ON BASIC HYPERGEOMETRIC SERIES, MOCK THETA FUNCTIONS, AND PARTITIONS (II)

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1. Introduction

IN this paper we shall examine further the consequences of a theory of certain basic hypergeometric series identities studied in (1). We shall apply the results of (1) to problems in additive number theory and in the theory of q-series.

In § 2 we derive two more identities in general basic hypergeometric series which will be useful to us in further developments. In § 3 new formulae for the family of third-order mock theta functions of (1) are found; several of the results of this section have been proved in special cases by N. J. Fine (3). In § 4 we prove several partition theorems; three of these theorems have been stated by N. J. Fine in (2) and proved by him in (3). Our theory allows us to give new proofs of Fine's results as well as other partition theorems. In § 5 we obtain new q-series identities for some of the false theta functions studied by L. J. Rogers [(7) 332–5]. Finally, in § 6 we prove some results analogous to the Rogers–Ramanujan identities by combining our theory with some formulae due to L. J. Slater (8).

At this point, I would like to thank N. J. Fine for allowing me to include several of his unpublished theorems in this work. I was motivated to find and prove 4a–4f of § 3 only after seeing Fine's corresponding identities for the original third-order mock theta functions.

2. Further theorems of basic hypergeometric type

In (1) six theorems of basic hypergeometric type were proved. In this section we prove two more such theorems.

THEOREM 7.
$$\sum_{n=0}^{\infty} \frac{\Pi_n(-a,q)\,\Pi_n(-b,q^2)}{\Pi_n(-q,q)\,\Pi_n(-atb,q^2)}\,t^n$$
$$= \frac{\Pi_\infty(-at,q^2)\,\Pi_\infty(-bt,q^2)}{\Pi_\infty(-t,q^2)\,\Pi_\infty(-atb,q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-a,q^2)\,\Pi_m(-b,q^2)}{\Pi_m(-q^2,q^2)\,\Pi_m(-bt,q^2)}\,(tq)^m.$$

In the notation of basic hypergeometric series, the theorem becomes

$$ {}_3\Phi_2\begin{bmatrix} a, b^{\frac12}, -b^{\frac12}; t; q \\ (atb)^{\frac12}, -(atb)^{\frac12} \end{bmatrix} = \frac{\Pi_\infty(-at, q^2)\,\Pi_\infty(-bt, q^2)}{\Pi_\infty(-t, q^2)\,\Pi_\infty(-atb, q^2)}\, {}_2\Phi_1\begin{bmatrix} a, b; tq; q^2 \\ bt \end{bmatrix}. $$

Proof.

$$ \sum_{n=0}^{\infty} \frac{\Pi_n(-a, q)\,\Pi_n(-b, q^2)}{\Pi_n(-q, q)\,\Pi_n(-atb, q^2)} t^n $$

$$ = \frac{\Pi_\infty(-at, q)\,\Pi_\infty(-b, q^2)}{\Pi_\infty(-t, q)\,\Pi_\infty(-atb, q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-at, q^2)\,\Pi_{2m}(-t, q)}{\Pi_m(-q^2, q^2)\,\Pi_{2m}(-at, q)} b^m $$

(by Theorem A3 of (1), interchange b with t, then replace a with at and replace c with at)

$$ = \frac{\Pi_\infty(-at, q)\,\Pi_\infty(-b, q^2)}{\Pi_\infty(-t, q)\,\Pi_\infty(-atb, q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-t, q^2)\,\Pi_m(-tq, q^2)}{\Pi_m(-q^2, q^2)\,\Pi_m(-atq, q^2)} b^m $$

$$ = \frac{\Pi_\infty(-at, q)\,\Pi_\infty(-b, q^2)\,\Pi_\infty(-tq, q^2)\,\Pi_\infty(-tb, q^2)}{\Pi_\infty(-t, q)\,\Pi_\infty(-atb, q^2)\,\Pi_\infty(-atq, q^2)\,\Pi_\infty(-b, q^2)} \times $$

$$ \times \sum_{m=0}^{\infty} \frac{\Pi_m(-a, q^2)\,\Pi_m(-b, q^2)}{\Pi_m(-q^2, q^2)\,\Pi_m(-bt, q^2)} (tq)^m $$

by Theorem A_2 of (1).

Simplifying this last expression, we obtain the desired result.

The next result (which will be needed later) [(6); 171 equation (2)] is proved here only to show that it falls within the scope of our theory.

