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APPLICATIONS OF BASIC HYPERGEOMETRIC FUNCTIONS*

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Abstract. This paper surveys recent applications of basic hypergeometric functions to partitions, number theory, finite vector spaces, combinatorial identities and physics.

1. Introduction. The theory of basic hypergeometric functions has until recently existed very much in the shadow of ordinary hypergeometric functions. The basic hypergeometric function is

$$(1.1)\qquad {}_m\phi_n\begin{bmatrix} a_1, \cdots, a_m; q, z \\ b_1, \cdots, b_n \end{bmatrix} = \sum_{j \geqq 0} \frac{(a_1)_j(a_2)_j \cdots (a_m)_j z^j}{(b_1)_j(b_2)_j \cdots (b_n)_j(q)_j},$$

where $(a)_n = (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$, and $|z| < 1$, $|q| < 1$, $b_i \neq q^{-n}$ for any nonnegative integer n.

The applications of the ordinary

$$(1.2)\qquad {}_mF_n\begin{bmatrix} a_1, \cdots, a_m; z \\ b_1, \cdots, b_n \end{bmatrix} = \sum_{j \geqq 0} \frac{z^j}{j!} \prod_{s=0}^{j-1} \frac{(a_1 + s)(a_2 + s) \cdots (a_m + s)}{(b_1 + s)(b_2 + s) \cdots (b_n + s)}$$

are so widespread that a survey of their applications would hardly be possible. Indeed we might summarize their application with the following paragraph of W. W. Sawyer [50, p. 63]:

> Besides the functions that occur in school work, there are many functions used by engineers or physicists—the Legendre polynomials and the Bessel functions, for example—which are particular cases of the hypergeometric function. In fact there must be many universities today where 95 per cent, if not 100 per cent, of the functions studied by physicists, engineering and even mathematics students are covered by this single symbol $F(a, b; c; x)$ $\left[= {}_2F_1\begin{bmatrix} a, b; x \\ c \end{bmatrix}\right]$.

Surprisingly then, basic hypergeometric functions have been for the most part the province of a few specialists. In fact prior to 1960, most of the known results on these functions had been obtained by E. Heine, C. R. Adams, F. H. Jackson, W. N. Bailey, D. B. Sears, W. Hahn, L. J. Slater and R. P. Agarwal. Perhaps the reason for this general neglect lay in the view that the theory of these functions was merely a generalization of the theory of ordinary hypergeometric functions; furthermore, since there were few applications of basic hypergeometric functions known, there was not widespread interest in investigating basic hypergeometric functions.

This paper is devoted to an exposition of several areas of pure and applied mathematics in which basic hypergeometric functions have assumed significant

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importance. We shall develop in detail only those properties of basic hypergeometric functions that are important to our applications. Extensive accounts of the general theory of basic hypergeometric functions can be found in books by L. J. Slater [52] and W. N. Bailey [15], and in the papers of W. Hahn [36], [37].

In § 2, we briefly discuss the application of basic hypergeometric functions to partitions; since there are recent surveys of partition theory that delve deeply into this application [4], [9], we shall restrict ourselves to only a few examples that are related to Heine's fundamental transformation of the ${}_2\phi_1$.

In § 3, we shall show how bilateral basic hypergeometric series have application in number theory (apart from partitions). In particular, we shall present a reasonably elementary proof of W. N. Bailey's summation of the well-poised ${}_6\psi_6$. This summation formula has many interesting applications in number theory; from it we shall derive Jacobi's triple product identity, the quintuple product identity, Ramanujan's congruence $p(5n + 4) \equiv 0 \pmod 5$ and the Jacobi formulas for $r_s(n)$, $s = 2, 4$ and 8, where $r_s(n)$ denotes the number of representations of n as a sum of s squares.

In § 4, we shall discuss the relationship between finite vector spaces and basic hypergeometric functions. In this section, we shall briefly describe the theory of Eulerian differential operators which arose from the combinatorial theory of finite vector spaces and which has interesting applications to basic hypergeometric series.

In § 5, we shall discuss the usefulness of both ordinary and basic hypergeometric functions in the proof of and classification of combinatorial identities such as those discussed in the books of J. Riordan [46] and H. W. Gould [35].

In § 6, we shall describe some of the recent applications of basic hypergeometric functions in physics. We shall relate in detail W. Hahn's application of ${}_m\phi_n$ to mechanics.

Neither this survey nor the bibliography is exhaustive. Each section of the paper provides only a sample of the work done in each area. It is my hope that these samples will be adequate to indicate the breadth of application of basic hypergeometric functions, and the bibliography should at least provide the basic leads to the extensive literature of this area.

2. Application to the theory of partitions. The starting point of this application lies in the following three simple theorems.

THEOREM 2.1. *Let $p_N(m, n)$ denote the number of partitions of n into m parts, each of which does not exceed N. Then*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_N(m, n) z^m q^n = \frac{1}{(zq)_N}, \qquad |z| < |q|^{-1}, \quad |q| < 1.$$

Proof.

$$\begin{aligned} \frac{1}{(zq)_N} &= \prod_{i=1}^{N} \frac{1}{1 - zq^i} \\ &= \prod_{i=1}^{N} \sum_{m_i=0}^{\infty} z^{m_i} q^{m_i \cdot i} \qquad \text{(by geometric series)} \\ &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} z^{m_1 + m_2 + \cdots + m_N} q^{m_1 \cdot 1 + m_2 \cdot 2 + \cdots + m_N \cdot N}. \end{aligned}$$

Hence the term $z^m q^n$ arises in the above series as many times as there are non-negative solutions to the Diophantine equation $n = m_1 \cdot 1 + m_2 \cdot 2 + \cdots + m_N \cdot N$ subject to the restriction $m_1 + m_2 + \cdots + m_N = m$. But this Diophantine equation may be viewed as a partition of n in which 1 appears m_1 times, 2 appears m_2 times, and so on, and where the total number of parts in the partition is just m. Thus the N-fold series given above equals

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_N(m, n) z^m q^n.$$

The conditions $|q| < 1$ and $|z| < |q|^{-1}$ guarantee absolute convergence of all N series $\sum z^{m_i} q^{m_i \cdot i}$; consequently our rearrangements of series were admissible. □

THEOREM 2.2. *Let $Q_N(m, n)$ denote the number of partitions of n into m distinct parts, each of which does not exceed N. Then*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q_N(m, n) z^m q^n = (-zq)_N.$$

Sketch of proof.

$$\begin{aligned}(-zq)_N &= \prod_{i=1}^{N} (1 + zq^i) \\ &= \prod_{i=1}^{N} \sum_{m_i=0}^{1} z^{m_i} q^{m_i \cdot i} \\ &= \sum_{m_1=0}^{1} \sum_{m_2=0}^{1} \cdots \sum_{m_N=0}^{1} z^{m_1+m_2+\cdots+m_N} q^{m_1 \cdot 1 + m_2 \cdot 2 + \cdots + m_N \cdot N} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q_N(m, n) z^m q^n. \quad \square\end{aligned}$$

Several remarks are in order about extensions of these theorems. First we note that they remain valid if we let $N \to \infty$ (for a justification of this, see [10, Chap. 13]). Therefore, with $p(m, n) = p_\infty(m, n)$ and $Q(m, n) = Q_\infty(m, n)$, we see that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p(m, n) z^m q^n = \frac{1}{(zq)_\infty}, \qquad |z| < |q|^{-1}, \quad |q| < 1, \tag{2.1}$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(m, n) z^m q^n = (-zq)_\infty, \qquad |q| < 1. \tag{2.2}$$

Also the same arguments may be applied mutatis mutandis to $p_N(S; m, n)$ and $Q_N(S; m, n)$, where S is some set of positive integers and $p_N(S; m, n)$ (resp. $Q_N(S; m, n)$) is the number of partitions of the type enumerated by $p_N(m, n)$ (resp. $Q_N(m, n)$) with the added condition that all the parts of the partition are in S. In this way, we see that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_N(S; m, n) = \prod_{\substack{j \in S \\ j \leqq N}} (1 - zq^j)^{-1}, \qquad |z| < |q|^{-1}, \quad |q| < 1, \tag{2.3}$$

$$(2.4)\qquad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Q_N(S;m,n) = \prod_{\substack{j\in S\\ j\leq N}} (1+zq^j),$$

$$(2.5)\qquad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P(S;m,n) = \prod_{j\in S}(1-zq^j)^{-1},\qquad |z|<|q|^{-1},\quad |q|<1,$$

$$(2.6)\qquad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Q(S;m,n) = \prod_{j\in S}(1+zq^j),\qquad |q|<1.$$

The next theorem allows the pasting together of different partition generating functions to produce more complex generating functions.

THEOREM 2.3. *Let $P_j(B_j, S_j; m, n)$ $(j = 1$ or $2)$ denote the number of partitions of n into m parts, where all the parts lie in S_j and each partition is subject to the condition B_j. Let $P_3(B_3, S_3; m, n)$ denote the number of partitions π of n into m parts, where $S_3 = S_1 \cup S_2$ and B_3 is the condition that the two subpartitions of π made up of parts belonging to S_1 (resp. S_2) satisfy condition B_1 (resp. B_2). Finally we let*

$$f_j(z,q) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_j(B_j, S_j; m,n) z^m q^n,$$

where $|z| < |q|^{-1}$, $|q| < 1$. Then if $S_1 \cap S_2 = \varnothing$,

$$(2.7)\qquad f_3(z,q) = f_1(z,q)f_2(z,q).$$

Proof. Absolute convergence of the series to be considered follows by comparison with the series in (2.1).

Suppose now that π is a partition of the type enumerated by $P_3(B_3, S_3; m, n)$. Then since $S_1 \cap S_2 = \varnothing$, π may be written uniquely in the form

$$a_1 + \cdots + a_{m_1} + b_1 + \cdots + b_{m_2}$$

where $n_1 = a_1 + \cdots + a_{m_1}$ is a partition of the type enumerated by $P_1(B_1, S_1; m_1, n_1)$ and $n_2 = b_1 + \cdots + b_{m_2}$ is a partition of the type enumerated by $P_2(B_2, S_2; m_2, n_2)$. Consequently $P_3(B_3, S_3; m, n)$, the total number of such partitions, is given by

$$\sum_{\substack{m_1+m_2=m\\ n_1+n_2=n\\ m_1,m_2,n_1,n_2\geq 0}} P_1(B_1, S_1; m_1, n_1)P_2(B_2, S_2; m_2, n_2),$$

and this establishes that the coefficients on either side of (2.7) are identical. □

We remark that Theorem 2.3 is an example of a large number of theorems of this nature. The disjointness of S_1 and S_2 is the essential feature that makes the theorem work.

We hope now to illustrate how partitions and basic hypergeometric functions interact. Theorem 2.4 is a basic hypergeometric function identity with a partition-theoretic proof. Theorems 2.5–2.8 are analytic identities that are corollaries of Theorem 2.4. The final three theorems are partition-theoretic consequences of Theorem 2.8.

THEOREM 2.4. *For $|z| < 1$, $|q| < 1$,*

$${}_1\phi_0[a;q,z] = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{(q)_n} = \frac{(az)_\infty}{(z)_\infty}.$$

Proof. We shall prove the equivalent identity:

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n z^n q^{2n}}{(q^2;q^2)_n} = \frac{(-azq^3;q^2)_\infty}{(zq^2;q^2)_\infty}. \tag{2.8}$$

Let $W_1(r, m, n)$ denote the number of partitions of n into m parts, of which exactly r are odd, subject to the condition that no odd parts are repeated and all odd parts are larger than 1. By splitting such partitions into two subpartitions containing odd parts (resp. even parts), we see that (2.5), (2.6) and Theorem 2.3 (slightly modified to account for the extra parameter r) imply

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} W_1(r,m,n)a^r z^m q^n = (-azq^3;q^2)_\infty \cdot \frac{1}{(zq^2;q^2)_\infty} = \frac{(-azq^3;q^2)_\infty}{(zq^2;q^2)_\infty}.$$

Let $W_2(r, m, n)$ denote the number of partitions of n with largest part $2m$, with exactly r parts odd, and with the condition that no odd parts are repeated. Arguments of the types used in Theorems 2.1–2.3 may be easily seen to establish that

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} W_2(r,m,n)a^r q^n = (-aq;q^2)_m \cdot \frac{q^{2m}}{(q^2;q^2)_m} = \frac{(-aq;q^2)_m q^{2m}}{(q^2;q^2)_m}.$$

Therefore

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} W_2(r,m,n)a^r z^m q^n = \sum_{m=0}^{\infty} \frac{(-aq;q^2)_m z^m q^{2m}}{(q^2;q^2)_m}.$$

Thus we see that (2.8) is equivalent to the following partition function identity:

$$W_1(r,m,n) = W_2(r,m,n). \tag{2.9}$$

To establish (2.9), we must utilize a modification of the Ferrars graph of a partition [39, p. 273]. Let π be a typical partition of the type enumerated by $W_2(r, m, n)$. Each even part $2v$ of π is to be represented in the graph by two rows of v nodes each, and each odd part $2v - 1$ of π is to be represented by one row of v nodes and one row of $v - 1$ nodes. Thus $8 + 8 + 5$ is represented graphically by

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We note that the graph of π contains n nodes, has m columns, has exactly r columns with an odd number of nodes. Furthermore, no two columns which each have an odd number of nodes can have the same number of nodes. Finally, every column must contain at least two nodes since the largest part of π was even. Let $\pi^1: b_1 + \cdots + b_m$ denote the partition of n in which b_i is the number of nodes in the

ith column of the graphical representation of π. From the above comments we see that π^1 is a partition of the type enumerated by $W_1(r, m, n)$, and indeed since the above procedure is reversible it establishes a bijection between the partitions enumerated by $W_2(r, m, n)$ and those enumerated by $W_1(r, m, n)$. Hence (2.9) is established, and so therefore are (2.8) and the theorem. □

THEOREM 2.5 (Heine [40, (5a), p. 106]). *Subject to obvious convergence conditions,*

$$ {}_2\phi_1\begin{bmatrix} a, b ; q, z \\ c \end{bmatrix} = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} {}_2\phi_1\begin{bmatrix} c/b, z ; q, b \\ az \end{bmatrix}. $$

Remark. The phrase "subject to obvious convergence conditions" means here $|z| < 1$, $|b| < 1$, $|q| < 1$, $c \neq q^{-n}$, $z \neq q^{-n}$, $az \neq q^{-n}$, for any nonnegative integer n. In general, conditions like $c \neq q^{-n}$ are needed to keep zeros out of denominators, while the conditions $|z| < 1$, $|b| < 1$, $|q| < 1$ guarantee absolute convergence of all the series and products appearing. For the most part, such conditions as these will be tacitly assumed; only in Theorem 3.3 will convergence problems be difficult enough to require mention.

Proof.

$$ \begin{aligned} {}_2\phi_1\begin{bmatrix} a, b ; q, z \\ c \end{bmatrix} &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(q)_n (c)_n} \\ &= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(q)_n} \cdot \frac{(cq^n)_\infty}{(bq^n)_\infty} \\ &= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(q)_n} \sum_{m=0}^{\infty} \frac{(c/b)_m}{(q)_m} b^m q^{nm} \qquad \text{(by Theorem 2.4)} \\ &= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m b^m}{(q)_m} \cdot \frac{(azq^m)_\infty}{(zq^m)_\infty} \qquad \text{(by Theorem 2.4)} \\ &= \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m (z)_m b^m}{(q)_m (az)_m} \\ &= \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} {}_2\phi_1\begin{bmatrix} c/b, z ; q, b \\ az \end{bmatrix}. \quad \square \end{aligned} $$

Our next result is the q-analogue of Gauss's summation formula [15, p. 2] for ${}_2F_1\begin{bmatrix} a, b ; 1 \\ c \end{bmatrix}$.

THEOREM 2.6 (Heine [40, (6), p. 107]).

$$ {}_2\phi_1\begin{bmatrix} a, b ; q, c/ab \\ c \end{bmatrix} = \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty}. $$

Proof. By Theorem 2.5, we see that

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, b; q, c/ab \\ c \end{matrix}\right] &= \frac{(b)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty} {}_1\phi_0[c/ab; q, b] \\
&= \frac{(b)_\infty (c/b)_\infty (c/a)_\infty}{(c)_\infty (c/ab)_\infty (b)_\infty} \qquad \text{(by Theorem 2.4)} \\
&= \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty}. \quad \square
\end{aligned}
$$

The following result is less well known than Theorem 2.6 and was found independently in the 1940's by W. N. Bailey [16] and J. A. Daum [29]; this theorem is the q-analogue of Kummer's summation [52, III. 5, p. 243] of ${}_2F_1\left[\begin{matrix} a, b; -1 \\ a+1-b \end{matrix}\right]$.

