq \lambda+2$

$$\begin{aligned} S_\ell(x) &= \sum_{r=0}^{k} \psi(\ell;k,\lambda,r,a;x;q)\ H^{\dagger}_{\lambda,k,r}(xq^{(\ell-1)(\lambda+1)}) - S_{\ell-1}(x) \\ &= -S_{\ell-1}(x) \end{aligned}$$

and the only way the functions here can each have nonnegative coefficients is if each is identically zero. Therefore

$$S_\ell(x) = 0 \quad \text{for} \quad \ell \geqq \lambda+1.$$

Thus (8.2) now follows from (8.11) when we take $\ell \geqq \lambda+1$ in (8.11). This completes the proof of theorem 8.1.

Theorem 8.2. For $|q| < 1$, $\frac{\lambda}{2} < a < \lambda$,

$$J_{\lambda,k,a}(1) = J_{\lambda,k,a}(1).$$

Remark. This result is weaker than the corresponding result in Theorem 6.2 for $a \geq \lambda$. In fact, however, replacing the argument 1 by x would make the assertion false.

Proof. First we observe that for $\ell \geqq 2$

$$\psi(\ell;k,\lambda,r,a;1;q) \tag{8.17}$$

$$= \sum_{E} \sigma_{2i-A_0-\dots-A_{\ell-1}+D_1+\dots+D_{\ell-1}}(\lambda)\sigma_{D_1}(\lambda)\sigma_{D_2}(\lambda)\dots\sigma_{D_{-1}}(\lambda)\, q^{\sum_{b=1}^{\ell-1}(k-D_b+A_b)b(\lambda+1)-(i-A_0-A_1-\dots-A_{\ell-1}+D_1+\dots+D_{\ell-1})(\lambda+1)},$$

where E is defined by (8.4)-$(8.4)_6$. Now many of the conditions in (8.4)-$(8.4)_6$ are redundant. In particular $(8.4)_4$ may be dropped since in order for a term to be nonzero we must have

$$0 \leqq D_j \leqq \lambda \leqq k,$$

and if $D_j = 0$, we see by $(8.4)_5$ and $(8.4)_6$, that

$$0 \leqq A_j \leqq D_j+D_{j+1}-k-1 \leqq \lambda-k-1 \leqq -1$$

which is impossible. The condition $A_j \leqq D_j$ in $(8.4)_5$ is also redundant since in order to produce a nonzero term we must have $D_{j+1} \leqq \lambda$ and hence

$$A_j \leqq D_j+D_{j+1}-k-1 \leqq D_j+ -k-1 \leqq D_j.$$

Furthermore $(8.4)_3$ is redundant since $(8.4)_1$ implies that (together with $D_1 \leqq \lambda$)

$$A_0 \leqq D_1-k+a-1 \leqq \lambda-k+\lambda-1 \leqq \lambda-1 < \lambda,$$

and also to produce a nonzero term we must have

$$2a-A_0-\dots-A_{\ell-1}+D_1+\dots+D_{\ell-1} \leqq \ell;$$

thus

$$\begin{aligned} A_0 &\geqq 2a-\lambda-A_1-\dots-A_{\ell-1}+D_1+\dots+D_{\ell-1} \\ &> 2 \cdot (\tfrac{\lambda}{2})-\lambda+(D_1-A_1)+\dots+(D_{\ell-1}-A_{\ell-1}) \\ &\geqq 0. \end{aligned}$$

Finally if we replace $A_{\ell-1}$ by $D_{\ell-1}-r$ we eliminate $(8.4)_2$, and if we replace A_0 by

$$2a-\mu-A_1-\dots-A_{\ell-1}+D_1+\dots+D_{\ell-1}$$

$$= 2a-\mu-A_1-\dots-A_{\ell-2}+r+D_1+\dots+D_{\ell-2}$$

then $(8.4)_1$ becomes

$$D_2+\dots+D_{\ell-2}-A_1-\dots-A_{\ell-2} \leqq \mu-a-r-k-1.$$

Hence we may rewrite (8.17) as

(8.18) $$\psi(\ell;k,\lambda,r,a;1;q)$$

$$= \sum_{E'} \sigma_{\mu}(\lambda)\sigma_{D_1}(\lambda)\sigma_{D_2}(\lambda)\dots\sigma_{D_{\ell-1}}(\lambda) q^{\sum_{b=1}^{\ell-2}(k-D_b+A_b)b(\lambda+1)-(\mu-i)(\lambda+1)+(k-r)(\ell-1)(\lambda+1)}$$

where E' is the set of those $(2\ell-2)$-tuples $(A_1,\dots,A_{\ell-2},\mu,D_1,\dots,D_{\ell-1})$ that satisfy