THEOREM 8.

$$ \sum_{n=0}^{\infty} \frac{\Pi_n(-a, q^2)\,\Pi_n(-b, q^2)}{\Pi_n(-q^2, q^2)\,\Pi_n(-c, q^2)} t^n $$

$$ = \frac{\Pi_\infty(-bt, q^2)\,\Pi_\infty(-c/b, q^2)}{\Pi_\infty(-t, q^2)\,\Pi_\infty(-c, q^2)} \sum_{n=0}^{\infty} \frac{\Pi_n(-b, q^2)\,\Pi_n(-abt/c, q^2)}{\Pi_n(-q^2, q^2)\,\Pi_n(-bt, q^2)} (c/b)^n. $$

In the notation of basic hypergeometric series the theorem becomes

$$ {}_2\Phi_1\begin{bmatrix} a, b; t; q^2 \\ c \end{bmatrix} = \frac{\Pi_\infty(-bt, q^2)\,\Pi_\infty(-c/b, q^2)}{\Pi_\infty(-t, q^2)\,\Pi_\infty(-c, q^2)}\, {}_2\Phi_1\begin{bmatrix} b, abt/c; c/b; q^2 \\ bt \end{bmatrix}. $$

Proof.

$$ \sum_{n=0}^{\infty} \frac{\Pi_n(-a, q^2)\,\Pi_n(-b, q^2)}{\Pi_n(-q^2, q^2)\,\Pi_n(-c, q^2)} t^n $$

$$ = \frac{\Pi_\infty(-b, q^2)\,\Pi_\infty(-at, q^2)}{\Pi_\infty(-c, q^2)\,\Pi_\infty(-t, q^2)} \sum_{m=0}^{\infty} \frac{\Pi_m(-t, q^2)\,\Pi_m(-c/b, q^2)}{\Pi_m(-q^2, q^2)\,\Pi_m(-at, q^2)} b^m $$

by Theorem A_2 of (1)

$$= \frac{\Pi_\infty(-b,q^2)\,\Pi_\infty(-at,q^2)\,\Pi_\infty(-c/b,q^2)\,\Pi_\infty(-tb,q^2)}{\Pi_\infty(-c,q^2)\,\Pi_\infty(-t,q^2)\,\Pi_\infty(-at,q^2)\,\Pi_\infty(-b,q^2)} \times$$

$$\times \sum_{n=0}^{\infty} \frac{\Pi_n(-b,q^2)\,\Pi_n(-abt/c,q^2)}{\Pi_n(-q^2,q^2)\,\Pi_n(-bt,q^2)} (c/b)^n$$

by Theorem A_2 of [1]

Simplifying this last expression, we obtain the desired result.

COROLLARY 1. $$\sum_{m=0}^{\infty} \frac{q^{m^2} x^m}{\Pi_{m+1}(-s,q^2)} = \sum_{m=0}^{\infty} \Pi_m(xq/s,q^2)s^m \quad (|s| < 1).$$

Proof. In Theorem 8 take

$$t = \tau x, \quad a = -q/\tau, \quad b = q^2, \quad c = sq^2,$$

and let $\tau \to 0$. This yields our result.

Replacing each of x, s and q by x^3 in Corollary 1, we obtain, upon multiplication of both sides by x and addition of 1 to both sides, that

$$1 + \sum_{m=0}^{\infty} \frac{x^{3m^2+3m+1}}{\Pi_{m+1}(-x^3,x^6)} = 1 + \sum_{m=0}^{\infty} \Pi_m(x^3,x^6)x^{3m+1}.$$

This identity was proved combinatorially by MacMahon [(5) 260–1].

3. The mock theta functions

In this section we shall further study the functions $f(\alpha;q)$, $\phi(\alpha;q)$, $\psi(\alpha;q)$, $\upsilon(\alpha;q)$, $\omega(\alpha;q)$ defined in (1). We shall derive new expressions for these functions which we shall use in studying partitions.