THEOREM 2.7.

$$
{}_2\phi_1\left[\begin{matrix} a, b; q, -q/b \\ qa/b \end{matrix}\right] = \frac{(aq; q^2)_\infty (-q)_\infty (q^2 a/b^2; q^2)_\infty}{(qa/b)_\infty (-q/b)_\infty}.
$$

Proof.

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, b; q, -q/b \\ qa/b \end{matrix}\right] &= {}_2\phi_1\left[\begin{matrix} b, a; q, -q/b \\ qa/b \end{matrix}\right] \\
&= \frac{(a)_\infty (-q)_\infty}{(qa/b)_\infty (-q/b)_\infty} \sum_{n=0}^{\infty} \frac{(q/b)_n (-q/b)_n a^n}{(q)_n (-q)_n} \qquad \text{(by Theorem 2.5)} \\
&= \frac{(a)_\infty (-q)_\infty}{(qa/b)_\infty (-q/b)_\infty} \sum_{n=0}^{\infty} \frac{(q^2/b^2; q^2)_n a^n}{(q^2; q^2)_n} \\
&= \frac{(a)_\infty (-q)_\infty (aq^2/b^2; q^2)_\infty}{(qa/b)_\infty (-q/b)_\infty (a; q^2)_\infty} \qquad \text{(by Theorem 2.4)} \\
&= \frac{(aq; q^2)_\infty (-q)_\infty (aq^2/b^2; q^2)_\infty}{(qa/b)_\infty (-q/b)_\infty}. \quad \square
\end{aligned}
$$

COROLLARY 2.7.1.

$$
\sum_{n=0}^{\infty} \frac{(a)_n q^{n(n+1)/2}}{(q)_n} = (aq; q^2)_\infty (-q)_\infty .
$$

Proof. The identity in question follows directly from letting $b \to \infty$ in Theorem 2.7. To justify the limit we need only observe that if we replace b by $1/\beta$ in Theorem 2.7, each side of the identity holds for β in a deleted neighborhood of zero. Since both sides are continuous functions of β at zero, both sides must be equal at $\beta = 0$. Alternatively one may justify this process by an appeal to Tannery's theorem [43, p. 371]. $\square$

Our first application to partitions will be to J. J. Sylvester's generalization of Euler's partition theorem [53, p. 293]. Our proof is based on one recently given by V. Ramamani and K. Venkatachaliengar [45].

THEOREM 2.8. *Let $A_k(n)$ denote the number of partitions of n into odd parts (repetitions allowed) with exactly k different parts in each partition. Let $B_k(n)$ denote the number of partitions of n into distinct parts such that exactly k sequences of consecutive integers appear in each partition. Then for each k and n, $A_k(n) = B_k(n)$.*

Remark. Euler's theorem merely asserts $\sum_{k=0}^{\infty} A_k(n) = \sum_{k=0}^{\infty} B_k(n)$.

Proof. First we note that

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_k(n)a^k q^n &= \prod_{j=1}^{\infty}(1 + aq^{2j-1} + aq^{2(2j-1)} + aq^{3(2j-1)} + \cdots)\\ &= \prod_{j=1}^{\infty}\left(1 + \frac{aq^{2j-1}}{1 - q^{2j-1}}\right)\\ &= \prod_{j=1}^{\infty}\frac{(1 + (a-1)q^{2j-1})}{(1 - q^{2j-1})}.\end{aligned}$$

The reasoning here exactly parallels that used in Theorem 2.1. Now since

$$\prod_{m=1}^{\infty}(1 + q^m) = \prod_{m=1}^{\infty}\frac{(1 - q^{2m})}{(1 - q^m)} = \prod_{j=1}^{\infty}\frac{1}{1 - q^{2j-1}},$$

we see that

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_k(n)a^k q^n = (-q)_\infty((1-a)q; q^2)_\infty. \tag{2.10}$$

On the other hand, let us consider the graphical representation of a partition of the type enumerated by $B_k(n)$. For example, $10 + 9 + 5 + 4 + 3 + 1$ is a partition of the type enumerated by $B_3(32)$, and it has the following graphical representation:

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If we examine the *conjugate* partition (i.e., the partition obtained by reading the graph vertically), we see that the conjugate partition (namely, $6 + 5 + 5 + 4 + 3 + 2 + 2 + 2 + 2 + 1$) has exactly two parts that are repeated. Indeed if π is any partition of the type enumerated by $B_k(n)$ in which 1 is a summand, then π', the conjugate of π, is a partition of n in which the largest part appears only once, all positive integers not exceeding the largest part appear and exactly $k - 1$ of these appear with repetition.

If the condition that 1 appears in π is removed, then π' is a partition of n in which the largest part appears more than once, all positive integers not exceeding the largest part appear and exactly k parts appear with repetition.

Since the union of the two classes of partitions described above is in one-to-one correspondence with the partitions enumerated by $B_k(n)$, we see that

$$
\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_k(n)a^k q^n &= 1 + \sum_{N=1}^{\infty} aq^N \prod_{j=1}^{N-1} (q^j + aq^{2j} + aq^{3j} + \cdots) \\
&\quad + \sum_{N=1}^{\infty} \left\{ \prod_{j=1}^{N-1} (q^j + aq^{2j} + aq^{3j} + \cdots) \right\} (aq^{2N} + aq^{3N} + \cdots) \\
&= 1 + \sum_{N=1}^{\infty} \left\{ \prod_{j=1}^{N-1} q^j \left(1 + \frac{aq^j}{1-q^j}\right) \right\} \frac{aq^N}{1-q^N} \\
&= 1 + \sum_{N=1}^{\infty} \frac{aq^{N(N+1)/2}((1-a)q)_{N-1}}{(q)_N} \\
&= 1 + \sum_{N=1}^{\infty} \frac{(1-(1-a))q^{N(N+1)/2}((1-a)q)_{N-1}}{(q)_N} \\
&= \sum_{N=0}^{\infty} \frac{((1-a))_N q^{N(N+1)/2}}{(q)_N}.
\end{aligned} \tag{2.11}
$$

Comparing (2.10) and (2.11), we see that Sylvester's theorem follows immediately from the identity

$$\sum_{N=0}^{\infty} \frac{((1-a))_N q^{N(N+1)/2}}{(q)_N} = (-q)_\infty ((1-a)q\,;q^2)_\infty,$$

which is merely Corollary 2.7.1 with a replaced by $1 - a$. □

Next we consider a result due to H. Göllnitz [31, Satz 2.3, p. 166] that may also be deduced from Corollary 2.7.1.

THEOREM 2.9. *Let $G_1(n)$ denote the number of partitions of n into parts, where each part is congruent to one of* 1, 5, *or* 6 (mod 8). *Let $H_1(n)$ denote the number of partitions of n of the form $b_1 + b_2 + \cdots + b_j$, where $b_i \geqq b_{i+1} + 2$ and strict inequality holds if b_i is odd. Then for each n, $G_1(n) = H_1(n)$.*

Proof. Let $h_a(m, n)$ ($a = 1$ or 2) denote the number of partitions of the type enumerated by $H_1(n)$ with the added restrictions that there be exactly m parts and that each part is $\geqq a$.

We shall now prove two elementary partition identities for the $h_a(m, n)$. First,

$$h_1(m, n) = h_2(m, n) + h_2(m-1, n-2m+1). \tag{2.12}$$

To see this identity, we split the partitions enumerated by $h_1(m, n)$ into two classes: (i) those partitions that contain 1 as a summand, (ii) those that do not. Clearly there are $h_2(m, n)$ elements of class (ii). We now transform the partitions in class (i) by deleting the summand 1 and subtracting 2 from all the remaining parts. This produces a partition of $n - 2(m-1) - 1 = n - 2m + 1$ into $m - 1$ parts, each of which is $\geqq 2$ (since originally the second smallest part was $\geqq 4$); furthermore, since the inequalities between the parts are not disturbed, we see that the transformed partition is of the type enumerated by $h_2(m - 1, n - 2m + 1)$. The above transformation clearly establishes a bijection between the partitions in class (i) and those enumerated by $h_2(m-1, n-2m+1)$. Thus identity (2.12) is established.

By considering those partitions of the type enumerated by $h_2(m, n)$ that contain a 2 and those that do not, we may in exactly the same way prove that

$$h_2(m,n) = h_1(m,n-2m) + h_2(m-1,n-2m). \tag{2.13}$$

Let

$$\eta_a(z,q) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_a(m,n))z^m q^n$$

Then identity (2.12) implies that

$$\begin{aligned}\eta_1(z,q) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_1(m,n)z^m q^n \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (h_2(m,n) + h_2(m-1,n-2m+1))z^m q^n \\ &= \eta_2(z,q) + zq\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_2(m-1,n-2m+1)(zq^2)^{m-1}q^{n-2m+1} \\ &= \eta_2(z,q) + zq\eta_2(zq^2,q).\end{aligned} \tag{2.14}$$

Similarly, identity (2.13) implies that

$$\eta_2(z,q) = \eta_1(zq^2,q) + zq^2\eta_2(zq^2,q). \tag{2.15}$$

We now substitute the formula for $\eta_1(z,q)$ from (2.14) into (2.15). Hence

$$\begin{aligned}\eta_2(z,q) &= \eta_2(zq^2,q) + zq^3\eta_2(zq^4,q) + zq^2\eta_2(zq^2,q) \\ &= (1+zq^2)\eta_2(zq^2,q) + zq^3\eta_2(zq^4,q).\end{aligned} \tag{2.16}$$

Therefore, if $\eta_2(z,q) = \sum_{n=0}^{\infty}\lambda_n z^n$, then by comparing coefficients of z^n on each side of (2.16), we see that $\lambda_n = \lambda_n q^{2n} + \lambda_{n-1}q^{2n} + \lambda_{n-1}q^{4n-1}$. Therefore

$$\lambda_n = \frac{q^{2n}(1+q^{2n-1})\lambda_{n-1}}{(1-q^{2n})}. \tag{2.17}$$

Iterating (2.17) n times and observing that $\lambda_0 = 1$, we see that

$$\lambda_n = \frac{q^{n(n+1)}(-q;q^2)_n}{(q^2;q^2)_n}. \tag{2.18}$$

Therefore

$$\eta_2(z;q) = \sum_{n=0}^{\infty}\lambda_n z^n = \sum_{n=0}^{\infty}\frac{z^n q^{n(n+1)}(-q;q^2)_n}{(q^2;q^2)_n}; \tag{2.19}$$

and by (2.14), we see that

$$\eta_1(z;q) = \sum_{n=0}^{\infty}\frac{z^n q^{n(n+1)}(-q;q^2)_n(1+zq^{2n+1})}{(q^2;q^2)_n}. \tag{2.20}$$

From (2.20) we may easily deduce our theorem by utilizing Corollary (2.7.1):

$$\sum_{n=0}^{\infty} H_1(n)q^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} h_1(m, n) \right) q^n = \eta_1(1, q)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q; q^2)_{n+1}}{(q^2; q^2)_n}$$

$$= (1 + q) \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^3; q^2)_n}{(q^2; q^2)_n}$$

$$= (1 + q)(-q^2; q^2)_\infty(-q^5; q^4)_\infty \qquad \text{(by Corollary 2.7.1)}$$

$$= \prod_{m=1}^{\infty} (1 + q^{2m})(1 + q^{4m-3})$$

$$= \prod_{m=1}^{\infty} \frac{(1 - q^{4m})(1 - q^{8m-6})}{(1 - q^{2m})(1 - q^{4m-3})}$$

$$= \prod_{m=1}^{\infty} \frac{(1 - q^{8m-4})(1 - q^{8m})(1 - q^{8m-6})}{(1 - q^{8m-6})(1 - q^{8m-4})(1 - q^{8m-2})(1 - q^{8m})(1 - q^{8m-7})(1 - q^{8m-3})}$$

$$= \prod_{m=1}^{\infty} \frac{1}{(-q^{8m-7})(1 - q^{8m-3})(1 - q^{8m-2})}$$

$$= \sum_{n=0}^{\infty} G_1(n)q^n.$$

Consequently $H_1(n) = G_1(n)$ for all n. □

It is now a very simple matter to prove a companion theorem for Theorem 2.9 that is also due to Göllnitz [31, Satz 2.4, p. 167].

THEOREM 2.10. *Let $G_2(n)$ denote the number of partitions of n into parts, where each part is congruent to one of* 2, 3, *or* 7 (mod 8). *Let $H_2(n)$ denote the number of partitions of n of the form $b_1 + b_2 + \cdots + b_j$, where $b_i \geqq b_{i+1} + 2$ and strict inequality holds if b_i is odd; in addition, $b_j \geqq 2$. Then for each n, $G_2(n) = H_2(n)$.*

Proof. From (2.19), we see that

$$\sum_{n=0}^{\infty} H_2(n)q^n = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} h_2(m, n) \right) q^n = \eta_2(1, q)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q; q^2)_n}{(q^2; q^2)_n}$$

$$= (-q^2; q^2)_\infty(-q^3; q^4)_\infty \qquad \text{(by Corollary 2.7.1)}$$

$$= \prod_{m=1}^{\infty} \frac{1}{(1 - q^{8m-6})(1 - q^{8m-5})(1 - q^{8m-1})}$$

$$= \sum_{n=0}^{\infty} G_2(n)q^n.$$

Consequently $G_2(n) = H_2(n)$ for all n. □

This concludes our sample of the relationship between partition identities and basic hypergeometric functions. The reader is referred to [4], [9] and [10, Chaps. 12–14] for a more extensive account of this subject.

3. Application to number theory. In this section we shall present an account of bilateral basic hypergeometric functions, i.e.,

$$ {}_m\psi_n\begin{bmatrix} a_1, a_2, \cdots, a_m; q, t \\ b_1, b_2, \cdots, b_n \end{bmatrix} = \sum_{j=-\infty}^{\infty} \frac{(a_1)_j(a_2)_j \cdots (a_m)_j t^j}{(b_1)_j(b_2)_j \cdots (b_n)_j}, $$

where

$$ (a)_j = (a;q)_j = (a)_\infty (aq^j)_\infty^{-1}, $$

or

$$ (a)_{-n} = \left(1 - \frac{a}{q^n}\right)^{-1} \cdots \left(1 - \frac{a}{q}\right)^{-1} = (-a)^{-n} q^{n(n+1)/2} (q/a)_n^{-1}. $$

Thus

$$ {}_m\psi_n\begin{bmatrix} a_1, \cdots, a_m; q, t \\ b_1, \cdots, b_n \end{bmatrix} $$

$$ = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_m)_j t^j}{(b_1)_j \cdots (b_n)_j} + \sum_{j=1}^{\infty} \frac{\left(\frac{q}{b_1}\right)_j \cdots \left(\frac{q}{b_n}\right)_j}{\left(\frac{q}{a_1}\right)_j \cdots \left(\frac{q}{a_m}\right)_j} \left(\frac{b_1 \cdots b_n}{a_1 \cdots a_m t}\right)^j (-1)^{j(m-n)} q^{j(j+1)(m-n)/2}. $$

Hence we see that to insure convergence we must require $n \leqq m$. Also $b_h \neq q^{-N}$, $a_h \neq q^{N+1}$ for any nonnegative integer N. Finally if $n < m$, we need only in addition require $|t| < 1$; however, if $n = m$, we need also

$$ \left|\frac{b_1 \cdots b_n}{a_1 \cdots a_m}\right| < |t| < 1. $$

Our first object is to provide a proof of the following identity due to W. N. Bailey [14, (4.7), p. 113]:

$$ {}_6\psi_6\begin{bmatrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e \\ \sqrt{a}, -\sqrt{a}, \frac{qa}{b}, \frac{qa}{c}, \frac{qa}{d}, \frac{qa}{e} ; q, \frac{a^2q}{bcde} \end{bmatrix} \tag{3.1} $$

$$ = \prod\begin{bmatrix} aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{be}, \frac{aq}{cd}, \frac{aq}{ce}, \frac{aq}{de}, q, \frac{q}{a} \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{a^2q}{bcde} \end{bmatrix}, $$

where

$$ \prod\begin{bmatrix} \alpha_1, \cdots, \alpha_r \\ \beta_1, \cdots, \beta_s \end{bmatrix} = \frac{(\alpha_1)_\infty \cdots (\alpha_r)_\infty}{(\beta_1)_\infty \cdots (\beta_s)_\infty}. $$

The identity (3.1) is probably the most general summation identity known for bilateral basic hypergeometric series. As we shall see, we shall be able to deduce many important and diverse results in number theory from (3.1).

