q-IDENTITIES OF AULUCK, CARLITZ, AND ROGERS

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1. Introduction. In [1] and [2], a large number of q-identities have been deduced from a single identity of basic hypergeometric type. In particular, most of the third order mock theta function identities [14; 63], all of the fifth order mock theta function identities [15; 277–279], several of the identities given by Fine in [7], and Heine's original transformation of basic hypergeometric series [8; 106] are all deducible from the Fundamental Lemma of [1]. The object of this paper is to show that many other q-identities which have not seemed to fit into any of the general theorems on basic hypergeometric series (c.f. Sears [11], [12], or Slater [13]) are actually deducible from the Fundamental Lemma of [1]. We shall utilize the following notation

$$\prod_m (x, q) = \prod_{j=0}^{m-1} (1 + xq^j)$$

$$\prod_\infty (x, q) = \prod_{j=0}^{\infty} (1 + xq^j).$$

We shall prove

$$\text{(A1)} \qquad \sum_{m=0}^{\infty} \frac{q^{(m+1)^2}}{\prod_m (-q, q) \prod_{m+1} (-q, q)} = q[\prod_\infty (-q, q)]^{-1} \sum_{p=0}^{\infty} (-1)^p q^{\frac{1}{2}p(p+3)},$$

$$\text{(A2)} \qquad \sum_{m=0}^{\infty} \frac{q^{m+1}}{\prod_m (-q, q) \prod_{m+1} (-q, q)} = q[\prod_\infty (-q, q)]^{-2} \sum_{p=0}^{\infty} (-1)^p q^{\frac{1}{2}p(p+3)},$$

$$\text{(A3)} \qquad \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}(m+1)(m+2)}}{\prod_m (-q, q) \prod_{m+1} (-q, q)} = q[\prod_\infty (-q, q)]^{-1} \sum_{m=0}^{\infty} \frac{q^{2m^2+3m}}{\prod_m (-q^2, q^2)}$$

$$\text{(C1)} \qquad \sum_{s=0}^{\infty} \frac{\prod_s (-b, q) x^s}{\prod_s (-q, q) \prod_s (-a, q)} = \frac{\prod_\infty (-bx, q)}{\prod_\infty (-a, q) \prod_\infty (-x, q)} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n-1)} a^n \prod_n (-x, q)}{\prod_n (-q, q) \prod_n (-bx, q)}$$

$$\text{(C2)} \qquad \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)} \prod_n (-a, q)}{\prod_n (-q, q)} = \prod_\infty (q, q) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} a^n}{\prod_n (-q^2, q^2)}$$

$$\text{(C3)} \qquad \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} \prod_n (-a, q^2)}{\prod_n (-q^2, q^2)} = \prod_\infty (-q, q^2) \sum_{n=0}^{\infty} \frac{q^{2n^2-n} a^n}{\prod_{2n} (-q, q)}$$

$$\text{(R1)} \qquad \sum_{n=0}^{\infty} \frac{q^{4n^2} z^{2n}}{\prod_n (-q^4, q^4)} = \prod_\infty (-zq, q^2) \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{\prod_n (-q^2, q^2) \prod_n (-zq, q^2)}$$

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$$\text{(R2)}\qquad \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{\prod_n (-q, q)} = \prod_\infty (zq^2, q^2) \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{\prod_n (-q^2, q^2) \prod_n (zq^2, q^2)}.$$

Identities (A1), (A2), and (A3) were proved by Auluck [3; 681 Equation (11); 683 Equation (20); 685 Equation (28)]. (C1) was proved by Carlitz in [5]; (C2) and (C3) were posed as problems by Carlitz [6] in the American Mathematical Monthly (actually there was a slight error in the original statement for (C3) which has been corrected here). (R1) and (R2) were originally proved in a slightly different form in L. J. Rogers's famous paper of 1894 [10; 330, Equations (2), (3), and (4)]. Also (R1) and (R2) were proved by Watson in [16; 46–47] while special cases of (R1) and (R2) had already been proved by Watson in [14; 59–60] and [15; 275].

We shall prove generalizations of all of these identities in §3. In §2, we shall show that all of the above identities are special cases of our more general identities.