(4a) $$\phi(\alpha;q) = 1 + q\sum_{m=0}^{\infty} \Pi_m(-q^2\alpha^{-1},q^2)(-\alpha q)^m$$

(set $s = -\alpha q$, $x = q^2$ in Corollary 1 of Theorem 8),

(4b) $$\phi(-\alpha q;-q) = (1+\alpha)\sum_{m=0}^{\infty} \Pi_m(\alpha^{-1}q,q^2)(-\alpha)^m$$

(set $s = -\alpha$, $x = -1$ in Corollary 1 of Theorem 8),

(4c) $$\psi(-\alpha q;-q) = -q\sum_{m=0}^{\infty} (-1)^m\alpha^m q^m \Pi_m(\alpha^{-1}q^2;q^2)$$

(by 4a since $\psi(-\alpha q;-q) = \phi(+\alpha;-q) - 1$),

(4d) $$\upsilon(\alpha;q) = \sum_{m=0}^{\infty} \Pi_m(-q^3\alpha^{-2};q^2)(-\alpha^2q^{-1})^m$$

(set $s = -\alpha^2q^{-1}$ and $x = q$ in Corollary 1 of Theorem 8),

(4e) $$f(\alpha q;q) = (1+\alpha)\sum_{n=0}^{\infty} \frac{(-1)^n\alpha^n}{\Pi_n(q,q)}.$$

This identity is slightly more difficult. In Theorem I of (1), we set $c = 0$, $b = q$, $t = -\alpha$. This yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^n}{\Pi_n(q,q)} = \sum_{m=0}^{\infty} \Pi_m(q\alpha^{-1}, q^2)(-\alpha)^m + q \sum_{m=0}^{\infty} \Pi_m(q^2\alpha^{-1}, q^2)(-\alpha q)^m$$

$$= (1+\alpha)^{-1}\phi(-\alpha q; -q) - \psi(-\alpha q; -q) \quad \text{(by (4}b\text{) and (4}c\text{))}.$$

Therefore,

$$(1+\alpha) \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^n}{\Pi_n(q,q)} = \phi(-\alpha q; -q) - (1+\alpha)\psi(-\alpha q; -q)$$

$$= f(\alpha q; q) \quad \text{(by (3}a\text{) of (1))}.$$

$$(4f) \qquad \omega(\alpha^2; q^2) = \sum_{m=0}^{\infty} \frac{(\alpha^2 q^{-1})^{2m}}{\Pi_{m+1}(-q^2, q^4)}.$$

In Theorem 2 of (1), we replace q throughout by q^2, then we set $t = \alpha^2 q^{-1}$, $b = q^2$, $c = 0$. This yields

$$\sum_{n=0}^{\infty} \Pi_n(q^2\alpha^{-2}, q^2)(\alpha^2 q^{-1})^n$$

$$= \frac{\Pi_\infty(-q^4, q^4)}{\Pi_\infty(-q^2, q^4)} \sum_{m=0}^{\infty} \frac{(\alpha^2 q^{-1})^{2m}}{\Pi_m(-q^4, q^4)} + \sum_{m=0}^{\infty} \frac{(\alpha^2 q^{-1})^{2m+1}}{\Pi_{m+1}(-q^2, q^4)}.$$

Thus

$$\nu(-\alpha; -q) = \tfrac{1}{2} q^{-\frac{1}{2}} \vartheta_2(0, q)[\Pi_\infty(-\alpha^4 q^{-2}, q^4)]^{-1} +$$

$$+ (\alpha^2 q^{-1}) \sum_{m=0}^{\infty} \frac{(\alpha^2 q^{-1})^{2m}}{\Pi_{m+1}(-q^2, q^4)}$$

by (4d) and identity (E) of (1).

But

$$\nu(-\alpha; -q) = \tfrac{1}{2} q^{-\frac{1}{2}} \vartheta_2(0, q)[\Pi_\infty(-\alpha^4 q^{-2}, q^4)]^{-1} + \alpha^2 q^{-1} \omega(\alpha^2; q^2)$$

by (3c) of (1).

Hence we deduce (4f).

The formulae (4a) through (4f) have all been found for the original mock theta functions by Fine (3) using a different method.

4. Partition theorems

We start our study of partitions with a generalization of the rank of a partition [(4) 290].

DEFINITION. *The (a, b)-rank of a partition is defined to be a times the largest summand minus b times the number of summands.*

For example, if we consider the partition $1+1+4+7+8$ of 21, the $(4,3)$-rank of this partition is $4\times 8-3\times 5 = 17$. The $(1,1)$-rank of a partition is just the original rank. In an unpublished monograph (3), N. J. Fine proves two generalizations of Euler's famous theorem that the number of partitions of n into distinct parts equals the number of partitions of n into odd parts. Since Fine's results follow naturally from our theory we include them.