There are three known proofs of (3.1). Bailey's original proof [14, § 4] relies on ingenious combinations of various transformation formulas he had developed for ordinary and basic hypergeometric series. L. J. Slater [51] uses an analogue of the Barnes-type integral, and A. Lakin [51] combines a q-difference equation technique with Carlson's theorem on entire functions.

We shall provide a more elementary proof; in fact we shall utilize q-difference equations together with the uniqueness of a Laurent series expansion about the origin. For the most part, our proof will rely on simple manipulations of series. In our proof, we shall need the following function:

$$\begin{aligned} K_{\lambda,k,i}(a_0, a_1, \cdots, a_\lambda; z; q) &= K_{\lambda,k,i}((a); z; q) \\ &= \sum_{n=-\infty}^{\infty} (-1)^{n(\lambda+1)} z^{(k+1)n} q^{(2k-\lambda+1)n^2/2 - in + (\lambda+1)n/2} \cdot \\ &\quad \cdot (a_0 a_1 \cdots a_\lambda)^{-n} \\ &\quad \cdot \frac{(a_0)_n (a_1)_n \cdots (a_\lambda)_n (1 - z^i q^{2ni})}{(zq/a_0)_n (zq/a_1)_n \cdots (zq/a_\lambda)_n (1 - z)}. \end{aligned} \tag{3.2}$$

I remark that $K_{3,1,1}(b, c, d, e; z, q)$ is the left-hand side of identity (3.1). In the following theorem we shall utilize a series rearrangement technique originally due to Cayley [27] that has also been effectively utilized in the study of the Rogers–Ramanujan identities [10] and more general basic hypergeometric functions [5].

THEOREM 3.1. *The following identities are valid:*

$$K_{\lambda,k,0}(a_0, a_1, \cdots, a_\lambda; z; q) = 0; \tag{3.3}$$

$$K_{\lambda,k,-i}(a_0, a_1, \cdots, a_\lambda; z; q) = -z^{-i} K_{\lambda,k,i}(a_0, a_1, \cdots, a_\lambda; z; q); \tag{3.4}$$

$$\begin{aligned} &K_{\lambda,k,i}(a_0, a_1, \cdots, a_\lambda; z; q) - \frac{z}{a_j} K_{\lambda,k,i-1}(a_0, a_1, \cdots, a_\lambda; z; q) \\ &= \frac{(1 - a_j) a_j^{-1} z^i (1 - zq)\left(1 - \dfrac{zq}{a_j}\right)}{(1 - z)\left(1 - \dfrac{zq}{a_0}\right) \cdots \left(1 - \dfrac{zq}{a_\lambda}\right)} \\ &\quad \cdot \sum_{r=0}^{\lambda} (-1)^r z^r q^r \sigma_r(j) K_{\lambda,k,k-i-r+1}(a_0, a_1, \cdots, a_j q, \cdots, a_\lambda; zq; q); \end{aligned} \tag{3.5}$$

$$\begin{aligned} &K_{\lambda,k,i}(a_0, a_1, \cdots, a_\lambda; z; q) \\ &= \frac{(-1)^{\lambda+1} z^{k+1} (a_0 a_1 \cdots a_\lambda)^{-1} q^{k-i+1} (1 - zq^2)(1 - a_0)(1 - a_1) \cdots (1 - a_\lambda)}{(1 - z)\left(1 - \dfrac{zq}{a_0}\right)\left(1 - \dfrac{zq}{a_1}\right) \cdots \left(1 - \dfrac{zq}{a_\lambda}\right)} \\ &\quad \cdot K_{\lambda,k,i}(a_0 q, a_1 q, \cdots, a_\lambda q; zq^2; q); \end{aligned} \tag{3.6}$$

above, and the second series contains the second curly braced expression. Therefore

$$K_{\lambda,k,i}(a_0, a_1, \cdots, a_\lambda; z; q) - \frac{z}{a_0} K_{\lambda,k,i}(a_0, a_1, \cdots, a_\lambda; z; q)$$

$$= \sum_{n=-\infty}^{\infty} (-1)^{n(\lambda+1)} z^{(k+1)n} q^{(2k-\lambda+1)n^2/2 - in + (\lambda+1)n/2} (a_0 a_1 \cdots a_\lambda)^{-n} \cdot \frac{(a_0)_n (a_1)_n \cdots (a_\lambda)_n}{\left(\frac{zq}{a_0}\right)_{n-1} \left(\frac{zq}{a_1}\right)_n \cdots \left(\frac{zq}{a_\lambda}\right)_n (1-z)}$$

$$+ z^i a_0^{-1} \sum_{n=-\infty}^{\infty} (-1)^{n(\lambda+1)} z^{(k+1)n} q^{(2k-\lambda+1)n^2/2 + n(i-1) + (\lambda+1)n/2} (a_0 a_1 \cdots a_\lambda)^{-n} \cdot \frac{(a_0)_{n+1} (a_1)_n \cdots (a_\lambda)_n}{\left(\frac{zq}{a_0}\right)_n \left(\frac{zq}{a_1}\right)_n \cdots \left(\frac{zq}{a_\lambda}\right)_n (1-z)}.$$

We now replace n by $n + 1$ in the first sum and then combine the two series term by term. Thus

$$K_{\lambda,k,i}((a); z; q) - \frac{z}{a_0} K_{\lambda,k,i-1}((a); z; q)$$

$$= a_0^{-1} z^i \sum_{n=-\infty}^{\infty} (-1)^{n(\lambda+1)} z^{(k+1)n} q^{(2k-\lambda+1)n^2/2 + n(i-1) + (\lambda+1)n/2} (a_0 a_1 \cdots a_\lambda)^{-n}$$

$$\cdot \frac{(a_0)_{n+1} (a_1)_n \cdots (a_\lambda)_n}{(zq/a_0)_n (zq/a_1)_n \cdots (zq/a_\lambda)_n (1-z)} \left\{ 1 + \frac{(-1)^{\lambda+1} z^{k-i+1} q^{(2k-\lambda-2i+2)n+k-i+1} (1-a_1 q^n)(1-a_2 q^n) \cdots (1-a_\lambda q^n)}{a_1 \cdots a_\lambda \left(1 - \frac{zq^{n+1}}{a_1}\right)\left(1 - \frac{zq^{n+1}}{a_2}\right) \cdots \left(1 - \frac{zq^{n+1}}{a_\lambda}\right)} \right\}.$$

We now combine fractions inside the curly brackets, and we make the observation that

$$\prod_{j=1}^{\lambda} \left(1 - \frac{X}{a_j}\right) = \sum_{r=0}^{\lambda} (-1)^r \sigma_r(0) X^r.$$

Consequently, we see that

$$K_{\lambda,k,i}((a); z; q) - \frac{z}{a_0} K_{\lambda,k,i-1}((a); z; q)$$

$$= \frac{a_0^{-1} z^i (1-zq)(1-a_0)}{(1-z)\left(1 - \frac{zq}{a_1}\right) \cdots \left(1 - \frac{zq}{a_\lambda}\right)}$$

(cont.)

$$\cdot \sum_{n=-\infty}^{\infty} (-1)^{n(\lambda+1)}(zq)^{(k+1)n} q^{(2k-\lambda+1)n^2/2-(k-i+1)n+(\lambda+1)n/2}((a_0q)a_1 \cdots a_\lambda)^{-n}$$

$$\cdot \frac{(a_0q)_n(a_1)_n \cdots (a_\lambda)_n}{(zq^2/a_0q)_n(zq^2/a_1)_n \cdots (zq^2/a_\lambda)_n(1-zq)}$$

$$\cdot \left\{ \sum_{r=0}^{\lambda} (-1)^r \sigma_r(0)(zq)^r q^{nr} - (zq)^{k-i+1} q^{2(k-i+1)n} \sum_{r=0}^{\lambda} (-1)^r \sigma_r(0) q^{-nr} \right\}$$

$$= \frac{a_0^{-1} z^i (1-zq)(1-a_0)}{(1-z)\left(1-\frac{zq}{a_1}\right)\cdots\left(1-\frac{zq}{a_\lambda}\right)} \sum_{r=0}^{\lambda} (-1)^r z^r q^r \sigma_r(0)$$

$$\cdot \sum_{n=-\infty}^{\infty} (-1)^{n(\lambda+1)}(zq)^{(k+1)n} q^{(2k-\lambda+1)n^2/2-(k-i-r+1)n+(\lambda+1)n/2}$$

$$\cdot ((a_0q)a_1 \cdots a_\lambda)^{-n}$$

$$\cdot \frac{(a_0q)_n(a_1)_n \cdots (a_\lambda)_n(1-(zq)^{k-i-r+1}q^{2(k-i-r+1)n})}{\left(\frac{zq^2}{a_0q}\right)_n\left(\frac{zq^2}{a_1}\right)_n \cdots \left(\frac{zq^2}{a_\lambda}\right)_n(1-zq)}$$

$$= \frac{a_0^{-1} z^i (1-zq)(1-a_0)}{(1-z)\left(1-\frac{zq}{a_1}\right)\cdots\left(1-\frac{zq}{a_\lambda}\right)} \sum_{r=0}^{\lambda} (-1)^r z^r q^r \sigma_r(0)$$

$$\cdot K_{\lambda,k,k-i-r+1}(a_0q, a_1, \cdots, a_\lambda; zq; q).$$

Thus we have established (3.5) for $j = 0$, and as we remarked earlier, the general case now follows from the symmetry of the a_j. □

Before we establish (3.1), we must first establish a special case of (3.1) known as the limiting form of Jackson's theorem [52, p. 96, (3.3.1.3)]. Our proof is essentially the one given in [5, Thm. 5].

THEOREM 3.2.

$$_6\phi_5\begin{bmatrix} z, qz^{1/2}, -qz^{1/2}, a_1, a_2, a_3; q, \frac{zq}{a_1a_2a_3} \\ z^{1/2}, -z^{1/2}, \frac{zq}{a_1}, \frac{zq}{a_2}, \frac{zq}{a_3} \end{bmatrix} = \prod \begin{bmatrix} zq, \frac{zq}{a_1a_2}, \frac{zq}{a_1a_3}, \frac{zq}{a_2a_3} \\ \frac{zq}{a_1}, \frac{zq}{a_2}, \frac{zq}{a_3}, \frac{zq}{a_1a_2a_3} \end{bmatrix}.$$

Proof. We let

(3.7) $$g_i(a_0, a_1, a_2, a_3; z) = K_{3,1,i}(a_0, a_1, a_2, a_3; z; q),$$

and we let

(3.8) $$f(z) = g_1(z, a_1, a_2, a_3; z).$$

From Theorem 3.1, we directly deduce the following relationships:

(3.9) $$g_0(a_0, a_1, a_2, a_3; z) = 0,$$

(3.10) $$g_{-i}(a_0, a_1, a_2, a_3; z) = -z^{-i} g_i(a_0, a_1, a_2, a_3; z),$$

$$g_i(a_0, a_1, a_2, a_3; z) - \frac{z}{a_j} g_{i-1}(a_0, a_1, a_2, a_3; z)$$

$$(3.11) \qquad = \frac{(1 - a_j)a_j^{-1}z^i(1 - zq)\left(1 - \frac{zq}{a_j}\right)}{(1 - z)\left(1 - \frac{zq}{a_0}\right)\left(1 - \frac{zq}{a_1}\right)\left(1 - \frac{zq}{a_2}\right)\left(1 - \frac{zq}{a_3}\right)} \sum_{r=0}^{3} (-1)^r z^r q^r \sigma_r(j)$$

$$\cdot g_{2-i-r}(a_0', a_1', a_2', a_3'; zq),$$

where $a_h' = a_h$ if $h \neq j$ and $a_j' = a_j q$.

Setting $i = 1$ and $j = 0$ in (3.11) and utilizing (3.9) and (3.8), we see that

$$(3.12) \qquad g_1(a_0, a_1, a_2, a_3; z) = \frac{(1 - a_0)a_0^{-1}z(1 - zq)}{(1 - z)\left(1 - \frac{zq}{a_1}\right)\left(1 - \frac{zq}{a_2}\right)\left(1 - \frac{zq}{a_3}\right)} \cdot (1 - zq\sigma_2(0)g_1(a_0q, a_1, a_2, a_3; zq) + zq\sigma_3(0)g_2(a_0q, a_1, a_2, a_3; zq)).$$

Setting $i = 0$ and $j = 0$ in (3.11) and utilizing (3.9) and (3.8), we see that

$$(3.13) \qquad g_1(a_0, a_1, a_2, a_3; z) = \frac{(1 - a_0)(1 - zq)}{(1 - z)\left(1 - \frac{zq}{a_1}\right)\left(1 - \frac{zq}{a_2}\right)\left(1 - \frac{zq}{a_3}\right)} \cdot (g_2(a_0q, a_1, a_2, a_3; zq) - (zq\sigma_1(0) - z^2q^2\sigma_3(0))g_1(a_0q, a_1, a_2, a_3; zq)).$$

Eliminating $g_2(a_0q, a_1, a_2, a_3; zq)$ from (3.11) and (3.12), we find that

$$(3.14) \qquad \left(\frac{a_0}{z} - zq\sigma_3(0)\right) g_1(a_0, a_1, a_2, a_3; z) = \frac{(1 - a_0)(1 - zq)}{(1 - z)\left(1 - \frac{zq}{a_1}\right)\left(1 - \frac{zq}{a_2}\right)\left(1 - \frac{zq}{a_3}\right)} \cdot (1 - zq\sigma_2(0) + zq\sigma_3(0)(zq\sigma_1(0) - z^2q^2\sigma_3(0)))g_1(a_0q, a_1, a_2, a_3; zq).$$

Simplifying (3.14), we obtain that

$$g_1(a_0, a_1, a_2, a_3; z)$$

$$(3.15) \qquad = \frac{a_0^{-1}z(1 - a_0)(1 - zq)\left(1 - \frac{zq}{a_1a_2}\right)\left(1 - \frac{zq}{a_1a_3}\right)\left(1 - \frac{zq}{a_2a_3}\right)}{(1 - z)\left(1 - \frac{zq}{a_1}\right)\left(1 - \frac{zq}{a_2}\right)\left(1 - \frac{zq}{a_3}\right)\left(1 - \frac{z^2q}{a_0a_1a_2a_3}\right)}$$

$$\cdot g_1(a_0q, a_1, a_2, a_3; zq).$$

We now set $a_0 = z$ and utilize (3.8) to deduce that

$$(3.16) \qquad f(z) = \frac{(1 - zq)\left(1 - \frac{zq}{a_1a_2}\right)\left(1 - \frac{zq}{a_1a_3}\right)\left(1 - \frac{zq}{a_2a_3}\right) f(zq)}{\left(1 - \frac{zq}{a_1}\right)\left(1 - \frac{zq}{a_2}\right)\left(1 - \frac{zq}{a_3}\right)\left(1 - \frac{zq}{a_1a_2a_3}\right)}.$$

Repeated iteration of (3.16) then implies that

$$f(z) = \frac{(zq)_N\left(\frac{zq}{a_1a_2}\right)_N\left(\frac{zq}{a_1a_3}\right)_N\left(\frac{zq}{a_2a_3}\right)_N f(zq^N)}{\left(\frac{zq}{a_1}\right)_N\left(\frac{zq}{a_2}\right)_N\left(\frac{zq}{a_3}\right)_N\left(\frac{zq}{a_1a_2a_3}\right)_N}. \tag{3.17}$$

Identity (3.7) now follows if we let $N \to \infty$ and observe that $\lim_{N\to\infty} f(zq^N) = f(0) = 1$ and

$$f(z) = {}_6\phi_5\begin{bmatrix} z, qz^{1/2}, -qz^{1/2}, a_1, a_2, a_3; q, \frac{zq}{a_1a_2a_3} \\ z^{1/2}, -z^{1/2}, \frac{zq}{a_1}, \frac{zq}{a_2}, \frac{zq}{a_3} \end{bmatrix}. \quad \square$$

The following result is a lemma that will be necessary in the proof of Theorem 3.3.