2. Derivation of original identities. This section is devoted to deducing the identities given in §1 from the following six identities.

$$\text{(I1)}\qquad \sum_{n=0}^{\infty} \frac{\prod_n (-\alpha, q) \prod_n (-\beta, q) \tau^n}{\prod_n (-q, q) \prod_n (-\gamma, q)} = \frac{\prod_\infty (-\beta, q) \prod_\infty (-\alpha\tau, q)}{\prod_\infty (-\gamma, q) \prod_\infty (-\tau, q)} \sum_{m=0}^{\infty} \frac{\prod_m (-\gamma/\beta, q) \prod_m (-\tau, q) \beta^m}{\prod_m (-q, q) \prod_m (-\alpha\tau, q)}$$

$$\text{(I2)}\qquad \sum_{n=0}^{\infty} \frac{\prod_n (-\alpha, q) \prod_n (-\beta, q) \tau^n}{\prod_n (-q, q) \prod_n (-\gamma, q)} = \frac{\prod_\infty (-\beta\tau, q) \prod_\infty (-\gamma/\beta, q)}{\prod_\infty (-\tau, q) \prod_\infty (-\gamma, q)} \cdot \sum_{n=0}^{\infty} \frac{\prod_n (-\beta, q) \prod_n (-\alpha\beta\tau/\gamma, q) (\gamma/\beta)^n}{\prod_n (-q, q) \prod_n (-\beta\tau, q)}$$

$$\text{(I3)}\qquad \sum_{n=0}^{\infty} \frac{\prod_n (q/\tau, q) \prod_n (-\beta, q) \tau^n}{\prod_n (-q, q)} = \prod_\infty (q, q) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} \beta^n}{\prod_n (-q^2, q^2)} + \prod_\infty (q, q) \prod_\infty (-\beta q, q^2) \sum_{m=1}^{\infty} \frac{\prod_m (-\beta, q^2) \tau^{2m}}{\prod_{2m} (-q, q)} + \prod_\infty (q, q) \prod_\infty (-\beta, q^2) \sum_{m=0}^{\infty} \frac{\prod_m (-\beta q, q^2) \tau^{2m+1}}{\prod_{2m+1} (-q, q)}$$

$$\text{(I4)}\qquad \sum_{n=0}^{\infty} \frac{\prod_n (-q/\tau, q^2) \prod_n (-\beta, q^2) \tau^n}{\prod_n (-q^2, q^2)} = \prod_\infty (-q, q^2) \sum_{m=0}^{\infty} \frac{q^{2m^2-m} \beta^m}{\prod_{2m} (-q, q)} + \tfrac{1}{2} \prod_\infty (-q, q^2) \prod_\infty (\beta^{\frac{1}{2}}, q) \sum_{n=1}^{\infty} \frac{\prod_n (-\beta^{\frac{1}{2}}, q) \tau^n}{\prod_n (-q^2, q^2)} + \tfrac{1}{2} \prod_\infty (-q, q^2) \prod_\infty (-\beta^{\frac{1}{2}}, q) \sum_{n=1}^{\infty} \frac{\prod_n (\beta^{\frac{1}{2}}, q) \tau^n}{\prod_n (-q^2, q^2)}$$

$$\text{(I5)}\qquad \sum_{n=0}^{\infty} \frac{\prod_{2n}(-\beta, q)\tau^{2n}}{\prod_n(-q^2, q^2)} = \frac{\prod_\infty(\tau\beta, q)}{\prod_\infty(\tau, q)} \sum_{m=0}^{\infty} \frac{\prod_m(-\beta, q)\tau^m}{\prod_m(-q, q)\prod_m(\tau\beta, q)}$$

$$\text{(I6)}\qquad \sum_{n=0}^{\infty} \frac{\prod_n(-\tau, q^2)\beta^n}{\prod_n(-q, q)} = \frac{\prod_\infty(-\beta q\tau, q^2)}{\prod_\infty(-\beta q, q^2)} \sum_{m=0}^{\infty} \frac{\prod_m(-\tau, q^2)\beta^m}{\prod_m(-q^2, q^2)\prod_m(-\beta q\tau, q^2)}.$$

(I1) was originally given by Heine [8; 106]. (I2) is easily deduced from (I1); however, it appears to have been first explicitly stated by Rogers [9; 171, Equation (2)]. (I3), (I4), (I5), and (I6) are new to my knowledge.

To obtain (A1) set $\alpha = -q^2/\tau^{\frac{1}{2}}$, $\beta = -q/\tau^{\frac{1}{2}}$, $\gamma = q^2$ in (I2) and let $\tau \to 0$. (A2) is a special case of (C1) obtained by setting $b = 0$ and $a = q^2$ in (C1). (A3) is a special case of (R1) obtained by replacing q by $q^{\frac{1}{2}}$ in (R1) and then setting $z = q^{\frac{3}{2}}$. (C1) is obtained from (I1) by setting $\tau = x$, $\alpha = b$, and $\gamma = a$ and letting $\beta \to 0$. (C2) is obtained from (I3) by setting $\beta = a$ and letting $\tau \to 0$. (C3) is obtained from (I4) by setting $\beta = a$ and letting $\tau \to 0$. (R1) is obtained from (I5) by replacing q throughout by q^2, then setting $\tau\beta = -zq$, and letting $\tau \to 0$. (R2) is obtained from (I6) by setting $\tau = -qz/\beta$ and letting $\beta \to 0$.