THEOREM F_1 (Fine (3)). *The number of partitions of N into distinct parts with maximal part M equals the number of partitions of N into odd parts with $(1,-2)$-rank equal to $2M+1$.*

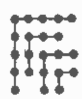
I give two proofs of this theorem.

First Proof. In Theorem 7, we take $t = \tau q$, $a = 0$, and $b = q^2$. This yields, upon multiplication by τq,

$$\sum_{n=0}^{\infty} \Pi_n(q,q)q^{n+1}\tau^{n+1} = \sum_{m=0}^{\infty} \frac{\tau^{m+1}q^{2m+1}}{\Pi_{m+1}(-\tau q, q^2)}.$$

Expanding both sides and comparing coefficients of $\tau^M q^N$ yields the desired result. This q-series identity was originally proved by Fine using a different method.

Second Proof. One of Sylvester's graph theoretic proofs of Euler's theorem can be modified to prove Fine's theorem [(5) 13]. Sylvester sets up a one-to-one correspondence between the two classes of partitions in the following way. The partitions into odd parts (e.g. $22 = 9+5+5+3$) he writes graphically as



He then reads the partition as

(i.e. $22 = 8+7+4+2+1$). This procedure is easily shown to establish the desired correspondence. Upon inspection, we see that each partition into distinct parts with maximal part M corresponds to a partition into odd parts with maximal part M' and number of parts ν such that $M = \frac{1}{2}(M'-1)+\nu$, or $2M+1 = M'+2\nu$; since $M'+2\nu$ is the $(1, -2)$-rank of the partition into odd parts, we see that Fine's theorem is again established.

We now prove a second refinement of Euler's theorem stated by Fine in (2) and proved in (3). This result is seen to be closely related to the mock theta function identity (3*c*) of (1).

THEOREM F_2 (Fine (3)). *If $U_m(n)$ denotes the number of partitions of n into odd parts with maximal part equal to m and $D_m(n)$ denotes the number of partitions of n into distinct parts with $(1, 1)$-rank equal to m, then*

$$U_{2r+1}(n) = D_{2r+1}(n)+D_{2r}(n).$$

Proof. From (3*c*) of (1) we deduce

$$v(-\alpha^{\frac{1}{2}}q^{\frac{1}{2}}; -q^{\frac{1}{2}})-v(\alpha^{\frac{1}{2}}q^{\frac{1}{2}}; q^{\frac{1}{2}}) = \alpha q\omega(\alpha q^{2}; q).$$

Thus, by (4*d*) and (4*f*),

$$\sum_{m=0}^{\infty} \Pi_m(\alpha^{-1}, q)\alpha^m q^m - \sum_{m=0}^{\infty} \Pi_m(-1)^m(-\alpha^{-1}, q)\alpha^m q^m$$
$$= 2\sum_{m=0}^{\infty} \frac{(\alpha q)^{2m+1}}{\Pi_{m+1}(-q, q^2)}.$$

After expanding both sides, we see that the coefficient of $\alpha^{2r+1}q^n$ on the right-hand side of the equation is $U_{2r+1}(n)$, and the coefficient of $\alpha^{2r+1}q^n$ on the left-hand side of the equation is $D_{2r+1}(n)+D_{2r}(n)$.

After stating this last theorem in (2) Fine announced the following double identity and deduced several results on partitions from it.

THEOREM F_3 (Fine (3)).

$$1-\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{\Pi_{n+1}(q, q^2)} = 1-\sum_{n=0}^{\infty} (-1)^n \Pi_n(q, q)q^{n+1}$$
$$= 1+\sum_{n=1}^{\infty} (-1)^n q^{\frac{1}{2}n(3n-1)}(1-q^n).$$

Proof. The first part of the identity is obtained by putting $\tau = -1$ in the q-series identity of Theorem F_1.

In the Corollary of Theorem 8, we replace q by $q^{\frac{1}{2}}$, then set $s = -q$, $x = -q^{\frac{1}{2}}$ and obtain

$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{1}{2}m(m+3)}}{\Pi_{m+1}(q, q)} = \sum_{m=0}^{\infty} (-1)^m \Pi_m(q, q)q^m.$$

Hence
$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{1}{2}m(m+1)}}{\Pi_m(q,q)} = 1 - \sum_{m=0}^{\infty} (-1)^m \Pi_m(q,q) q^{m+1}.$$

Now L. J. Rogers has proved [(7) 333, equation (6)] that
$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{1}{2}m(m+1)}}{\Pi_m(q,q)} = 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{1}{2}n(3n-1)}(1-q^n).$$

From this the second part of the double identity follows.