LEMMA 3.1. *$g_1(a_0, a_1, a_2, a_3; q^m) = 0$ for $m = 1, 2, 3, \cdots$.*

Proof. First we see that

$$\begin{aligned}
&\lim_{z\to 1}(1 - z)g_1(a_0, a_1, a_2, a_3; z) \\
&= \lim_{z\to 1}\sum_{n=-\infty}^{\infty}(1 - zq^{2n})\frac{(a_0)_n(a_1)_n(a_2)_n(a_3)_n\left(\frac{z^2q}{a_0a_1a_2a_3}\right)^n}{\left(\frac{zq}{a_0}\right)_n\left(\frac{zq}{a_1}\right)_n\left(\frac{zq}{a_2}\right)_n\left(\frac{zq}{a_3}\right)_n} \\
&= \sum_{n=-\infty}^{\infty}\frac{(a_0)_n(a_1)_n(a_2)_n(a_3)_n\left(\frac{q}{a_0a_1a_2a_3}\right)^n}{\left(\frac{q}{a_0}\right)_n\left(\frac{q}{a_1}\right)_n\left(\frac{q}{a_2}\right)_n\left(\frac{q}{a_3}\right)_n} - \sum_{n=-\infty}^{\infty}\frac{(a_0)_n(a_1)_n(a_2)_n(a_3)_n\left(\frac{q^3}{a_0a_1a_2a_3}\right)^n}{\left(\frac{q}{a_0}\right)_n\left(\frac{q}{a_1}\right)_n\left(\frac{q}{a_2}\right)_n\left(\frac{q}{a_3}\right)_n} \\
&= 0,
\end{aligned}$$

since the second sum may be transformed into the first by replacing n by $-n$. Consequently if in (3.15) we replace a_0 by a_0q^{-1}, multiply by $(1 - z)$ and let $z \to 1$, we see that $g_1(a_0, a_1, a_2, a_3; q) = 0$. In general, in (3.15) we replace a_0 by a_0q^{-1}, z by q^{k-1} and from the truth of the theorem for $m = k - 1$, we deduce that $g_1(a_0, a_1, a_2, a_3; q^m) = 0$. $\square$

As we shall see, the theorems on two squares and four squares are implied by Theorem 3.2; its main usefulness to us, however, lies in its application in the following proof of (3.1).

THEOREM 3.3. *Identity* (3.1) *holds.*

Proof. In order to avoid convergence problems, we shall initially assume that $|q| < 1$, $|a_h| \geqq 1$ for $0 \leqq h \leqq 3$, $0 < |z| < |q|^{-1/2}$. These conditions will simplify our work and at the conclusion of the proof we shall show how to weaken them to the obvious minimal conditions required.

We note that $g_1(b, c, d, e; z)$ is the left-hand side of (3.1). We have already translated most of the identities of Theorem 3.1 into identities for $g_i(a_0, a_1, a_2,$

$a_3; z)$, namely (3.18), (3.19) and (3.20). There remains identity (3.6), and when $\lambda = 3$, $k = 1$, we see that (3.6) yields the following result:

$$\begin{aligned} &g_i(a_0, a_1, a_2, a_3; z) \\ (3.18)\qquad &= \frac{z^2(a_0a_1a_2a_3)^{-1}q^{2-i}(1-zq^2)(1-a_0)(1-a_1)(1-a_2)(1-a_3)}{(1-z)\left(1-\frac{zq}{a_0}\right)\left(1-\frac{zq}{a_1}\right)\left(1-\frac{zq}{a_2}\right)\left(1-\frac{zq}{a_3}\right)} \\ &\cdot g_i(a_0q, a_1q, a_2q, a_3q; zq^2). \end{aligned}$$

If we let

$$(3.19)\qquad \mathscr{P}(y; b, c, d; z) = \frac{y^{-1}z(1-y)(1-zq)\left(1-\frac{zq}{bc}\right)\left(1-\frac{zq}{bd}\right)\left(1-\frac{zq}{cd}\right)}{(1-z)\left(1-\frac{zq}{b}\right)\left(1-\frac{zq}{c}\right)\left(1-\frac{zq}{d}\right)\left(1-\frac{z^2q}{ybcd}\right)},$$

then we may write (3.15) in the following simple form:

$$(3.20)\qquad g_1(a_0, a_1, a_2, a_3; z) = \mathscr{P}(a_0; a_1, a_2, a_3; z)g_1(a_0q, a_1, a_2, a_3; zq).$$

We note that the left-hand side of (3.20) is symmetric in a_0, a_1, a_2, a_3, and therefore (3.20) is valid under any permutation of the a_j. Consequently by (3.20), we see that

$$\begin{aligned} &g_1(a_0, a_1, a_2, a_3; z) \\ &= \mathscr{P}(a_0; a_1, a_2, a_3; z)\mathscr{P}(a_1; a_0q, a_2, a_3, zq)g_1(a_0q, a_1q, a_2, a_3; zq^2) \\ &= \mathscr{P}(a_0; a_1, a_2, a_3; z)\mathscr{P}(a_1; a_0q, a_2, a_3; zq) \\ (3.21)\qquad &\quad\cdot \mathscr{P}(a_2; a_0q, a_1q, a_3; zq^2)g_1(a_0q, a_1q, a_2q, a_3; zq^3) \\ &= \mathscr{P}(a_0; a_1, a_2, a_3; z)\mathscr{P}(a_1; a_0q, a_2, a_3; zq) \\ &\quad\cdot \mathscr{P}(a_2; a_0q, a_1q, a_3; zq^2)\mathscr{P}(a_3; a_0q, a_1q, a_2q; zq^3) \\ &\quad\cdot g_1(a_0q, a_1q, a_2q, a_3q; zq^4). \end{aligned}$$

To simplify the final result in (3.21), we write it in the following form:

$$\begin{aligned} &g_1(a_0, a_1, a_2, a_3; z) \\ &= \Lambda(a_0, a_1, a_2, a_3; z)g_1(a_0q, a_1q, a_2q, a_3q; zq^4) \\ (3.22)\qquad &= \frac{(1-zq^2)\left(1-\frac{zq^3}{a_0}\right)\left(1-\frac{zq^3}{a_1}\right)\left(1-\frac{zq^3}{a_2}\right)\left(1-\frac{zq^3}{a_3}\right)\Lambda(a_0, a_1, a_2, a_3; z)}{z^2q^5(a_0a_1a_2a_3)^{-1}(1-zq^4)(1-a_0)(1-a_1)(1-a_2)(1-a_3)} \\ &\cdot g_1(a_0, a_1, a_2, a_3; zq^2), \end{aligned}$$

where the last line follows from (3.18) with z replaced by zq^2. Thus when (3.22) is completely written out, we have

$$g_1(a_0, a_1, a_2, a_3; z) = \frac{z^2 q \left(\frac{zq}{a_0a_1}\right)_2 \left(\frac{zq}{a_0a_2}\right)_2 \left(\frac{zq}{a_0a_3}\right)_2 \left(\frac{zq}{a_1a_3}\right)_2 \left(\frac{zq}{a_2a_3}\right)_2 (1 - zq^2)}{\left(\frac{zq}{a_0}\right)_2 \left(\frac{zq}{a_1}\right)_2 \left(\frac{zq}{a_2}\right)_2 \left(\frac{zq}{a_3}\right)_2 \left(\frac{z^2q}{a_0a_1a_2a_3}\right)_4 (1 - z)} \cdot g_1(a_0, a_1, a_2, a_3; zq^2). \tag{3.23}$$

Hence if

$$h(z) = \frac{\left(\frac{zq}{a_0}\right)_\infty \left(\frac{zq}{a_1}\right)_\infty \left(\frac{zq}{a_2}\right)_\infty \left(\frac{zq}{a_3}\right)_\infty \left(\frac{z^2q}{a_0a_1a_2a_3}\right)_\infty}{\left(\frac{zq}{a_0a_1}\right)_\infty \left(\frac{zq}{a_0a_2}\right)_\infty \left(\frac{zq}{a_0a_3}\right)_\infty \left(\frac{zq}{a_1a_2}\right)_\infty \left(\frac{zq}{a_1a_3}\right)_\infty \left(\frac{zq}{a_2a_3}\right)_\infty (zq)_\infty \left(\frac{q}{z}\right)_\infty} \cdot g_1(a_0, a_1, a_2, a_3; z) \tag{3.24}$$

then we deduce directly from (3.23) that

$$h(z) = h(zq^2). \tag{3.25}$$

It is clear from our initial conditions (namely, $|q| < 1$, $|a_j| \geqq 1$, $|z| < |q|^{-1/2}$) that $h(z)$ is analytic in any annulus centered on the origin which lies inside the circle $|z| = |q|^{-1/2}$ except possibly at $z = q^m$ for $m = 1, 2, 3, \cdots$; however, by Lemma 3.1 these are removable singularities. Thus

$$h(z) = \sum_{n=-\infty}^{\infty} A_n z^n, \qquad 0 < |z| < |q|^{-1/2}. \tag{3.26}$$

Substituting (3.26) into (3.25) and comparing coefficients of z^n, we see that

$$A_n = q^{2n} A_n. \tag{3.27}$$

Thus $A_n = 0$ for all $n \neq 0$, and $h(z) = A_0$. Since $h(z)$ is a constant function, we need only know one value of it. We thus obtain from (3.24) that

$$\begin{aligned} h(z) &= h(a_0) \\ &= \frac{(q)_\infty \left(\frac{a_0q}{a_1}\right)_\infty \left(\frac{a_0q}{a_2}\right)_\infty \left(\frac{a_0q}{a_3}\right)_\infty \left(\frac{a_0q}{a_1a_2a_3}\right)_\infty}{\left(\frac{q}{a_1}\right)_\infty \left(\frac{q}{a_2}\right)_\infty \left(\frac{q}{a_3}\right)_\infty \left(\frac{a_0q}{a_1a_2}\right)_\infty \left(\frac{a_0q}{a_1a_3}\right)_\infty \left(\frac{a_0q}{a_2a_3}\right)_\infty (a_0q)_\infty \left(\frac{q}{a_0}\right)_\infty} \\ &\quad \cdot \frac{(a_0q)_\infty \left(\frac{a_0q}{a_1a_2}\right)_\infty \left(\frac{a_0q}{a_1a_3}\right)_\infty \left(\frac{a_0q}{a_2a_3}\right)_\infty}{\left(\frac{a_0q}{a_1}\right)_\infty \left(\frac{a_0q}{a_2}\right)_\infty \left(\frac{a_0q}{a_3}\right)_\infty \left(\frac{a_0q}{a_1a_2a_3}\right)_\infty}, \end{aligned} \tag{3.28}$$

where we have evaluated $g_1(a_0, a_1, a_2, a_3; a_0) = f(a_0)$ from Theorem 3.2. Hence

$$h(z) = \frac{(q)_\infty}{\left(\frac{q}{a_0}\right)_\infty \left(\frac{q}{a_1}\right)_\infty \left(\frac{q}{a_2}\right)_\infty \left(\frac{q}{a_3}\right)_\infty}. \tag{3.29}$$

Comparing the expression in (3.29) for $h(z)$ with that in (3.24) we see that

$$g_1(a_0, a_1, a_2, a_3; z) = \prod \begin{bmatrix} zq, \frac{zq}{a_0a_1}, \frac{zq}{a_0a_2}, \frac{zq}{a_0a_3}, \frac{zq}{a_1a_2}, \frac{zq}{a_1a_3}, \frac{zq}{a_2a_3}, q, \frac{q}{z} \\ \frac{q}{a_0}, \frac{q}{a_1}, \frac{q}{a_2}, \frac{q}{a_3}, \frac{zq}{a_0}, \frac{zq}{a_1}, \frac{zq}{a_2}, \frac{zq}{a_3}, \frac{z^2q}{a_0a_1a_2a_3} \end{bmatrix}, \tag{3.30}$$

and this identity is easily seen to be (3.1). By analytic continuation, we see that (3.1) is valid in any region containing the one we have treated in which both sides of the identity converge; however the conditions $|q| < 1$, $|z| < |q|^{-1/2}$, $|a_j| \geqq 1$ will be adequate for most of our purposes. □

We shall now utilize (3.1) and Theorem 3.2 (which may be deduced from (3.1) by setting $b = z$) to prove various results important in number theory. We start with Jacobi's triple product identity [10, Thm. 13-8, pp. 169–170].

THEOREM 3.4. *For* $|q| < 1$, $z \neq 0$,

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2; q^2)_\infty (-zq; q^2)_\infty (-z^{-1}q; q^2)_\infty.$$

Proof. Let b, c, d, e all $\to \infty$ in (3.1) (such limit-taking may be justified by an appeal to Tannery's theorem; see [43, p. 371] for details). Then

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n^2-n} a^{2n} (1 - aq^{2n})}{(1-a)} = (aq)_\infty (q)_\infty \left(\frac{q}{a}\right)_\infty.$$

Replacing q by q^2 and then a by $-zq$ in this result, we see that

$$\sum_{n=-\infty}^{\infty} q^{4n^2} z^{2n} (1 + zq^{4n+1}) = (-zq; q^2)_\infty (q^2; q^2)_\infty (-z^{-1}q; q^2)_\infty. \tag{3.31}$$

Now

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{4n^2} z^{2n} (1 + zq^{4n+1}) &= \sum_{n=-\infty}^{\infty} a^{(2n)^2} z^{2n} + \sum_{n=-\infty}^{\infty} q^{(2n+1)^2} z^{2n+1} \\ &= \sum_{n=-\infty}^{\infty} q^{n^2} z^n. \end{aligned} \tag{3.32}$$

We may now combine (3.31) and (3.32) to obtain the desired result. □

COROLLARY 3.4.1. *For* $|q| < 1$,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q)_\infty}{(-q)_\infty}.$$

Proof. Setting $z = -1$ in Theorem 3.4 and recalling from the proof of Theorem 2.8 that $(-q)_\infty = 1/(q; q^2)_\infty$, we see that

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = (q^2; q^2)_\infty (q; q^2)_\infty (q; q^2)_\infty = (q)_\infty (q; q^2)_\infty = \frac{(q)_\infty}{(-q)_\infty}. \quad □$$

THEOREM 3.5. *For* $n \geqq 1$, $r_2(n) = 4(d_1(n) - d_3(n))$, *where* $r_2(n)$ *is the number of representations of n as a sum of two squares and* $d_a(n)$ *is the number of divisors of n congruent to a* (mod 4).

Remark. We say that $n = a^2 + b^2 = c^2 + d^2$ are two different representations of n as a sum of two squares if $a \neq c$ or $b \neq d$. Thus 1 has the four representations $1^2 + 0^2, (-1)^2 + 0, 0^2 + 1^2, 0^2 + (-1)^2$.

Proof. First we note

$$\left(\sum_{-\infty}^{\infty} (-1)^n q^{n^2}\right)^2 = \sum_{n=0}^{\infty} r_2(n)(-q)^n. \tag{3.33}$$

$$\begin{aligned}
1 + 4 \sum_{n=1}^{\infty} &(-1)^n (d_1(n) - d_3(n)) q^n \\
&= 1 + 4 \sum_{m=0}^{\infty} \sum_{r=1}^{\infty} (-1)^r (q^{(4m+1)r} - q^{(4m+3)r}) \\
&= 1 + 4 \sum_{m=0}^{\infty} \sum_{r=1}^{\infty} (-1)^{m+r} q^{(2m+1)r} \\
&= 1 + 4\left(\sum_{m=0}^{\infty} \sum_{r=m+1}^{\infty} (-1)^{m+r} q^{(2m+1)r} + \sum_{m=0}^{\infty} \sum_{r=1}^{m} (-1)^{m+r} q^{(2m+1)r}\right) \qquad (3.34) \\
&= 1 - 4 \sum_{m=0}^{\infty} \frac{q^{(2m+1)(m+1)}}{1 + q^{2m+1}} + 4 \sum_{r=1}^{\infty} \sum_{m=r}^{\infty} (-1)^{m+r} q^{(2m+1)r} \\
&= 1 - 4 \sum_{m=0}^{\infty} \frac{q^{(2m+1)(m+1)}}{1 + q^{2m+1}} + 4 \sum_{r=1}^{\infty} \frac{q^{(2r+1)r}}{1 + q^{2r}} \\
&= 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 + q^n}.
\end{aligned}$$

Comparing (3.33) with (3.34) and using Corollary 3.4.1, we see that our theorem is equivalent to the following analytic identity:

$$1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 + q^n} = \left(\frac{(q)_\infty}{(-q)_\infty}\right)^2. \tag{3.35}$$

In Theorem 3.2 set $a_1 = a_2 = -1$, let $a_3 \to \infty$, and $z \to 1$ (equivalently, in (3.1) set $b = a$, then $c = d = -1$, and let $e \to \infty$, $a \to 1$). This yields

$$1 + \sum_{n=1}^{\infty} \frac{(q)_{n-1}(1 - q^{2n})(-1)_n^2 (-1)^n q^{n(n+1)/2}}{(q)_n (-q)_n^2} = \left(\frac{(q)_\infty}{(-q)_\infty}\right)^2. \tag{3.36}$$

Now we may simplify the terms of the series as follows:

$$\frac{(q)_{n-1}(1 - q^{2n})(-1)_n^2 (-1)^n q^{n(n+1)/2}}{(q)_n (-q)_n^2} = \frac{4(-1)^n q^{n(n+1)/2}}{1 + q^n}.$$

Therefore (3.36) reduces to (3.35), and our theorem is established. □

THEOREM 3.6. *For* $n \geqq 1$, $r_4(n)$ *equals* 8 *times the sum of the divisors of n not divisible by* 4.