3. Derivation of (I1)–(I6). We shall need the following identity

$$\text{(H1)}\qquad \sum_{n=0}^{\infty} \frac{\prod_n(-a, q)z^n}{\prod_n(-q, q)} = \frac{\prod_\infty(-az, q)}{\prod_\infty(-z, q)} \qquad [4; 66].$$

The identities (I1)–(I6) are obtained by repeated application of the following three lemmas. Since all these lemmas are special cases of the Fundamental Lemma of [1], we shall give a detailed proof only of Lemma 2; the other lemmas are proved similarly.

LEMMA 1. *If $k \geq 1$ is an integer,*

$$\sum_{n=0}^{\infty} \frac{\prod_n(-a, q^k)\prod_{kn}(-b, q)t^n}{\prod_n(-q^k, q^k)\prod_{kn}(-c, q)} = \frac{\prod_\infty(-b, q)\prod_\infty(-at, q^k)}{\prod_\infty(-c, q)\prod_\infty(-t, q^k)} \cdot \sum_{m=0}^{\infty} \frac{\prod_m(-c/b, q)\prod_m(-t, q^k)b^m}{\prod_m(-q, q)\prod_m(-at, q^k)}$$

Proof. In Theorem A of [1], take $r = 1$, $s = l = 0$, $e = k$, $d = 1$.

LEMMA 2.

$$\sum_{m=0}^{\infty} \frac{\prod_{2m}(-a, q)\prod_m(-b, q^2)t^{2m}}{\prod_{2m}(-q, q)\prod_m(-c, q^2)} = \frac{1}{2} \frac{\prod_\infty(-b, q^2)\prod_\infty(-at, q)}{\prod_\infty(-c, q^2)\prod_\infty(-t, q)} \sum_{n=0}^{\infty} \frac{\prod_n(-c/b, q^2)\prod_n(-t, q)b^n}{\prod_n(-q^2, q^2)\prod_n(-at, q)} + \frac{1}{2} \frac{\prod_\infty(-b, q^2)\prod_\infty(at, q)}{\prod_\infty(-c, q^2)\prod_\infty(t, q)} \sum_{n=0}^{\infty} \frac{\prod_n(-c/b, q^2)\prod_n(t, q)b^n}{\prod_n(-q^2, q^2)\prod_n(at, q)}$$

Proof.

$$\sum_{m=0}^{\infty} \frac{\prod_{2m}(-a, q) \prod_m (-b, q^2) t^{2m}}{\prod_{2m}(-q, q) \prod_m (-c, q^2)}$$

$$= \frac{\prod_\infty (-b, q^2)}{\prod_\infty (-c, q^2)} \sum_{m=0}^{\infty} \frac{\prod_{2m}(-a, q) \prod_\infty (-cq^{2m}, q^2) t^{2m}}{\prod_{2m}(-q, q) \prod_\infty (-bq^{2m}, q^2)}$$

$$= \frac{1}{2} \frac{\prod_\infty (-b, q^2)}{\prod_\infty (-c, q^2)} \sum_{m=0}^{\infty} \frac{(1 + (-1)^m) \prod_m (-a, q) \prod_\infty (-cq^m, q^2) t^m}{\prod_m (-q, q) \prod_\infty (-bq^m, q^2)}$$

$$= \frac{1}{2} \frac{\prod_\infty (-b, q^2)}{\prod_\infty (-c, q^2)} \sum_{n=0}^{\infty} \frac{\prod_n (-c/b, q^2) b^n}{\prod_n (-q^2, q^2)} \sum_{m=0}^{\infty} \frac{\prod_m (-a, q) t^m q^{mn} (1 + (-1)^m)}{\prod_m (-q, q)}$$

(by (H1))

$$= \frac{1}{2} \frac{\prod_\infty (-b, q^2)}{\prod_\infty (-c, q^2)} \sum_{n=0}^{\infty} \frac{\prod_n (-c/b, q^2) b^n}{\prod_n (-q^2, q^2)} \left\{ \frac{\prod_\infty (-atq^n, q)}{\prod_\infty (-tq^n, q)} + \frac{\prod_\infty (atq^n, q)}{\prod_\infty (tq^n, q)} \right\}$$