For our next theorem, we need the following definitions.

DEFINITION. *By $P_a(N, M)$ we shall denote the number of partitions of N into parts of which the largest is at most M and such that the number of parts is congruent to a* (mod 2). *For $P_a(N, N)$, we write $P_a(N)$.*

DEFINITION. *By $Q(N, M)$, we shall denote the number of partitions of N into distinct odd parts such that the* (1, 2)*-rank of the partition is less than or equal to $2M-1$. Partitions of this type will be referred to as Q_{MN}-partitions. For $Q(N, N)$, we write $Q(N)$; $Q(N)$ thus denotes the number of partitions of N into distinct odd parts.*

DEFINITION. *By $R(N, M)$, we shall denote the number of partitions of N with unique largest part such that every other part occurs exactly twice and such that the* (2, 1)*-rank of the partition is equal to $2M+1$. Partitions of this type will be referred to as R_{MN}-partitions.*

If we consider Euler's identity written as
$$\Pi_\infty(-q, q^2) = \{\Pi_\infty(q,q)\}^{-1},$$
we obtain
$$\sum_{n=0}^{\infty} (-1)^n Q(n) q^n = \sum_{n=0}^{\infty} (P_0(n) - P_1(n)) q^n.$$
Hence
$$(-1)^n Q(n) = P_0(n) - P_1(n).$$

We shall prove a refinement of this result. Since our theorem is completely new, we include most of the details in the proof.

THEOREM. $P_0(N, M) - P_1(N, M) = (-1)^N (Q(N, M) - (-1)^M R(N, M))$.

Proof. Just as the second theorem of this section hinged on one of the generalized mock theta function identities of (1) so also does this theorem.

From identity (3*a*) of (1) we have
$$f(\alpha q; q) = \phi(-\alpha q; -q) - (1+\alpha)\psi(-\alpha q; -q).$$
Hence by (4*b*), (4*c*), and (4*f*), we have
$$\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^n}{\Pi_n(q,q)} = \sum_{m=0}^{\infty} \Pi_m(\alpha^{-1}q, q^2)(-\alpha)^m + q \sum_{m=0}^{\infty} (-1)^m \Pi_m(\alpha^{-1}q^2, q^2) \alpha^m q^m.$$

Now

$$\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^n}{\Pi_n(q,q)} = \sum_{n=0}^{\infty} \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (-1)^{n+k_1+\ldots+k_n} \alpha^n q^{k_1.1+\ldots+k_n.n}$$
$$= \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} (-1)^M (P_0(N,M)-P_1(N,M))\alpha^M q^N.$$

Also

$$\sum_{m=0}^{\infty} \Pi_m(\alpha^{-1}q, q^2)(-\alpha)^m = \sum_{m=0}^{\infty} \sum_{k_1=0}^{1} \cdots \sum_{k_m=0}^{1} (-1)^m \alpha^{m-k_1-\ldots-k_m} \times$$
$$\times q^{k_1.1+\ldots+k_m(2m-1)}$$
$$= \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \left(\sum_{\substack{2m-1-2k_1-\ldots-2k_m=2M-1 \\ k_1.1+\ldots+k_m(2m-1)=N}} (-1)^{k_1+\ldots+k_m} \right) q^N \alpha^M (-1)^M.$$

The finite sum in this last expression gives the excess of Q_{MN}-partitions with an even number of parts over Q_{MN}-partitions with an odd number of parts. Since the number of parts of a Q_{MN}-partition must be even if N is even and odd if N is odd, we see that the finite sum we are considering is $(-1)^N Q(M,N)$.

Finally,

$$\sum_{m=0}^{\infty} (-1)^m \Pi_m(\alpha^{-1}q^2, q^2)\alpha^m q^{m+1}$$
$$= \sum_{m=0}^{\infty} \sum_{k_1=0}^{1} \cdots \sum_{k_m=0}^{1} (-1)^m \alpha^{m-k_1-\ldots-k_m} q^{m+1+2k_m.m+\ldots+2k_1.1}$$
$$= \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \left(\sum_{\substack{2(m+1)-2k_1-\ldots-2k_m-1=2M+1 \\ m+1+2k_m.m+\ldots+2k_1.1=N}} (-1)^m \right) \alpha^M q^N.$$