Proof. As in the previous theorem,

$$\left(\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\right)^4 = \sum_{n=0}^{\infty} r_4(n)(-q)^n. \tag{3.37}$$

Next we note that if $N = 2^\alpha m$, where m is odd, then

$$\begin{aligned}\sum_{d|n}(-1)^{d+Nd^{-1}}d &= \sum_{d|m}\sum_{j=0}^{\alpha}(-1)^{2^jd+N2^{-j}d^{-1}}2^jd \\ &= -\sum_{d|m}(-1)^N d + \sum_{d|m}\sum_{j=1}^{\alpha}(-1)^{N2^{-j}d^{-1}}2^jd \\ &= \left(\sum_{d|m} d\right)\left(-(1)^N + \sum_{j=1}^{\alpha-1}2^j - 2^\alpha\right) \\ &= \begin{cases}\sum_{d|m} d & \text{if } \alpha = 0 \text{ (i.e., } N \text{ odd)}, \\ -3\sum_{d|m} d & \text{if } \alpha \geqq 1.\end{cases}\end{aligned}$$

Recalling that the σ function $\sigma(n) = \sum_{d|N} d$ is multiplicative and that $\sigma(2) = 3$, we see that

$$-\sum_{d|N}(-1)^{d+Nd^{-1}}d = (-1)^N \sum_{\substack{d|N \\ 4\nmid d}} d. \tag{3.38}$$

Therefore by (3.38)

$$\begin{aligned}1 + 8\sum_{N=1}^{\infty}\left(\sum_{d|N} d\right)(-q)^N &= 1 - 8\sum_{r=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{r+m}rq^{rm} \\ &= 1 + 8\sum_{m=1}^{\infty}\frac{(-1)^m q^m}{(1+q^m)^2}\end{aligned} \tag{3.39}$$

(here we used $z/(1+z)^2 = -\sum m(-1)^m z^m$). Comparing (3.37) with (3.39) and using Corollary 3.4.1, we see that our theorem is equivalent to the following analytic identity:

$$1 + 8\sum_{m=1}^{\infty}\frac{(-1)^m q^m}{(1+q^m)^2} = \left(\frac{(q)_\infty}{(-q)_\infty}\right)^4. \tag{3.40}$$

In Theorem 3.2 set $a_1 = a_2 = a_3 = -1$ and let $z \to 1$ (equivalently, in (3.1) set $b = a$, $c = d$, $e = -1$, and let $a \to 1$). This yields

$$1 + \sum_{n=1}^{\infty}\frac{(q)_{n-1}(1-q^{2n})(-1)_n^3(-1)^n q^n}{(q)_n(-q)_n^3} = \left(\frac{(q)_\infty}{(-q)_\infty}\right)^4. \tag{3.41}$$

Now we may simplify the terms of this series as follows:

$$\frac{(q)_{n-1}(1-q^{2n})(-1)_n^3(-1)^n q^n}{(q)_n(-q)_n^3} = \frac{8(-1)^n q^n}{(1+q^n)^2}.$$

Therefore (3.41) reduces to (3.40) and our theorem is established. □

To simplify our work on eight squares, we require a minor lemma.

LEMMA 3.2. *For* $N \geqq 1$,

$$\lim_{z\to 1} \frac{(-q)_n^N(-1)_n^N - z^L(-z^{-1})_n^N(-zq)_n^N}{(1-z)} = (-1)_n^N(-q)_n^N\left(L - \frac{N}{2} + \frac{Nq^n}{1+q^n}\right).$$

Proof. Let $Q(z) = z^L(-z^{-1})_n^N(-zq)_n^N$. Then the above limit is just $Q'(1)$. Hence by the product rule for derivatives,

$$\begin{aligned} Q'(1) &= (-1)_n^N(-q)_n^N\left\{\frac{L}{z} - N\sum_{j=0}^{n-1}\frac{q^jz^{-2}}{1+z^{-1}q^j} + N\sum_{j=0}^{n-1}\frac{q^{j+1}}{1+zq^{j+1}}\right\}_{z=1} \\ &= (-1)_n^N(-q)_n^N\left(L - \frac{N}{2} + \frac{Nq^n}{1+q^n}\right); \end{aligned}$$

the simplification in the last equation occurs because the two sums cancel at $z = 1$ except for first and last terms. □

THEOREM 3.7. *For* $n \geqq 1$, $r_8(n) = 16(-1)^n \sum_{d|n}(-1)^d d^3$.

Proof. As before,

$$\left(\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\right)^8 = \sum_{n=0}^{\infty} r_8(n)(-q)^n. \tag{3.42}$$

Utilizing the elementary summation

$$\sum_{m=1}^{\infty}(-1)^m m^3 z^m = \frac{-z + 4z^2 - z^3}{(1+z)^4},$$

we see that

$$\begin{aligned} 1 + 16\sum_{n=1}^{\infty}\left(\sum_{d|n}(-1)^d d^3\right)q^n &= 1 + 16\sum_{r=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m m^3 q^{mr} \\ &= 1 + 16\sum_{r=1}^{\infty}\frac{(-q^r + 4q^{2r} - q^{3r})}{(1+q^r)^4}. \end{aligned} \tag{3.43}$$

Comparing (3.42) with (3.43) and using Corollary 3.4.1, we see that our theorem is equivalent to the following analytic identity:

$$1 + 16\sum_{n=1}^{\infty}\frac{(-q^n + 4q^{2n} - q^{3n})}{(1+q^n)^4} = \left(\frac{(q)_\infty}{(-q)_\infty}\right)^8. \tag{3.44}$$

In (3.1) set $b = c = d = e = -1$, and let $z \to 1$. This yields

$$\begin{aligned} \left(\frac{(q)_\infty}{(-q)_\infty}\right)^8 &= \lim_{z\to 1}\sum_{-\infty}^{\infty}\frac{(1 - zq^{2n})z^{2n}q^n(-1)_n^4}{(1-z)(-zq)_n^4} \\ &= \lim_{z\to 1}\left\{\sum_{-\infty}^{\infty}\frac{z^{2n}q^n(-1)_n^4}{(1-z)(-zq)_n^4} - z\sum_{n=-\infty}^{\infty}\frac{z^{-2n}q^{-3n}(-z^{-1})_n^4 z^{4n}q^{2n}}{(1-z)(-q)_n^4}\right\} \end{aligned}$$

(where $-n$ has replaced n in the second sum)

$$= \sum_{-\infty}^{\infty}\frac{q^n}{(-q)_n^8}\lim_{z\to 1}\frac{(-1)_n^4(-q)_n^4 - z(-z^{-1})_n^4(-zq)_n^4}{(1-z)}$$

(where $\lim \sum = \sum \lim$ by Tannery's theorem [43, p. 371])

(*cont.*)

$$(3.45)\qquad = \sum_{-\infty}^{\infty} \frac{q^n}{(-q)_n^8}(-1)_n^4(-q)_n^4\left(-1 + \frac{4q^n}{1+q^n}\right) \qquad \text{(by Lemma 3.2)}$$

$$= 1 + 16\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{q^n(3q^n-1)}{(1+q^n)^5}$$

$$= 1 + 16\sum_{n=1}^{\infty} \frac{q^n(3q^n-1)}{(1+q^n)^5} + 16\sum_{n=1}^{\infty} \frac{q^{-n}(3q^{-n}-1)q^{5n}}{(1+q^n)^5}$$

$$= 1 + 16\sum_{n=1}^{\infty} \frac{(-q^n + 3q^{2n} + 3q^{3n} - q^{4n})}{(1+q^n)^5}$$

$$= 1 + 16\sum_{n=1}^{\infty} \frac{(-q^n + 4q^{2n} - q^{3n})}{(1+q^n)^4}.$$

Thus (3.45) is identical with (3.44), and so our theorem is proved. □

A similar treatment may be given to $r_6(n)$, where in (3.1) we set $b = c = d = -1$ and let $e \to \infty$, $z \to 1$. Algebraic manipulations of the resulting series are somewhat messy however.

There are a number of other applications of (3.1) to number theory. Our next theorem is originally due to S. Ramanujan. We shall give a proof due to W. N. Bailey [17].

THEOREM 3.8. *For* $n \geqq 0$, $p(5n+4) \equiv 0 \pmod 5$, *where* $p(n)$ *is the number of partitions of* n.

Remark. It might seem that this result belongs in § 2; however, since it is deducible from (3.1) and since it is a congruence, we have included it here.

Proof.

$$\sum_{n=0}^{\infty}\left\{\frac{q^{5n+1}}{(1-q^{5n+1})^2} - \frac{q^{5n+2}}{(1-q^{5n+2})^2} - \frac{q^{5n+3}}{(1-q^{5n+3})^2} + \frac{q^{5n+4}}{(1-q^{5n+4})^2}\right\}$$

$$= \sum_{n=-\infty}^{\infty}\left\{\frac{q^{5n+1}}{(1-q^{5n+1})^2} - \frac{q^{5n+3}}{(1-q^{5n+3})^2}\right\}$$

$$= \sum_{n=-\infty}^{\infty} \frac{q^{5n+1}(1-q^2)(1+q^{5n+2})(1-q^{5n+2})}{(1-q^{5n+1})(1-q^{5n+3})^2}$$

$$(3.46)\qquad = \frac{q(1-q^2)(1-q^4)}{(1-q)^2(1-q^3)^2}\,{}_6\psi_6\left[\begin{matrix} q^7, -q^7, q, q, q^3, q^3; q^5, q^5 \\ q^2, -q^2, q^8, q^8, q^6, q^6 \end{matrix}\right]$$

$$= \frac{q(1-q^2)(1-q^4)}{(1-q)^2(1-q^3)^2}\,\frac{(q^9;q^5)_\infty(q^7;q^5)_\infty(q^5;q^5)_\infty^4(q^3;q^5)_\infty(q;q^5)_\infty}{(q^4;q^5)_\infty^2(q^2;q^5)_\infty^2(q^8;q^5)_\infty^2(q^6;q^5)_\infty^2}$$

$$= q\frac{(q^5;q^5)_\infty^5}{(q)_\infty}.$$

Now from (2.1) we see that $\sum_{n=0}^{\infty} p(n)q^n = 1/(q)_\infty$. Using this fact and comparing terms in the extreme members of (3.46) that involve powers of q^5, we see that

$$(q^5;q^5)_\infty^5 \sum_{n=0}^{\infty} p(5n+4)q^{5n+5}$$

$$= \text{terms involving } q^5 \text{ in } \sum_{n=1}^{\infty} \frac{(n/5)q^n}{(1-q^n)^2}$$

(where $(n/5) = 0, 1, -1, 1$ for $n \equiv 0, 1, 2, 3, 4 \pmod 5$, respectively)

$$= \text{terms involving } q^5 \text{ in } \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(\frac{n}{5}\right) mq^{mn}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(\frac{n}{5}\right) 5mq^{5mn}$$

$$= 5\sum_{n=1}^{\infty} \frac{(n/5)q^{5n}}{(1-q^{5n})^2}.$$

Therefore

$$\sum_{n=0}^{\infty} p(5n+4)q^{n+1} = \frac{5\sum_{n=1}^{\infty} \frac{(n/5)q^n}{(1-q^n)^2}}{(q)_\infty^5}. \tag{3.47}$$

Hence comparing coefficients of q^{n+1} on each side of (3.47), we see that $p(5n+4) \equiv 0 \pmod 5$ for each $n \geqq 0$. □

To conclude this section we shall derive the quintuple product identity from (3.1). This identity has been rediscovered many times (see [25] for a brief history), and has been of use in several problems in number theory (see Bailey [18], Gordon [32], Andrews [7]).

THEOREM 3.9. *For* $|q| < 1, |z| \neq 0$,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} z^{3n}(1+zq^n) = (-qz^{-1})_\infty(-z)_\infty(qz^{-2};q^2)_\infty(qz^2;q^2)_\infty(q)_\infty.$$

Proof. In (3.1) let $b, c, d \to \infty$ and set $e = q^{1/2}a^{1/2}$; then

$$\sum_{-\infty}^{\infty} (1-zq^{2n})a^{3/2n}q^{n(3n-2)/2}(-1)^n$$

$$= (1-a)\frac{(aq)_\infty(q)_\infty(qa^{-1})_\infty}{(q^{1/2}a^{-1/2})_\infty(a^{1/2}q^{1/2})_\infty}$$

$$= (-a^{1/2};q^{1/2})_\infty(a^{1/2};q)_\infty(q)_\infty(-q^{1/2}a^{-1/2};q^{1/2})_\infty(a^{-1/2}q;q)_\infty$$

$$= (-a^{1/2})_\infty(-a^{1/2}q^{1/2})_\infty(a^{1/2})_\infty(q)_\infty(-q^{1/2}a^{-1/2})_\infty(-qa^{-1/2})_\infty(a^{-1/2}q)_\infty$$

$$= (a;q^2)_\infty(-a^{1/2}q^{1/2})_\infty(q)_\infty(-q^{1/2}a^{-1/2})_\infty(a^{-1}q^2;q^2)_\infty.$$

Replacing a by $z^{-2}q$, we see that

$$\sum_{-\infty}^{\infty}(1 - z^{-2}q^{2n+1})q^{n(3n+1)/2}z^{-3n}(-1)^n = (qz^{-2};q^2)_\infty(-z^{-1}q)_\infty(q)_\infty(-z)_\infty(qz^2;q^2)_\infty. \tag{3.48}$$

Therefore

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}z^{3n}(1+zq^n)\\
&= \sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n+1)/2}z^{-3n} + \sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n+1)/2}z^{3n+1}\\
&\qquad\qquad (n \text{ replaced by } -n \text{ in first sum})\\
&= \sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n+1)/2}z^{-3n} - \sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n+1)/2+2n+1}z^{-3n-2}\\
&\qquad\qquad (n \text{ replaced by } -n-1 \text{ in second sum})\\
&= \sum_{n=-\infty}^{\infty}(1-z^{-2}q^{2n+1})q^{n(3n+1)/2}z^{-3n}(-1)^n\\
&= (-qz^{-1})_\infty(-z)_\infty(qz^{-2};q^2)_\infty(qz^2;q^2)_\infty(q)_\infty \quad (\text{by } (3.48)),
\end{aligned}$$

and this is precisely the desired result. □

It has been our object in this section merely to scratch the surface of applications of basic hypergeometric series to number theory. Apart from Theorems 3.2 and 3.8, none of the proofs given in this section have appeared before. Therefore the techniques introduced here appear to hold promise for many further number-theoretic investigations. We remark that M. Jackson has found significant number-theoretic application of formulas involving a well-poised ${}_8\psi_8$ (i.e., $K_{5,2,2}(a_0, a_1, a_2, a_3, a_4, a_5; z; q)$), and it should be possible to extend our q-difference equation techniques to prove the following identity of W. N. Bailey [19, (3.2), p. 197]:

$$\begin{aligned}
{}_2\psi_2\begin{bmatrix} e, f; \dfrac{aq}{ef} \\ \dfrac{aq}{c}, \dfrac{aq}{d} \end{bmatrix} &= \prod\begin{bmatrix} \dfrac{q}{c}, \dfrac{q}{d}, \dfrac{aq}{e}, \dfrac{aq}{f} \\ aq, \dfrac{q}{a}, \dfrac{aq}{cd}, \dfrac{aq}{ef} \end{bmatrix}\\
&\cdot \sum_{s=-\infty}^{\infty}\frac{(q\sqrt{a})_s(-q\sqrt{a})_s(c)_s(d)_s(e)_s(f)_s}{(\sqrt{a})_s(-\sqrt{a})_s\left(\dfrac{aq}{c}\right)_s\left(\dfrac{aq}{d}\right)_s\left(\dfrac{aq}{e}\right)_s\left(\dfrac{aq}{f}\right)_s}\left(\frac{a^3q}{cdef}\right)^s q^{s^2}.
\end{aligned} \tag{3.49}$$

This identity may be used to prove the Rogers–Ramanujan identities (as Bailey remarks), and it should be applicable to various fifth order mock theta function identities [6].