(by (H1))

$$= \frac{1}{2} \frac{\prod_\infty (-b, q^2) \prod_\infty (-at, q)}{\prod_\infty (-c, q^2) \prod_\infty (-t, q)} \sum_{n=0}^{\infty} \frac{\prod_n (-c/b, q^2) \prod_n (-t, q) b^n}{\prod_n (-q^2, q^2) \prod_n (-at, q)}$$
$$+ \frac{1}{2} \frac{\prod_\infty (-b, q^2) \prod_\infty (at, q)}{\prod_\infty (-c, q^2) \prod_\infty (t, q)} \sum_{n=0}^{\infty} \frac{\prod_n (-c/b, q^2) \prod_n (t, q) b^n}{\prod_n (-q^2, q^2) \prod_n (at, q)}.$$

This concludes the proof of Lemma 2.

LEMMA 3.

$$\sum_{n=0}^{\infty} \frac{\prod_n (-a, q^2) \prod_n (-b, q) t^n}{\prod_n (-q^2, q^2) \prod_n (-c, q)}$$

$$= \frac{\prod_\infty (-b, q) \prod_\infty (-at, q^2)}{\prod_\infty (-c, q) \prod_\infty (-t, q^2)} \sum_{m=0}^{\infty} \frac{\prod_{2m} (-c/b, q) \prod_m (-t, q^2) b^{2m}}{\prod_{2m} (-q, q) \prod_m (-at, q^2)}$$
$$+ \frac{\prod_\infty (-b, q) \prod_\infty (-atq, q^2)}{\prod_\infty (-c, q) \prod_\infty (-tq, q^2)} \sum_{m=0}^{\infty} \frac{\prod_{2m+1} (-c/b, q) \prod_m (-tq, q^2) b^{2m+1}}{\prod_{2m+1} (-q. q) \prod_m (-atq, q^2)}.$$

Proof. This is just Theorem A_1 of [1] which in turn is a special case of the Fundamental Lemma of [1].

We can now derive the identities (I1)–(I6). (I1) is obtained directly from Lemma 1 by taking $k = 1$ (actually (I1) appears as Theorem A_2 of [1]). (I2) is obtained by a double application of (I1) (actually (I2) is Theorem 8 of [2]).

The derivation of (I3) is slightly more involved.

$$\sum_{n=0}^{\infty} \frac{\prod_n (q/\tau, q) \prod_n (-\beta, q) \tau^n}{\prod_n (-q, q)} = \frac{\prod_\infty (-\beta, q) \prod_\infty (q, q)}{\prod_\infty (-\tau, q)} \sum_{n=0}^{\infty} \frac{\prod_n (-\tau, q) \beta^n}{\prod_n (-q^2, q^2)}$$

(by Lemma 1 with $a = -q/\tau$, $k = 1$, $b = \beta$, $c = 0$)

$$= \frac{\prod_\infty(-\beta, q)\prod_\infty(q, q)}{\prod_\infty(-\tau, q)}\left\{\frac{\prod_\infty(-\tau, q)}{\prod_\infty(-\beta, q^2)}\sum_{m=0}^{\infty}\frac{\prod_m(-\beta, q^2)\tau^{2m}}{\prod_{2m}(-q, q)} + \frac{\prod_\infty(-\tau, q)}{\prod_\infty(-q, q^2)}\sum_{m=0}^{\infty}\frac{\prod_m(-\beta q, q^2)\tau^{2m+1}}{\prod_{2m+1}(-q, q)}\right\}$$

(by Lemma 3 with $a = c = 0$, $b = \tau$, $t = \beta$)

$$= \prod_\infty(q, q)\prod_\infty(-\beta q, q^2) + \prod_\infty(q, q)\prod_\infty(-\beta q, q^2)\sum_{m=1}^{\infty}\frac{\prod_m(-\beta, q^2)\tau^{2m}}{\prod_{2m}(-q, q)} + \prod_\infty(q, q)\prod_\infty(-\beta, q^2)\sum_{m=0}^{\infty}\frac{\prod_m(-\beta q, q^2)\tau^{2m+1}}{\prod_{2m+1}(-q, q)}$$

$$= \prod_\infty(q, q)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}\beta^n}{\prod_n(-q^2, q^2)} + \prod_\infty(q, q)\prod_\infty(-\beta q, q^2)\sum_{m=1}^{\infty}\frac{\prod_m(-\beta, q^2)\tau^{2m}}{\prod_{2m}(-q, q)} + \prod_\infty(q, q)\prod_\infty(-\beta, q^2)\sum_{m=0}^{\infty}\frac{\prod_m(-\beta q, q^2)\tau^{2m+1}}{\prod_{2m+1}(-q, q)}$$