The finite sum considered here gives the excess of R_{MN}-partitions with maximal part odd over R_{MN}-partitions with maximal part even. Since the maximal part of an R_{MN}-partition must be even if N is even and odd if N is odd, we see that the finite sum we are considering is

$$(-1)^{N+1} R(N,M).$$

Whence comparing coefficients of $(-1)^M \alpha^M q^N$ in our original identity we obtain

$$P_0(N,M)-P_1(N,M) = (-1)^N Q(N,M)+(-1)^{M+N+1} R(N,M)$$
$$= (-1)^N (Q(N,M)-(-1)^M R(N,M)).$$

5. False theta functions

The false theta functions were originally defined and studied by L. J. Rogers [(7) 328]. We shall show that for every integer $r \geq 0$ there exists a polynomial $Q_r(q)$ of degree at most $2r$ such that

$$f(q^{2r+1}; q^2) = (-1)^{r-1}\,\Pi_r(q, q^2) \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(3n-1)}(1-q^n) + Q_r(q). \tag{5a}$$

The sum on the right-hand side of this equation is a false theta function. From (3*a*) and (3*b*) of (1) it is easy, in virtue of (5*a*), to deduce false theta identities for $\phi(-q^{2r+1}; -q^2)$ and $\psi(-q^{2r+1}; -q^2)$. Our proof of (5*a*) involves the false theta identity of Rogers used to establish Theorem F_3.

We first treat $r = 0$.

$$f(q; q^2) = \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{\Pi_{2n}(q,q)} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(1+(-1)^n)q^{\frac{1}{2}n(n-1)}}{\Pi_n(q,q)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{\Pi_n(q,q)} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{\Pi_n(q,q)}.$$

Now

$$\sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{\Pi_n(q,q)} + \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}}{\Pi_n(q,q)} = \sum_{n=0}^{\infty} \frac{(1+q^n)q^{\frac{1}{2}n(n-1)}}{\Pi_n(q,q)}$$

$$= 2 + \sum_{n=1}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{\Pi_{n-1}(q,q)} = 2 + \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}}{\Pi_n(q,q)}.$$

Hence

$$\sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{\Pi_n(q,q)} = 2.$$

Also

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{\Pi_n(q,q)} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{\Pi_n(q,q)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n-1)}(1+q^n)}{\Pi_n(q,q)}$$

$$= 2 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{\Pi_{n-1}(q,q)} = 2 - \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{\Pi_n(q,q)}.$$

Therefore

$$f(q; q^2) = 2 - \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{\Pi_n(q,q)}$$

$$= 1 - \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(3n-1)}(1-q^n) \quad [(7)\ 333,\ \text{equation (6)}].$$

Thus (5*a*) holds for $r = 0$ with $Q_0(q) = 1$.

Assume now that (5a) is true for a given $r \geqslant 0$. Then

$$f(q^{2r+3}; q^2)$$

$$(1+q^{2r+1})(2-f(q^{2r+1}; q^2)) \quad \text{(by Lemma 3.2 of (1))}$$

$$= (1+q^{2r+1})\Big(2-(-1)^{r-1}\Pi_r(q,q^2)\sum_{n=0}^{\infty}(-1)^n q^{\frac{1}{2}n(3n-1)}(1-q^n)-Q_r(q)\Big)$$

$$= (-1)^r\,\Pi_{r+1}(q,q^2)\sum_{n=0}^{\infty}(-1)^n q^{\frac{1}{2}n(3n-1)}(1-q^n)+Q_{r+1}(q),$$

where $$Q_{r+1}(q) = (1+q^{2r+1})(2-Q_r(q)).$$

Thus (5a) is completely established.

6. Further identities of the Rogers-Ramanujan-Slater type

In (8), L. J. Slater proved 130 formulae similar to the Rogers-Ramanujan identities. All of her results relate expressions involving infinite products to q-series. The results that we have derived for basic hypergeometric series allow us to prove further identities related to those of Slater. We prove the following three identities although probably still others may be obtained.