It is quite conceivable that a more extensive study of $K_{\lambda,k,i}(a_0, \cdots, a_\lambda; z; q)$ could lead to new proofs of Mordell's formulas [44] for $r_{2s}(n)$, to new proofs of more of the Ramanujan congruences [13], and to other number-theoretic results related to elliptic functions.

Finally, it should be mentioned that Theorem 2.7 is a special case of Theorem 3.2 (which is in turn the special case of (3.1) when $b = a$); to see this, set $a_0 = z^{1/2}$, $a_1 = -z^{1/2}$ in Theorem 3.2. Thus Theorems 2.8, 2.9 and 2.10 may be seen as corollaries of (3.1) also.

Theorems on sums of squares are often treated as corollaries of identities involving elliptic functions. Indeed, L. Carlitz (Nieuw Arch. Wisk., 3 (1955), pp. 193–196, and Proc. Amer. Math. Soc., 8 (1957), pp. 120–124) has treated the sums of four, six and eight squares in a very elegant manner utilizing certain elliptic function identities. It is quite likely that these more general elliptic function identities are also special cases of (3.1).

4. Application to finite vector spaces. The relationship between basic hypergeometric series and finite vector spaces arises from the following well-known theorem (see, for example, [48, p. 240]).

THEOREM 4.1. *Let V_n be a finite-dimensional vector space of dimension n over $GF(q)$, the finite field of q elements. Then there are exactly*

$$\binom{n}{k}_q = \frac{(q)_n}{(q)_k(q)_{n-k}}$$

subspaces of V_n of dimension k.

Remark. $\binom{n}{k}_q$ is called a *q-binomial coefficient* or a *Gaussian polynomial.*

Proof. First we determine the number of k-tuples of linearly independent vectors $\{v_1, v_2, \cdots, v_k\}$ that exist in V_n. We may select any nonzero vector for v_1, and since there are q^n vectors in V_n, we may select v_1 in $q^n - 1$ ways. The vector v_2 must be selected to lie outside the subspace spanned by v_1, and so v_1 may be selected in $q^n - q$ ways. In general, v_i must lie outside the subspace spanned by $\{v_1, \cdots, v_{i-1}\}$, and so v_i may be chosen in $q^n - q^{i-1}$ ways. Hence the number of k-tuples of linearly independent vectors is

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1}). \tag{4.1}$$

Each such k-tuple spans a k-dimensional subspace; however, two different k-tuples may span the same subspace. In fact the number of k-tuples spanning the same subspace is just the number of linearly independent k-tuples that exist in a k-dimensional space, and by (4.1) this number is

$$(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1}). \tag{4.2}$$

Therefore the number of k-dimensional subspaces of V_n is just

$$\begin{aligned}&\frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}\\ &\quad= \frac{q^{k(k-1)/2}(-1)(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-k+1})}{q^{k(k-1)/2}(-1)^k(1 - q^k)(1 - q^{k-1}) \cdots (1 - q)}\\ &\quad= \frac{(q)_n}{(q)_k(q)_{n-k}} = \binom{n}{k}_q. \quad \square\end{aligned}$$

To illustrate how finite vector spaces may be used in the theory of basic hypergeometric functions, we prove the following q-analogue of Vandermonde's theorem [52, (3.3.2.6), p. 97]; E. A. Bender [21] has proved a more general result of this nature using finite vector spaces.

THEOREM 4.2. *For nonnegative integers m, n and h,*

$$\sum_{l=0}^{h} \binom{n}{l}_q \binom{m}{h-l}_q q^{l(m-h+l)} = \binom{m+n}{h}_q .$$

Remark. From the definition of $\binom{n}{l}_q$, we may rewrite the above result as

$${}_2\phi_1\left[\begin{matrix} q^{-h}, q^{-n}; q, q^{n+m+1} \\ q^{m-h+1} \end{matrix}\right] = \frac{(q)_{m-h}(q)_{n+m}}{(q)_m(q)_{n+m-h}},$$

and if we assume $|q| < 1$, then this result follows directly from Theorem 2.6, the q-analogue of Gauss's theorem. In the proof below we assume that q is a prime power.

Proof. Let us consider V_{m+n} (a finite vector space of dimension $m + n$ over $GF(q)$), and let V_m be a chosen subspace of dimension m. We note from Theorem 4.1 that there are $\binom{m+n}{h}$ subspaces of V_{m+n} of dimension h.

We now ask for the number T_l of h-dimensional subspaces of V_{m+n} that intersect V_m in an l-dimensional subspace. To count these spaces, we ask for linearly independent h-tuples $(x_1, x_2, \cdots, x_l, y_1, y_2, \cdots, y_{h-l})$ such that the x's lie in V_m and the y's outside V_m. Just as in Theorem 4.1, the x's may be chosen in $(q^m - 1)(q^m - q) \cdots (q^m - q^{l-1})$ ways, and the y's may be chosen in $(q^{m+n} - q^m) \cdots (q^{m+n} - q^{m+(h-l)-1})$ ways. Thus the number of h-tuples of the desired type is

$$(4.3)\qquad \begin{aligned} &(q^m - 1)(q^m - q) \cdots (q^m - q^{l-1}) \\ &\cdot (q^{m+n} - q^m)(q^{m+n} - q^{m+1}) \cdots (q^{m+n} - q^{m+h-l-1}). \end{aligned}$$

As in Theorem 4.1, several h-tuples may span the same space; indeed exactly those h-tuples lying in the same V_h with the first l entries in V_l and the rest outside V_l are the ones spanning the same space. Consequently to obtain T_l, we must divide (4.3) by $(q^l - 1) \cdots (q^l - q^{l-1})(q^h - q^l) \cdots (q^h - q^{l+(h-l)-1})$. Therefore

$$T_l = \frac{(q^m-1)(q^m-q)\cdots(q^m-q^{l-1})(q^{m+n}-q^m)(q^{m+n}-q^{m+1})\cdots(q^{m+n}-q^{m+h-l-1})}{(q^l-1)(q^l-q)\cdots(q^l-q^{l-1})(q^h-q^l)(q^h-q^{l+1})\cdots(q^h-q^{h-1})}$$

$$= \binom{m}{l}_q \frac{q^{m(h-l)}(q^n-1)(q^n-q)\cdots(q^n-q^{h-l-1})}{q^{l(h-l)}(q^{h-l}-1)(q^{h-l}-q)\cdots(q^{h-l}-q^{h-l-1})}$$

$$= \binom{m}{l}_q \binom{n}{h-l}_q q^{(m-l)(h-l)}.$$

Hence

$$\binom{n+m}{h}_q = \sum_{l=0}^{h} T_l = \sum_{l=0}^{h} \binom{m}{l}_q \binom{n}{h-l}_q q^{(m-l)(h-l)} = \sum_{l=0}^{h} \binom{m}{h-l}_q \binom{n}{l}_q q^{(m-h+l)l},$$

where we have replaced l by $h - l$ in the last sum. □

G.-C. Rota and J. Goldman in [47] and [48] have shown how fruitful the relationship between basic hypergeometric functions and finite vector spaces is; at the conclusion of [48] they pose a number of problems that appear to be quite important in this study. In a short note [42] D. Knuth presented canonical mappings between certain sets of partitions, the lattice of subspaces of V_n, and the lattice of subsets of $\{1, 2, \cdots, n\}$; Knuth in this way answered one of the central problems posed by Rota and Goldman [48, p. 258].

A second problem posed by Rota and Goldman concerns the extension of the theory of binomial enumeration [49] to finite vector spaces. This extension was made in [8]. Below we present a brief introduction to this topic.

Our consideration will be devoted to certain linear operators acting on $\mathbf{P} = \mathbf{R}[X]$, the ring of polynomials with real coefficients. If $f(X) \in \mathbf{P}$, we define η, the *Eulerian shift operator*, by

$$\eta^a f(X) = f(q^a X) = f(AX), \tag{4.4}$$

where $A = q^a$.

An *Eulerian differential operator* τ is a linear operator on $\mathbf{P}$ satisfying the following two conditions:

$$q^{-a}\tau\eta^a = \eta^a\tau, \quad \text{and} \quad \tau X^n \not\equiv 0 \quad \text{for each } n > 0. \tag{4.5}$$

An *Eulerian family of polynomials* is a sequence of polynomials $\{p_n(X)\}_{n=0}^{\infty}$ such that $p_0(X) = 1$, $p_n(X)$ is of degree n, and

$$p_n(XY) = \sum_{j \geqq 0} \begin{pmatrix} n \\ j \end{pmatrix}_q p_j(X) Y^j p_{n-j}(Y). \tag{4.6}$$

Given an Eulerian differential operator τ, a sequence of polynomials $\{p_n(X)\}_{n=0}^{\infty}$ is the sequence of *Eulerian basic polynomials for* τ if $p_0(X) = 1$, $p_n(1) = 0$ for $n > 0$, and

$$\tau p_n(X) = (1 - q^n) p_{n-1}(X). \tag{4.7}$$

We now present the central theorems related to these concepts. Since the proofs are somewhat lengthy and are readily available in [8], we shall for the most part omit them.

THEOREM 4.3. (a) *If* $\{p_n(X)\}_{n=0}^{\infty}$ *is an Eulerian basic sequence for some Eulerian differential operator, then it is an Eulerian family of polynomials.*

(b) *If* $\{p_n(X)\}_{n=0}^{\infty}$ *is an Eulerian family of polynomials, then it is an Eulerian basic sequence for some Eulerian differential operator.*

Proof. See [8, pp. 351–352].

THEOREM 4.4. *If* $\{p_n(X)\}_{n=0}^{\infty}$ *is an Eulerian family of polynomials and if* C_n *is the leading coefficient of* $p_n(X)$, *then*

$$\sum_{n \geqq 0} \frac{p_n(X) t^n}{(q)_n} = \frac{f(Xt)}{f(t)}, \tag{4.8}$$

where

$$f(t) = \sum_{n \geqq 0} \frac{C_n t^n}{(q)_n}. \tag{4.9}$$

Proof. See [8, pp. 356–357].

THEOREM 4.5. *Let* $\{p_n(X)\}_{n=0}^{\infty}$ *and* $f(t)$ *be as in Theorem* 4.4. *Then*

$$f(t) = \exp\left\{\sum_{n \geqq 0} \frac{p_n'(1)t^n}{(q)_n n}\right\}.$$

We choose a simple example to show how this theory is applied. Let

$$D_q = (1/X)(1 - \eta). \tag{4.10}$$

Then one trivially verifies (4.5) for D_q; hence D_q is an Eulerian differential operator (called the *q-derivative*).

Next we define

$$P_n(X, Y) = (X - Y)(X - Yq) \cdots (X - Yq^{n-1}), \qquad P_0(X, Y) = 1. \tag{4.11}$$

Since

$$\begin{aligned} D_q P_n(X, 1) &= \frac{(X-1)(X-q)\cdots(X-q^{n-1}) - (Xq-1)\cdots(Xq-q^{n-1})}{X} \\ &= (X-1)(X-q)\cdots(X-q^{n-2})\ \frac{X - q^{n-1} - q^n(X - 1/q)}{X} \\ &= (1-q^n)P_{n-1}(X, 1) \end{aligned}$$

and $P_0(X, 1) = 1$, $P_n(1, 1) = 0$, we see (by Theorem 4.3) that $\{P_n(X, 1)\}_{n=0}^{\infty}$ is the Eulerian family of polynomials associated with D_q.

Therefore, by Theorem 4.4,

$$\sum_{n \geqq 0} \frac{P_n(X, 1)t^n}{(q)_n} = \frac{e(Xt)}{e(t)}, \tag{4.12}$$

where $e(t) = \sum_{n \geqq 0} t^n/(q)_n$.

Since

$$P_n'(1, 1) = \lim_{X \to 1} \frac{P_n(X, 1)}{X - 1} = (q)_{n-1},$$

we see by Theorem 4.5 that

$$\begin{aligned} e(t) &= \exp\left\{\sum_{n \geqq 1} \frac{(q)_{n-1}t^n}{(q)_n n}\right\} = \exp\left\{\sum_{n \geqq 1} \frac{t^n}{(1-q^n)n}\right\} \\ &= \exp\left\{\sum_{n \geqq 1}\sum_{m \geqq 0} \frac{t^n q^{mn}}{n}\right\} = \exp\left\{-\sum_{m \geqq 0} \log(1 - tq^m)\right\} = (t)_{\infty}^{-1}. \end{aligned} \tag{4.13}$$

Combining (4.13) with (4.12), we see that

$$\sum_{n \geqq 0} \frac{P_n(X, 1)t^n}{(q)_n} = \frac{(t)_{\infty}}{(tX)_{\infty}},$$

or

$$\sum_{n \geqq 0} \frac{(X^{-1})_n (tX)^n}{(q)_n} = \frac{(t)_{\infty}}{(tX)_{\infty}}, \tag{4.14}$$

a result equivalent to Theorem 2.4.

If we define

$$\gamma = \frac{1}{X} \frac{(1-\eta)}{1-b\eta q^{-1}},$$

then γ is also an Eulerian differential operator, and if $\{g_n(X)\}_{n=0}^{\infty}$ is the related Eulerian family of polynomials, then it turns out that [8, (9.4), p. 365]

$$\sum_{n \geqq 0} \frac{g_n(X)t^n}{(q)_n} = \frac{(bXt)_\infty (t)_\infty}{(Xt)_\infty (bt)_\infty}. \tag{4.15}$$

If we expand $g_n(X)$ in terms of $P_n(X, 1)$ so that

$$\sum_{n \geqq 0} \frac{g_n(X)t^n}{(q)_n} = \sum_{n \geqq 0} \frac{C_n(t)P_n(X, 1)}{(q)_n}, \tag{4.16}$$

then it is possible to deduce [8, first Eq. (9.9), p. 367] that

$$C_n(t) = \frac{t^n(b)_n}{(bt)_n}. \tag{4.17}$$

Combining (4.15), (4.16) and (4.17), we see that [8, second Eq. (9.9), p. 367]

$$\sum_{n \geqq 0} \frac{(X^{-1})_n (b)_n (tX)^n}{(q)_n (bt)_n} = \frac{(bXt)_\infty (t)_\infty}{(Xt)_\infty (bt)_\infty}, \tag{4.18}$$

a result equivalent to Theorem 2.6.

Other important results in basic hypergeometric functions are deducible from this theory including the Rogers–Ramanujan identities, which are related to the Eulerian differential operator $(1/X)(\eta^{-2} - \eta^{-1})$.

Finally, we note [48, p. 252] that when $X = q^x$, $Y = q^y$ and $x > y$, then $P_n(X, Y)$ is the number of nonsingular linear transformations of V_n into V_x where the image of V_n has only 0 in common with V_y, a fixed subspace of V_x. This is easily seen, for if $v_1, \cdots, v_n$ form a basis for V_n, we need only determine the action of a linear transformation f on this basis. There are clearly $X - Y$ choices for $f(v_1)$. Now $f(v_2)$ must lie outside the space spanned by V_y and $f(v_1)$; hence there are $X - Yq$ choices for $f(v_2)$. Continuing in this manner, we see that the total number of such mappings is $(X - Y)(X - Yq) \cdots (X - Yq^{n-1}) = P_n(X, Y)$. Thus we see the relationship between properties of finite vector spaces and Eulerian families of polynomials.