$(8.19)_1$ $$D_2+\dots+D_{\ell-2}-A_1-\dots-A_{\ell-2} \leqq \mu-a-r-k-1,$$

$(8.19)_2$ $$0 \leqq A_c \leqq D_c+D_{c+1}-k-1, \quad \text{for } 1 \leqq c \leqq \ell-1.$$

We now compare the right hand side of (8.18) with the right hand side of (6.29) and we see that the mapping of E' onto $\mathcal{D}''$ given by $(A_1,\dots,A_{\ell-2},\mu,D_1,\dots,D_{\ell-1}) \to (\alpha_1,\dots,\alpha_{\ell-2},\mu,\delta_1,\dots,\delta_{\ell-1})$ shows that

(8.20) $$\psi(\ell;k,\lambda,r,a;1;q)$$

$$= \gamma(\ell+1;k,\lambda,r,k-a+1;q^{-\lambda-1};q)$$

$$- \gamma(\ell+1;k,\lambda,r,k-a;q^{-\lambda-1};q).$$

Equation (8.20) now makes the establishment of Theorem 8.2 a straightforward matter.

$$J_{\lambda,k,a}(1) = J^{\dagger}_{\lambda,k,a}(1) - \sum_{r=0}^{a-1} \psi(1;k,\lambda;r,a;1;q)H^{\dagger}_{\lambda,k,a-r}(1)$$

$$+ \sum_{\ell\geq 2} (-1)^{\ell} \sum_{r=0}^{k} \psi(\ell;k,\lambda,r,a;1;q)H^{\dagger}_{\lambda,k,r}(q^{(\ell-1)(\lambda+1)})$$

(by Theorem 8.1)

$$= \sum_{j=0}^{a} \sigma_j(\lambda)H^{\dagger}_{\lambda,k,a-j}(1) - \sum_{r=0}^{a-1} q^{(r-a)(\lambda+1)}\sigma_{2a-r}(\lambda)H^{\dagger}_{\lambda,k,a-r}(1)$$

$$+ \sum_{\ell\geq 2} (-1)^{\ell} \sum_{r=0}^{k} \psi(\ell;k,\lambda,r,a;1;q)H^{\dagger}_{\lambda,k,r}(q^{(\ell-1)(\lambda+1)})$$

$$+ \sum_{\ell\geq 2} (-1)^{\ell-1} \sum_{r=0}^{k} g_1(\ell;k,\lambda,r,a;1;q)H^{\dagger}_{\lambda,k,r}(q^{(\ell-1)(\lambda+1)})$$

(by (3.9) and Theorem 7.1)

$$= \sum_{j=0}^{a} \sigma_j(\lambda)\{H^{*}_{\lambda,k,a-j}(1)$$

$$- \sum_{\ell\geq 2} (-1)^{\ell-1} \sum_{r=0}^{k} H^{\dagger}_{\lambda,k,r}(q^{(\ell-1)(\lambda+1)})\gamma(\ell;k,\lambda,r,a-j;1;q)\}$$

$$- \sum_{\rho=1}^{\lambda-a} q^{-\rho(\lambda+1)}\sigma_{a+\rho}(\lambda)\ H^{\dagger}_{\lambda,k,\rho}(1)$$

$$+ \sum_{\ell\geq 2} (-1)^{\ell} \sum_{r=0}^{k} \psi(\ell;k,\lambda,r,a;1;q)H^{\dagger}_{\lambda,k,r}(q^{(\ell-1)(\lambda+1)})$$

$$+ \sum_{\ell\geq 2} (-1)^{\ell-1} \sum_{r=0}^{k} g_1(\ell;k,\lambda,r,a;1;q)H^{\dagger}_{\lambda,k,r}(q^{(\ell-1)(\lambda+1)})$$