(6a) $$\sum_{m=0}^{\infty}\frac{(-1)^m q^{2m^2}\,\Pi_m(-q^2,q^4)}{\Pi_m(-q^8,q^8)}$$

$$= [\Pi_\infty(-q,q^2)]^2\,\Pi_\infty(-q^2,q^4)J(q)-q[\Pi_\infty(q^2,q^4)]^2K(-q^2),$$

where $$J(q) = \frac{\Pi_\infty(q^5,q^6)\,\Pi_\infty(q,q^6)\,\Pi_\infty(-q^6,q^6)}{\Pi_\infty(-q^3,q^3)},$$

$$K(q) = \frac{\Pi_\infty(-q^{10},q^{12})\,\Pi_\infty(-q^2,q^{12})\,\Pi_\infty(-q^{12},q^{12})}{\Pi_\infty(-q,q)}.$$

(6b) $$\sum_{m=0}^{\infty}\frac{q^{4m^2}\,\Pi_m(1,q^4)}{\Pi_{2m}(-q^2,q^2)\,\Pi_m(q^2,q^4)} = \frac{L(q)+L(-q)}{2[\Pi_\infty(q^2,q^4)]^2}$$

(6c) $$\sum_{m=0}^{\infty}\frac{q^{4m^2+4m}\,\Pi_m(q^2,q^4)}{\Pi_{2m+1}(-q^2,q^2)\,\Pi_m(q^4,q^4)} = \frac{L(-q)-L(q)}{2[\Pi_\infty(q^4,q^4)]^2},$$

where $$L(q) = \frac{\Pi_\infty(-q,q^3)\,\Pi_\infty(q^2,-q^3)\,\Pi_\infty(-q,q^3)\,\Pi_\infty(q^2,-q^3)}{\Pi_\infty(-q^4,q^4)}.$$

To obtain (6*a*) we replace q by q^2 in Theorem 1 of (1), then we set $b = -q$, $c = q^3$ and let $t \to 0$. This yields

$$\sum_{n=0}^{\infty} \frac{\Pi_n(q, q^2) q^{2n^2+2n}}{\Pi_n(-q^4, q^4)\, \Pi_n(-q^3, q^2)}$$

$$= \frac{[\Pi_\infty(q^4, q^4)]^2}{\Pi_\infty(-q^3, q^2)\, \Pi_\infty(-q^2, q^4)\, \Pi_\infty(-q, q^2)} \sum_{m=0}^{\infty} \frac{(-1)^m q^{2m^2}\, \Pi_m(-q^2, q^4)}{\Pi_m(-q^4, q^4)\, \Pi_m(q^4, q^4)} -$$

$$- \frac{q\, \Pi_\infty(q^2, q^4)\, \Pi_\infty(q^6, q^4)}{\Pi_\infty(-q^3, q^2)\, \Pi_\infty(-q^2, q^4)\, \Pi_\infty(-q, q^2)} \sum_{m=0}^{\infty} \frac{(-1)^m q^{2m^2+4m}\, \Pi_m(-q^2, q^4)}{\Pi_m(-q^4, q^4)\, \Pi_m(q^6, q^4)}.$$

Slater has proved [(8) 154, equation (27)] that the sum on the left-hand side of this identity is $(1-q)J(q)$ and that the second sum on the right-hand side is $K(-q^2)$ [(8) 157, equation (50)]. Simplifying we obtain (6*a*).

We obtain (6*b*) and (6*c*) together. In Theorem 2 of (1), we replace q by q^2 throughout and then set $b = -1$, $c = q$. This yields

$$\sum_{n=0}^{\infty} \frac{\Pi_n(1, q^2) q^{n(n+1)}}{\Pi_{2n}(-q, q)}$$

$$= \Pi_\infty(q^2, q^2) \left[\frac{\Pi_\infty(q^2, q^4)}{\Pi_\infty(-q, q^2)}\right]^2 \sum_{m=0}^{\infty} \frac{q^{4m^2}\, \Pi_m(1, q^4)}{\Pi_{2m}(-q^2, q^2)\, \Pi_m(q^2, q^4)} -$$

$$-2q\, \Pi_\infty(q^2, q^2) \left[\frac{\Pi_\infty(q^4, q^4)}{\Pi_\infty(-q, q^2)}\right]^2 \sum_{m=0}^{\infty} \frac{q^{4m^2+4m}\, \Pi_m(q^2, q^4)}{\Pi_{2m+1}(-q^2, q^2)\, \Pi_m(q^4, q^4)}.$$

Slater has proved [(8) 156, equation (48)] that the sum on the left-hand side of this identity is

$$\frac{\Pi_\infty(q^2, q^2)}{[\Pi_\infty(-q, q^2)]^2} L(q).$$

Utilizing this formula to simplify our result, we obtain (6*b*) and (6*c*).

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