Another more complicated type of nonsingular linear transformation of a finite vector space is also studied in [8, § 11]; here the Eulerian family is

$$h_n(X) = \sum_{j \geqq 0} \begin{pmatrix} n \\ j \end{pmatrix}_q P_j(X, 1)P_{n-j}(X, 1)U^{n-j},$$

where U is a fixed parameter. The $h_n(X)$ are seen to be closely related to what are called the *q-Hermite polynomials* [24], $\sum_{j \geqq 0} \begin{pmatrix} n \\ j \end{pmatrix}_q U^{n-j}$.

We should add that recent work by M. Henle [41] suggests that both the Rota–Mullin theory of binomial enumeration [49] and the theory of Eulerian differential operators [8] may be subsumed in a more general structure with diverse combinatorial applications.

5. Application to combinatorial identities. With the rekindling of interest in combinatorial analysis has come a large amount of work on binomial coefficient summations such as

$$\sum_{k \geqq 0} \binom{2n+1}{2p+2k+1}\binom{p+k}{k} = \binom{2n-p}{p} 2^{2n-2p} \tag{5.1}$$

(see [30, p. 71], [34]), and

$$\sum_{k \geqq 0} \binom{x+y+k}{k}\binom{y}{a-k}\binom{x}{b-k} = \binom{x+a}{b}\binom{y+b}{a} \tag{5.2}$$

(see [33]).

The impression one gains from treatises on the subject (cf. Riordan's book [46] or Gould's table [35]) is that, for the most part, such identities cannot be put into a coherent setting. Riordan's statement is, perhaps, the most pessimistic [46, p. vii]: "The central fact developed is that identities are both inexhaustible and unpredictable; the age-old dream of putting order in this chaos is doomed to failure."

This situation seems quite unfortunate. Indeed, binomial coefficient identities arise in many different areas of mathematics, and it would be quite useful if it were possible for mathematicians generally to be able to prove such identities in short order rather than being diverted into a general study of binomial identities just for the purpose of cracking one particular problem. H. W. Gould [35] has made a significant contribution to mitigating this situation. He has tabulated 555 combinatorial identities, of which over 450 are purely binomial coefficient identities. However, there still remains the difficulty of finding one's own identity among a vast number of others, and as Gould points out [35, p. viii], the situation may be further complicated through changes of variable and introduction of redundant factors.

The object of this section is to emphasize the close relationship between binomial coefficient identities and ordinary hypergeometric series (and simultaneously that between q-binomial coefficient identities and basic hypergeometric series). There are several important reasons for studying these relationships: first, by translating binomial coefficient identities into hypergeometric series identities, we then need only check the 4-page, 32-entry table of L. J. Slater [52, Appendix III] rather than the 450 entries of Gould. Second, we not only avoid the need to fashion ad hoc methods for summing our series, but we have our result completely established since we are quoting a known theorem. Third, the number of known q-binomial coefficient identities is much smaller than the number of known ordinary binomial coefficient identities. The approach used here allows the production of q-analogues of many known combinatorial identities by merely replacing the hypergeometric series translation by its q-analogue.

Theorem 5.1 indicates the procedure for translating binomial coefficient identities into hypergeometric series identities, and Theorem 5.2 illustrates why hypergeometric series identities are not easily disguised. The remaining theorems of this section show how our technique applies to some of the more popular binomial coefficient summations.

We define

$$[a]_n = a(a+1)\cdots(a+n-1). \tag{5.3}$$

The standard notation for this product is $(a)_n$; however, this last symbol has already been utilized extensively in this paper with another meaning.

THEOREM 5.1. *Let A, B, C, D be fixed nonnegative integers with $A > C \geqq 0$. Then*

$$(An + DA + C)! = (DA + C)!A^{An}\left[D + \frac{C+1}{A}\right]_n\left[D + \frac{C+2}{A}\right]_n \cdots \left[D + \frac{C}{A} + 1\right]_n; \tag{5.4}$$

$$(B - An)! = \frac{(-1)^{An}B!}{A^{An}\left[-\frac{B}{A}\right]_n\left[-\frac{B-1}{A}\right]_n \cdots \left[-\frac{B-A+1}{A}\right]_n}, \tag{5.5}$$

provided $B - An \geqq 0$;

$$(q)_{An+DA+C} = (q)_{DA+C}(q^{DA+C+1}; q^A)_n \cdots (q^{DA+C+A}; q^A)_n; \tag{5.6}$$

$$(q)_{B-An} = \frac{q^{-BAn+An(An-1)/2}(-1)^{An}(q)_B}{(q^{-B}; q^A)_n(q^{-B+1}; q^A)_n \cdots (q^{-B+A-1}; q^A)_n}. \tag{5.7}$$

Proof. These results are all obtained by simple algebraic manipulation. First,

$$\begin{aligned}(An + DA + C)! &= (DA + C)![DA + C + 1]_{An} \\ &= (DA + C)!A^{An}\left[D + \frac{C+1}{A}\right]_n\left[D + \frac{C+2}{A}\right]_n \cdots \left[D + \frac{C}{A} + 1\right]_n.\end{aligned}$$

Next,

$$\begin{aligned}(B - An)! &= \frac{B!}{B(B-1)\cdots(B-An+1)} = \frac{(-1)^{An}B!}{[-B]_{An}} \\ &= \frac{(-1)^{An}B!A^{-An}}{\left[-\frac{B}{A}\right]_n\left[-\frac{B-1}{A}\right]_n \cdots \left[-\frac{B-A+1}{A}\right]_n}.\end{aligned}$$

Since (5.6) and (5.7) are proved in an entirely analogous manner, we shall omit the proofs. □

We now need some standard definitions from the theory of ordinary hypergeometric functions.

Consider

$${}_nH_n\left[\begin{matrix}a_1, \cdots, a_n; z \\ b_1, \cdots, b_n\end{matrix}\right] = \sum_{m=-\infty}^{\infty} \frac{[a_1]_m \cdots [a_n]_m}{[b_1]_m \cdots [b_n]_m} z^m, \tag{5.8}$$

where $[\alpha]_{-n} = (-1)^n[1-\alpha]_n^{-1}$. This series reduces to ${}_nF_{n-1}$ (see (1.2)) if $b_1 = 1$.

We say that the series in (5.8) is *Saalschutzian* if $a_1 + a_2 + \cdots + a_n + 2 = b_1 + b_2 + \cdots + b_n$. We say that the series in (5.8) is *well-poised* if $a_1 + b_1 = a_2 + b_2 = \cdots = a_n + b_n$, and we say *nearly-poised* if only $n - 1$ of these equalities hold.

Analogously we may consider (as in § 3)

$$(5.9) \qquad {}_n\psi_n\left[\begin{matrix}\alpha_1, \cdots, \alpha_n; q, t\\ \beta_1, \cdots, \beta_n\end{matrix}\right] = \sum_{m=-\infty}^{\infty} \frac{(\alpha_1)_m \cdots (\alpha_n)_m t^m}{(\beta_1)_m \cdots (\beta_n)_m}.$$

This series reduces to ${}_n\phi_{n-1}$ (see (1.1)) if $\beta_1 = q$. We say that the series in (5.9) is *Saalschutzian* if $\alpha_1\alpha_2 \cdots \alpha_n q^2 = \beta_1\beta_2 \cdots \beta_n$; *well-poised* if $\alpha_1\beta_1 = \alpha_2\beta_2 = \cdots = \alpha_n\beta_n$, and *nearly-poised* if only $n - 1$ of these equalities hold.

In Slater's table of hypergeometric summations [52, Appendix III] only three identities [52, (III. 23), (III. 24) and (III. 28), p. 245] do not involve either a Saalschutzian, well-poised, or nearly-poised series; furthermore, all of the basic hypergeometric summations in [52, Appendix IV] involve basic hypergeometric series that are either Saalschutzian or well-poised.

THEOREM 5.2. *The properties "Saalschutzian", "well-poised" and "nearly-poised" are invariant under shifting the index of summation or reversing the order of summation.*

Remark. This result provides evidence for why Gould's table of binomial coefficient identities is so much longer than Slater's; namely, altering the index of summation now does not alter the most important properties of the series in question, and redundant factors are easily spotted for they will appear identically in both the numerators and denominators of the given series.

Proof. Replacing m by $-m$ in (5.8), we find

$$(5.10) \qquad \sum_{m=-\infty}^{\infty} \frac{[1 - b_1]_m \cdots [1 - b_n]_m z^{-m}}{[1 - a_1]_m \cdots [1 - a_n]_m};$$

hence $2 + \sum a_i = \sum b_i$ implies $2 + \sum (1 - b_i) = \sum (1 - a_i)$, and $a_1 + b_1 = \cdots = a_n + b_n$ implies $(1 - b_1) + (1 - a_1) = \cdots = (1 - b_n) + (1 - a_n)$. Also if only $n - 1$ of the latter equations hold for $a_i + b_i$, then the corresponding $n - 1$ equations hold for $(1 - a_i) + (1 - b_i)$. Thus the three properties listed in the theorem are preserved when m is replaced by $-m$.

If m is replaced by $m + k$ in (5.8), we obtain

$$(5.11) \qquad \frac{[a_1]_k \cdots [a_n]_k z^k}{[a_1]_k \cdots [a_n]_k} \sum_{m=-\infty}^{\infty} \frac{[a_1 + k]_m \cdots [a_n + k]_m z^m}{[b_1 + k]_m \cdots [b_n + k]_m}.$$

In this case $2 + \sum a_i = \sum b_i$ implies $2 + \sum (a_i + k) = \sum (b_i + k)$, and $a_1 + b_1 = \cdots = a_n + b_n$ implies $(a_1 + k) + (b_1 + k) = \cdots = (a_n + k) + (b_n + k)$. Similarly the nearly-poised condition holds, and thus the theorem is established for ordinary hypergeometric series.

The same approach applied to the basic series (5.9) produces first

$$\sum_{m=-\infty}^{\infty} \frac{\left(\frac{q}{\beta_1}\right)_m \cdots \left(\frac{q}{\beta_n}\right)_m \left(\frac{\beta_1\beta_2 \cdots \beta_n}{\alpha_1\alpha_2 \cdots \alpha_n t}\right)^n}{(q/\alpha_1)_m \cdots (q/\alpha_n)_m},$$

and next

$$\frac{(\alpha_1)_k \cdots (\alpha_n)_k t^k}{(\beta_1) \cdots (\beta_n)_k} \sum_{m=-\infty}^{\infty} \frac{(\alpha_1 q^k)_m \cdots (\alpha_n q^k)_m t^m}{(\beta_1 q^k)_m \cdots (\beta_n q^k)_m}.$$

The invariance of the three properties is again easily checked, and so our theorem is established. □

We shall now illustrate the applicability of our method by proving some identities of widespread combinatorial interest. In each instance, we easily produce the corresponding q-analogue.

THEOREM 5.3.

$$\sum_{k \geqq 0} \binom{2n+1}{2p+2k+1}\binom{p+k}{k} = \binom{2n-p}{p} 2^{2n-2p}, \tag{5.12}$$

$$\sum_{k \geqq 0} \binom{2n}{2p+2k}\binom{p+k}{k} = \frac{n}{2n-p}\binom{2n-p}{p} 2^{2n-2p}. \tag{5.13}$$

Remark. H. T. Davis [30] attributes these identities to the fictional, arch-criminal James Moriarty of the Sherlock Holmes short story, "The Final Problem". The reference is to Holmes's description of Moriarty in which he says, "At the age of twenty-one he wrote a treatise upon the binomial theorem, which has had a European vogue." Apparently Moriarty had turned to crime even earlier than had been thought previously (except by Gould [34, (15)]); for as we shall see below, these two results are merely disguised forms of Gauss's summation of ${}_2F_1\left[\begin{matrix} a, b; 1 \\ c \end{matrix}\right]$.

Proof. For (5.12), we see that

$$\sum_{k \geqq 0} \binom{2n+1}{2p+2k+1}\binom{p+k}{k}$$

$$= \binom{2n+1}{2p+1} \sum_{k \geqq 0} \frac{[p-n]_k[p-n+\frac{1}{2}]_k[p+1]_k}{[p+1]_k[p+\frac{3}{2}]_k k!} \qquad \text{(by Theorem 5.1)}$$

$$= \binom{2n+1}{2p+1} {}_2F_1\left[\begin{matrix} p-n, p-n+\frac{1}{2}; 1 \\ p+\frac{3}{2} \end{matrix}\right]$$

$$= \binom{2n+1}{2p+1} \frac{[n+1]_{n-p}}{[p+\frac{3}{2}]_{n-p}} \qquad \text{(by Gauss's theorem [52, III. 3 (or III. 4), p. 243])}$$

$$= \frac{2^{2n-2p}[1]_n[\frac{3}{2}]_2(2n-p)![\frac{3}{2}]_p}{(2n-2p)![1]_p[\frac{3}{2}]_p[1]_n[\frac{3}{2}]_n} \qquad \text{(by Theorem 5.1)}$$

$$= \binom{2n-p}{p} 2^{2n-2p}.$$

For (5.13), we see that

$$\sum_{k\geqq 0}\binom{2n}{2p+2k}\binom{p+k}{k}$$

$$=\binom{2n}{2p}\sum_{k\geqq 0}\frac{[p-n]_k[p-n+\frac{1}{2}]_k[p+1]_k}{[p+\frac{1}{2}]_k[p+1]_k k!}\qquad\text{(by Theorem 5.1)}$$

$$=\binom{2n}{2p}{}_2F_1\begin{bmatrix}p-n,p-n+\frac{1}{2};1\\ p+\frac{1}{2}\end{bmatrix}$$

$$=\binom{2n}{2p}\frac{[n]_{n-p}}{[p+\frac{1}{2}]_{n-p}}\qquad\text{(by Gauss's theorem [52, III. 3 (or III. 4), p. 243])}$$

$$=\frac{2^{2n-2p}[1]_n[\frac{3}{2}]_{n-1}(2n-p-1)![\frac{1}{2}]_p}{(2n-2p)![1]_p[\frac{3}{2}]_{p-1}(n-1)![\frac{1}{2}]_n}$$

$$=2^{2n-2p}\binom{2n-p}{p}\frac{n}{n-2p}.\qquad\square$$

If we consider the q-analogue of the above application of Gauss's theorem, we find the following q-analogues of the Moriarty identities.

THEOREM 5.3.