(by (6.16))

$$= \sum_{j=0}^{a} \sigma_j(\lambda)\ H^{*}_{\lambda,k,a-j}(1)$$

$$- \sum_{\ell\geq 2} (-1)^{\ell-1} \sum_{r=0}^{k} H^{\dagger}_{\lambda,k,r}(q^{(\ell-1)(\lambda+1)}) \sum_{j=0}^{a}\sigma_j(\lambda)\gamma(\ell;k,\lambda,r,a-j;1;q)$$

$$- \sum_{\rho=1}^{\lambda-a} q^{-\rho(\lambda+1)}\sigma_{a+\rho}(\lambda)\{H^{*}_{\lambda,k,\rho}(1)$$

$$- \sum_{\ell\geq 2} (-1)^{\ell-1} \sum_{r=0}^{k} H^{\dagger}_{\lambda,k,r}(q^{(\ell-1)(\lambda+1)})\gamma(\ell;k,\lambda,r,\rho;1;q)\}$$

$$+ \sum_{\ell\geq 2} (-1)^{\ell} \sum_{r=0}^{k} \psi(\ell;k,\lambda,r,a;1;q)\ H^{\dagger}_{\lambda,k,r}(q^{(\ell-1)(\lambda+1)})$$

$$+ \sum_{\ell\geq 2} (-1)^{\ell-1} \sum_{r=0}^{k-1} g_1(\ell;k,\lambda,r,a;1;q)\ H^{\dagger}_{\lambda,k,r}(q^{(\ell-1)(\lambda+1)}) \qquad \text{(by (6.16))}$$

$$= \sum_{j=0}^{a} \sigma_j(\lambda)\ H^{*}_{\lambda,k,a-j}(1) - \sum_{\rho=1}^{\lambda-a} q^{-\rho(\lambda+1)}\sigma_{\rho+a}(\lambda)H^{*}_{\lambda,k,\rho}(1)$$

$$+ \sum_{\ell\geq 2} (-1)^{\ell-1} \sum_{r=1}^{k} H^{\dagger}_{\lambda,k,r}(q^{(\ell-1)(\lambda+1)}).\{ - \sum_{j=0}^{a} \sigma_j(\lambda)\gamma(\ell;k,\lambda,r,a-j;1;q)$$

$$+ g_1(\ell;k,\lambda,r,a;1;q)$$

$$+ \sum_{\rho=1}^{\lambda-a} q^{-\rho(\lambda+1)}\sigma_{\rho+a}(\lambda)\gamma(\ell;k,\lambda,r,\rho;1;q)$$

$$-\gamma(\ell+1;k,\lambda,r,k-a+1;q^{-\lambda-1};q)$$

$$+ \gamma(\ell+1;k,\lambda,r,k-a;q^{-\lambda-1};q)\} \qquad \text{(by (8.20))}$$

$$= J^{*}_{\lambda,k,a}(1) \qquad \text{(by (6.23))}$$

$$= J_{\lambda,k,a}(1) \qquad \text{(by Theorem 4.2)}.$$

Theorem 8.3. For $k \geqq \lambda$, $k \geqq a > \frac{\lambda}{2}$, we have that $A_{\lambda,k,a}(n) = B_{\lambda,k,a}(n)$ for all n.

Proof. We proceed exactly as in Theorem 6.3. We assume $a < \lambda$ since the case $a \geqq \lambda$ is covered by Theorem 6.3.

$$\begin{aligned} 1 + \sum_{n=1} B_{\lambda,k,a}(n)q^n &= J_{\lambda,k,a}(1) \\ &= J_{\lambda,k,a}(1) \qquad \text{(by Theorem 8.2)} \\ &= 1 + \sum_{n=1}^{\infty} A_{\lambda,k,a}(n)q^n, \end{aligned}$$

just as in Theorem 6.3. Hence if we compare coefficients in the extremes of this equation we derive Theorem 8.3. □

9. Conclusion. Several remarks should be made about the work just completed. First of all the technique utilized in proving results like equation (3.9) is quite general. It is a sieving process that should yield solutions to some of the unsolved problems described in [8].

It was conjectured in Section 5 of [4], that Theorem 8.3 of this paper is the best possible result obtainable. That is, if we allow $k < \lambda$, then Theorem 8.3 supposedly no longer holds. We have in this paper established that $k \geqq \lambda$ is sufficient for the truth of Theorem 8.3; in [4], the result was proved only for $k \geqq 2\lambda-1$. Evidence from the computer strongly suggests the following strengthened conjecture.