(5.14)
$$\sum_{k\geqq 0}\binom{2n+1}{2p+2k+1}_q\binom{p+k}{k}_{q^2}q^{k(2p+2k+1)}=\binom{2n-p}{p}_{q^2}(-q)_{2n-2p},$$

(5.15)
$$\sum_{k\geqq 0}\binom{2n}{2p+2k}_q\binom{p+k}{k}_{q^2}q^{k(2p+2k-1)}=\binom{2n-p}{p}_{q^2}(-q)_{2n-2p}\frac{(1-q^{2n})}{(1-q^{4n-2p})}.$$

Proof. For (5.14), we see that

$$\sum_{k\geqq 0}\binom{2n+1}{2p+2k+1}_q\binom{p+k}{k}_{q^2}q^{k(2p+2k+1)}$$

$$=\binom{2n+1}{2p+1}_q\sum_{k\geqq 0}\frac{q^{(2n-2p)2k-k(2k-1)}(q^{2p-2n})_{2k}(q^{2p+2};q^2)_k q^{k(2p+2k+1)}}{(q^{2p+2})_{2k}(q^2;q^2)_k}$$

(by Theorem 5.1)

$$=\binom{2n+1}{2p+1}_q\sum_{k\geqq 0}\frac{(q^{2p-2n};q^2)_k(q^{2p-2n+1};q^2)_k(q^{2p+2};q^2)_k q^{4nk-2kp+2k}}{(q^{2p+2};q^2)_k(q^{2p+3};q^2)_k(q^2;q^2)_k}$$

$$=\binom{2n+1}{2p+1}_q{}_2\phi_1\begin{bmatrix}q^{2p-2n},q^{2p-2n+1};q^2,q^{4n-2p+2}\\ q^{2p+3}\end{bmatrix}\qquad(cont.)$$

$$= \begin{pmatrix} 2n+1 \\ 2p+1 \end{pmatrix}_q \frac{(q^{2n+2};q^2)_{n-p}}{(q^{2p+3};q^2)_{n-p}} \quad \text{(by the } q\text{-analogue of Gauss's theorem [52, IV. 2 or IV. 3, p. 247], i.e., our Theorem 2.6)}$$

$$= \frac{(q;q^2)_{n+1}(q^2;q^2)_n(q^2;q^2)_{2n-p}(q;q^2)_{p+1}}{(q;q^2)_{p+1}(q^2;q^2)_p(q)_{2n-2p}(q^2;q^2)_n(q;q^2)_{n+1}}$$

$$= \begin{pmatrix} 2n-p \\ p \end{pmatrix}_{q^2} (-q)_{2n-2p}.$$

For (5.14), we see that

$$\sum_{k\geq 0} \begin{pmatrix} 2h \\ 2p+2k \end{pmatrix}_q \begin{pmatrix} p+k \\ k \end{pmatrix}_{q^2} q^{k(2p+2k-1)}$$

$$= \begin{pmatrix} 2n \\ 2p \end{pmatrix}_q \sum_{k\geq 0} \frac{q^{(2n-2p)2k-k(2k-1)}(q^{2p-2n})_{2k}(q^{2p+2};q^2)_k q^{k(2p+2k-1)}}{(q^{2p+1})_{2k}(q^2;q^2)_k}$$

$$= \begin{pmatrix} 2n \\ 2p \end{pmatrix}_q \sum_{k\geq 0} \frac{(q^{2p-2n+1};q^2)_k(q^{2p-2n};q^2)_k q^{(4n-2p)k}}{(q^2;q^2)_k(q^{2p+1};q^2)_k}$$

$$= \begin{pmatrix} 2n \\ 2p \end{pmatrix}_q {}_2\phi_1\left[\begin{matrix} q^{2p-2n+1}, q^{2p-2n}; q^2, q^{4n-2p} \\ q^{2p+1} \end{matrix}\right]$$

$$= \begin{pmatrix} 2n \\ 2p \end{pmatrix}_q \frac{(q^{2n};q^2)_{n-p}}{(q^{2p+1};q^2)_{n-p}} \quad \text{(by the } q\text{-analogue of Gauss's theorem [52; IV. 2 or IV. 3, p. 247], i.e., our Theorem 2.6)}$$

$$= \frac{(q;q^2)_n(q^2;q^2)_n(q;q^2)_p(q^2;q^2)_{2n-p-1}}{(q;q^2)_p(q^2;q^2)_p(q)_{2n-2p}(q;q^2)_n(q^2;q^2)_{n-1}}$$

$$= \begin{pmatrix} 2n-p \\ p \end{pmatrix}_{q^2} (-q)_{2n-2p}\frac{(1-q^{2n})}{(1-q^{4n-2p})}. \quad \square$$

We next treat a slightly harder problem, namely the Reed Dawson identities [46, p. 71]; these results show that sometimes it is necessary to reverse summation when treating a ${}_2F_1$ (or ${}_2\phi_1$). In this case (as we shall see), the results are corollaries of Gauss's second summation of a ${}_2F_1$.

THEOREM 5.4

$$\sum_{k\geq 0} (-1)^k \binom{n}{k} 2^{-k} \binom{2k}{k} = \begin{cases} 2^{-2v}\binom{2v}{v} & \text{if } n = 2v, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

Proof. First we see that

$$\sum_{k\geq 0} (-1)^k \binom{n}{k} 2^{-k} \binom{2k}{k} = \sum_{k\geq 0} \frac{[-n]_k[1]_k[\frac{1}{2}]_k 2^k}{k!k!k!} \quad \text{(by Theorem 5.1)}$$

$$= {}_2F_1\left[\begin{matrix} -n, \frac{1}{2}; 2 \\ 1 \end{matrix}\right].$$

A check of Slater's Appendix III shows no ${}_2F_1$ series with argument 2; however, since a number have argument $\frac{1}{2}$, we should clearly reverse the order of summation (i.e., replace k by $n-k$). Hence

$$\sum_{k\geqq 0}(-1)^k\binom{n}{k}2^{-k}\binom{2k}{k}$$

$$=\sum_{k\geqq 0}(-1)^{n-k}\binom{n}{k}2^{-n+k}\binom{2n-2k}{n-k}$$

$$=\sum_{k\geqq 0}(-1)^{n-k}\frac{(-n)_k(-1)^k2^{-n+k}(-n)_k(-n)_k(2n)!}{k!n!n!2^{2k}(-n)_k(-n+\frac{1}{2})_k}\qquad\text{(by Theorem 5.1)}$$

$$=(-1)^n2^{-n}\binom{2n}{n}{}_2F_1\left[\begin{matrix}-n,\,-n;\frac{1}{2}\\ -n+\frac{1}{2}\end{matrix}\right]$$

$$=(-1)^n2^{-n}\binom{2n}{n}\frac{\Gamma(\frac{1}{2})\Gamma(-n+\frac{1}{2})}{\Gamma\left(\frac{1}{2}-\frac{n}{2}\right)^2}\qquad\text{(by Gauss's second theorem [52, III. 6, p. 243])}$$

$$=\begin{cases}0 & \text{if } n \text{ is odd (since } \Gamma(z)^{-1} \text{ has zeros when } z \text{ is a negative integer)},\\ 2^{-2v}\binom{4v}{2v}\dfrac{[\frac{1}{2}-v]_v}{[\frac{1}{2}-2v]_v} & \text{if } n=2v \text{ since } [a]_n=\Gamma(a+n)/\Gamma(a)\end{cases}$$

$$=\begin{cases}0 & \text{if } n \text{ is odd},\\ \dfrac{2^{-2v}(4v)!(2v)!2^{2v}(2v)!2^v[\frac{1}{2}]_v}{(2v)!(2v)!2^v v!(4v)!} & \text{if } n=2v\end{cases}$$

$$=\begin{cases}0 & \text{if } n \text{ is odd},\\ 2^{-2v}\binom{2v}{v} & \text{if } n=2v.\quad\square\end{cases}$$

In passing to the q-analogue of the Reed Dawson identities, we encounter the problem of finding the q-analogue of Gauss's second theorem; it is not listed by Slater and has in fact only been discovered recently [11]. Namely,

$$\sum_{k=0}^{\infty}\frac{(a)_k(b)_kq^{k(k+1)/2}}{(q)_k(abq;q^2)_k}=\frac{(-q)_\infty(aq;q^2)_\infty(bq;q^2)_\infty}{(qab;q^2)_\infty}.\tag{5.16}$$

THEOREM 5.5.

$$\sum_{k\geqq 0}(-1)^k\binom{n}{k}_q\frac{1}{(-q)_k}\binom{2k}{k}_q q^{(n-k)(n-k-1)/2}=\begin{cases}0 & \textit{if } n \textit{ is odd}\\ \dfrac{q^{2v^2}}{(-q)_v^2}\binom{2v}{v}_q & \textit{if } n=2v.\end{cases}\tag{5.17}$$

Proof. We begin by replacing k by $n-k$.

Hence

$$\sum_{k\geqq 0} (-1)^{n-k}\binom{n}{k}_q \frac{1}{(-q)_{n-k}}\binom{2n-2k}{n-k}_q q^{k(k-1)/2}$$

$$= (-1)^n \sum_{k\geqq 0} \frac{(q^{-n})_k q^{kn-k(k-1)/2}(q)_{2n-2k} q^{k(k-1)/2}}{(q)_k (q)_{n-k}(q^2;q^2)_{n-k}} \qquad \text{(by Theorem 5.1)}$$

$$= (-1)^n \sum_{k\geqq 0} \frac{(q^{-n})_k q^{kn}(q;q^2)_{n-k}}{(q)_k(q)_{n-k}}$$

$$= \frac{(-1)^n(q;q^2)_n}{(q)_n} \sum_{k\geqq 0} \frac{(q^{-n})_k(q^{-n})_k q^{2kn-k^2/2+k/2-k2n+k^2}}{(q)_k(q^{-2n+1};q^2)_k}$$

$$= \frac{(-1)^n(q;q^2)_n}{(q)_n} \sum_{k\geqq 0} \frac{(q^{-n})_k(q^{-n})_k q^{k(k+1)/2}}{(q)_k(q^{-2n+1};q^2)_k}$$

$$= \frac{(-1)^n(q;q^2)_n(-q)_\infty(q^{-n+1};q^2)_\infty^2}{(q)_n(q^{-2n+1};q^2)_\infty} \qquad \text{(by (5.16), the } q\text{-analogue of Gauss's second theorem)}$$

$$= \begin{cases} 0 \quad \text{if } n \text{ is odd, for then } (q^{-n+1};q^2)_\infty = 0, \\ \dfrac{(q;q^2)_{2\nu}(q^{-2\nu+1};q^2)_\nu^2}{(q)_{2\nu}(q^{-4\nu+1};q^2)_{2\nu}} \quad \text{if } n = 2\nu \end{cases}$$

$$= \begin{cases} 0 \quad \text{if } n \text{ is odd}, \\ \dfrac{q^{2\nu^2}}{(-q)_\nu^2}\dbinom{2\nu}{\nu} \quad \text{if } n = 2\nu. \end{cases}$$

Theorems 5.3–5.5 illustrate well how our method works. As theorems, 5.4 and 5.5 are new; it should be clear that this technique is extremely useful in producing q-analogues. As far as ordinary binomial coefficient identities go, we shall have to wait and see how useful this approach is. There is a great deal of evidence to support our belief that the use of hypergeometric series is useful in such problems. For example, of the first 50 entries in Table 3 of Gould's book [35], over 40 may be proved by our method using only Gauss's first and second theorems, Kummer's theorem, Saalschutz's theorem, and Dixon's theorem [52, III.3, III.6, III.5, III.2, III.8, p. 243]. Another example is the Chinese identity

$$\binom{n+p}{p}^2 = \sum_{k\geqq 0}\binom{p}{k}^2\binom{n+2p-k}{2p}; \tag{5.18}$$

Riordan points out [46, p. 16] that many proofs of (5.18) were given (at least 8). As Carlitz pointed out in [23], this is merely a corollary of Saalschutz's theorem [52, III.2, p. 243]. Similarly the binomial identity obtained by Cartier and Foata in their study of graphs on three vertices [26, Chap. 6] and (5.2) above are both a direct corollary of Saalschutz's theorem.

Since the approach described in this section is like an algorithm for summing series of binomial coefficients, we are preparing a computer program [12] that should greatly simplify work on such problems.

6. Application to physics. There have been several applications of basic hypergeometric series to physics in recent years. M. Baker and D. D. Coon [20] have successfully utilized basic hypergeometric functions in a series of papers (in the *Physical Review* D) on particle physics. Also N. G. van Kampen [54] has applied basic hypergeometric functions to fluctuations in an electric circuit consisting of a condenser and a diode.

We have chosen as an illustration W. Hahn's solution to the equation of motion of a weightless elastic cable that carries n point masses, where both the weights and mutual distances of successive masses decrease in geometric progression with ratio q, $0 < q < 1$. The solution is a q-analogue of the Laguerre polynomials; in fact, if the weights and distances are constant (i.e., $q = 1$) the solution reduces to the ordinary Laguerre polynomials. The classical case $q = 1$ is treated in detail by O. Bottema [22].

We consider n-point masses P_i $(1 \leqq i \leqq n)$ on a weightless elastic cable acting only under the force of gravity. We let k_i denote the distance from P_i to P_{i+1}, m_i the mass of P_i, and $u_i(t) = u_i$ the displacement of the point P_i at time t. We denote the tension in the portion of cable from P_i to P_{i+1} by S_i. The force which acts on m_i in the positive u-direction is then given by

$$-\frac{S_{i-1}}{k_{i-1}}(u_i - u_{i-1}) + \frac{S_i}{k_i}(u_{i+1} - u_i) \qquad (1 \leqq i \leqq n, u_0 = u_{n+1} = 0).$$

Consequently the equations of motion are

$$m_i u_i''(t) = -\frac{S_{i-1}}{k_{i-1}}(u_i - u_{i-1}) + \frac{S_i}{k_i}(u_{i+1} - u_i).$$

One now applies the method of Lagrange, making the substitution

$$u_i = y_i \cos(\sqrt{u}\, t + \varphi),$$

which leads to the following set of linear equations for the y_i:

$$\mu m_1 y_1 = -\frac{S_1}{k_1}(y_2 - y_1) \tag{6.1}$$

$$\mu m_i y_i = \frac{S_{i-1}}{k_{i-1}}(y_i - y_{i-1}) - \frac{S_i}{k_i}(y_{i+1} - y_1) \qquad (2 \leqq i \leqq n). \tag{6.2}$$

If we now assume

$$m_i = q^{i-1} m_1, \qquad k_i = q^{i-1} k, \qquad 0 < q < 1, \tag{6.3}$$

and for simplicity

$$m_1 = g^{-1}, \qquad k_1 = k/(1 - q), \tag{6.4}$$

then utilizing the substitution

$$T_i(\mu) = (-1)^i y_{i+1}(\mu) \prod_{r=1}^{i} \frac{S^r}{k^r m^r}, \qquad T_{-1} = 0, \qquad T_0 = 1,$$

we deduce that

$$T_i(\mu) = \left(\mu - \frac{1-q^{i-1}}{kq^{2i-3}} - \frac{1-q^i}{kq^{2i-2}}\right) T_{i-1}(\mu) - \frac{(1-q^{i-1})^2}{k^2 q^{4i-5}} T_{i-2}(\mu).$$

It is now a simple matter to check (by comparing coefficients of μ^r) that

$$(6.5)\qquad \begin{aligned} T_i(\mu) &= (-1)^i q^{-i(i-1)/2}(q)_i\, {}_2\phi_1\left[\begin{matrix} q^{-i}, 0; q, q^i k\mu \\ q \end{matrix}\right] \\ &= (-1)^i q^{-i(i-1)/2}(q)_i \sum_{r=0}^{i} (-1)^r \frac{k^r \mu^r q^{r(r-1)/2}}{(q)_r} \binom{i}{r}_q . \end{aligned}$$

We note that

$$\lim_{q\to 1} \frac{(-1)^i q^{i(i-1)/2}}{(q)_i} T_i(\mu(1-q)) = \sum_{r=0}^{i} (-1)^r \frac{k^r \mu^r}{r!}\binom{i}{r} = L_i(k\mu),$$

the ith Laguerre polynomial [22, (3), p. 46].

If we now assume that our cable possesses infinitely many point masses subject to (6.3) and (6.4), we shall see that a q-difference equation arises whose solution is $\lim_{i\to\infty}(-1)^i q^{i(i-1)/2} T_i(q)/(q)_i$. The length of our cable is in this case $L = k(1-q)^{-2}$. The rest point is taken to be P_1, and now $S_i = g(m_{i+1} + m_{i+2} + \cdots) = q^i/(1-q)$. Then instead of (6.2) we are led to

$$(6.6)\qquad \mu q^{i-1} y_i = \frac{1}{k}(y_i - y_{i-1}) - \frac{1}{k}(y_{i+1} - y_i).$$

If we write $x = q^{i-1}$ and $V(x) = y_{i-1}$, we see that (6.6) becomes

$$(6.7)\qquad V(q^2 x) + (k\mu x - 2)V(qx) + V(x) = 0.$$

The simplest solution of (6.7) analytic in x at 0 may easily be determined by substituting $V(x) = \sum_{j=0}^{\infty} A_j x^j$ into (6.7). As a result, we find

$$(6.8)\qquad A_j(q^j - 1)^2 = -k\mu q^{j-1} A_{j-1},$$

or

$$(6.9)\qquad V(x) = A_0 \sum_{r=0}^{\infty} \frac{(-1)^r q^{r(r-1)/2}(\mu k x)^r}{(q)_r^2},$$

a q-analogue of the Bessel function

$$J_0(2\sqrt{\mu k x}) = \sum_{r=0}^{\infty} \frac{(-1)^r(\mu k x)^r}{(r!)^2}.$$

By the above simple example, we hope to make clear that many problems related to arithmetic differences have q-analogues in which geometric differences replace arithmetic differences. Just as solutions in the arithmetic case many times are hypergeometric series, so are solutions in the geometric case basic hypergeometric series.

7. Conclusion. As was stated in the Introduction, we have in this paper only briefly sampled several areas of mathematics in which basic hypergeometric functions play an important role. Not only are there many further applications in each of the areas we have mentioned, but there are many unexplored possibilities for applications still open. I would hope that the variety and novelty of the results already found might provide inspiration for further research. The interesting work of R. P. Agarwal and his students [2], [3] on bi-basic hypergeometric series should also yield interesting applications in the future.

The following list of references is by no means exhaustive; in fact, we only list those papers cited in the text. The reader is referred to [1], [2], [4], [9], [15], [28], [35], [36] and [52] for much more extensive bibliographies.

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