Conjecture 1. For $\frac{\lambda}{2} < a \leq k < \lambda$,

$$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$$

for $0 \leqq n < \binom{k+\lambda-a+1}{2} + (k-\lambda+1)(\lambda+1)$

while

$$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)+1,$$

when $n = \binom{k+\lambda-a+1}{2} + (k-\lambda+1)(\lambda+1)$.

Conjecture 1 has been verified for $3 \leqq \lambda \leqq 7$, $\frac{\lambda}{2} < k \leqq \min(\lambda-1,5)$, $\frac{\lambda}{2} < a \leqq k$.

Also a question was raised in [4] concerning the possibility of modifying the conditions on the partitions enumerated by $B_{\lambda,k,a}(n)$ so that values of $k < \lambda$ would be admissible. It was pointed out there that I.J. Schur has proved [14; p. 495] that

$$A_{3,2,2}(n) = B^0_{3,2,2}(n),$$

where $B^0_{3,2,2}(n)$ denotes the number of partitions enumerated by $B_{3,2,2}(n)$ with the added restriction that no parts are $\equiv 2 \pmod 4$.

Computer evidence suggests the truth of the following

Conjecture 2.

$$A_{4,3,3}(n) = B^0_{4,3,3}(n)$$

where $B^0_{4,3,3}(n)$ denotes the number of partitions $\sum_{j=1}^{\infty} f_j \cdot j = n$ enumerated by $B_{4,3,3}(n)$ with the added restrictions:

$$f_{5j+2} + f_{5j+3} \leqq 1, \quad \text{for} \quad j \geqq 0$$

$$f_{5j+4} + f_{5j+6} \leqq 1, \quad \text{for} \quad j \geqq 0$$

$$f_{5j-1} + f_{5j} + f_{5j+5} + f_{5j+6} \leqq 3, \quad \text{for} \quad j \geqq 1.$$

Conjecture 3 has been verified for $n \leqq 59$ (note $A_{4,3,3}(59) = 2938 = B^0_{4,3,3}(59)$).

Unfortunately the assumption $k \geqq \lambda$ so permeates the work in this paper that Conjecture 2 seems well beyond the techniques herein introduced.

If Conjecture 2 is in fact correct, the methods of proof should have interesting ramifications in the theory of partition identities.

References

1. G.E. Andrews, An analytic proof of the Rogers-Ramanujan-Gordon identities, Amer. J. Math., 88(1966), 844-846.

2. G.E. Andrews, A generalization of the Göllnitz-Gordon partition theorems, Proc. Amer. Math. Soc., 19(1967), 945-952.

3. G.E. Andrews, q-difference equations for certain well-poised basic hypergeometric series, Quart. J. Math. Oxford Ser. (2), 19(1968), 433-447.

4. G.E. Andrews, A generalization of the classical partition theorems, Trans. Amer. Math. Soc., 145(1969), 205-221.

5. G.E. Andrews, Sieves for theorems of Euler, Rogers, and Ramanujan, in The Theory of Arithmetic Functions, Lecture Notes in Math., No. 251, Springer, New York, 1971.

6. G.E. Andrews, Number Theory, W.B. Saunders, Philadelphia, 1971.

7. G.E. Andrews, Partition identities, Advances in Math., 9(1972), 10-51.

8. G.E. Andrews, A general theory of identities of the Rogers-Ramanujan type, Amer. Math. Soc. Audio Recording No. 75 and Supplementary Manual, Providence, 1972.

9. G.E. Andrews, Sieves in the theory of partitions, Amer. J. Math., 94(1972), 1214-1230.

10. H. Göllnitz, Parititionen mit Differenzenbedingungen, J. reine und angew. Math., 225(1967), 154-190.

11. B. Gordon, A combinatorial generalization of the Rogers-Ramanujan identities, Amer. J. Math., 83(1961), 393-399.

12. G.H. Hardy and E.M. Wright, An Introduction to the Theory of Numbers, 4th ed., Oxford University Press, Oxford, 1960.

13. H. Rademacher, Lectures on Elementary Number Theory, Blaisdell, New York, 1964.

14. I.J. Schur, Zur additiven Zahlentheorie, Sitzungsber. Deutsch. Adak. Wissensch., Berlin, Phys.-Math. Kl., 1926, 488-495.

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