(by (H1) applied to $\prod_\infty(-\beta q, q^2)$). We now derive (I4).

$$\sum_{n=0}^{\infty}\frac{\prod_n(-q/\tau, q^2)\prod_n(-\beta, q^2)\tau^n}{\prod_n(-q^2, q^2)} = \frac{\prod_\infty(-\beta, q^2)\prod_\infty(-q, q^2)}{\prod_\infty(-\tau, q^2)}\sum_{m=0}^{\infty}\frac{\prod_m(-\tau, q^2)\beta^m}{\prod_{2m}(-q, q)}$$

(replacing q by q^2 in Lemma 1, then setting $k = 1$, $a = q/\tau$, $b = \beta$, $c = 0$, $t = \tau$)

$$= \frac{\prod_\infty(-\beta, q^2)\prod_\infty(-q, q^2)}{\prod_\infty(-\tau, q^2)}\left\{\frac{1}{2}\frac{\prod_\infty(-\tau, q^2)}{\prod_\infty(-\beta^{\frac{1}{2}}, q)}\sum_{n=0}^{\infty}\frac{\prod_n(-\beta^{\frac{1}{2}}, q)\tau^n}{\prod_n(-q^2, q^2)} + \frac{1}{2}\frac{\prod_\infty(-\tau, q^2)}{\prod_\infty(\beta^{\frac{1}{2}}, q)}\sum_{n=0}^{\infty}\frac{\prod_n(\beta^{\frac{1}{2}}, q)\tau^n}{\prod_n(-q^2, q^2)}\right\}$$

(by Lemma 2 with $a = c = 0$, $b = \tau$, $t = \beta$)

$$= \tfrac{1}{2}\prod_\infty(-q, q^2)\{\prod_\infty(\beta^{\frac{1}{2}}, q) + \prod_\infty(-\beta^{\frac{1}{2}}, q)\} + \tfrac{1}{2}\prod_\infty(-q, q^2)\prod_\infty(\beta^{\frac{1}{2}}, q)\sum_{n=1}^{\infty}\frac{\prod_n(-\beta^{\frac{1}{2}}, q)\tau^n}{\prod_n(-q^2, q^2)} + \tfrac{1}{2}\prod_\infty(-q, q^2)\prod_\infty(-\beta^{\frac{1}{2}}, q)\sum_{n=1}^{\infty}\frac{\prod_n(\beta^{\frac{1}{2}}, q)\tau^n}{\prod_n(-q^2, q^2)}.$$

(I4) is now obtained by noting that (H1) implies

$$\prod_\infty(\beta^{\frac{1}{2}}, q) + \prod_\infty(-\beta^{\frac{1}{2}}, q) = 2\sum_{m=0}^{\infty}\frac{q^{2m^2-m}\beta^m}{\prod_{2m}(-q, q)}.$$

We now prove (I5).

$$\sum_{n=0}^{\infty} \frac{\prod_{2n}(-\beta, q)\tau^{2n}}{\prod_n(-q^2, q^2)} = \frac{\prod_\infty(-\beta, q)}{\prod_\infty(-\tau^2, q^2)} \sum_{m=0}^{\infty} \frac{\prod_m(-\tau^2, q^2)\beta^m}{\prod_m(-q, q)}$$

(by Lemma 1 with $k = 2$, $a = c = 0$ $t = \tau^2$, $b = \beta$)

$$= \frac{\prod_\infty(-\beta, q)}{\prod_\infty(-\tau^2, q^2)} \sum_{m=0}^{\infty} \frac{\prod_m(\tau, q)\prod_m(-\tau, q)\beta^m}{\prod_m(-q, q)}$$

$$= \frac{\prod_\infty(-\beta, q)\prod_\infty(-\tau, q)\prod_\infty(\tau\beta, q)}{\prod_\infty(-\tau^2, q^2)\prod_\infty(-\beta, q)} \sum_{m=0}^{\infty} \frac{\prod_m(-\beta, q)\ \tau^m}{\prod_m(-q, q)\prod_m(\tau\beta, q)}$$

(by Lemma 1 with $k = 1$, $a = -\tau$, $b = \tau$, $c = 0$, $t = \beta$). Simplifying this last expression, we obtain (I5).

Finally for (I6), we have

$$\frac{\prod_\infty(-\beta, q)}{\prod(-\tau, q^2)} \sum_{m=0}^{\infty} \frac{\prod_m(-\tau, q^2)\beta^m}{\prod_m(-q, q)} = \sum_{n=0}^{\infty} \frac{\prod_{2n}(-\beta, q)\tau^n}{\prod_n(-q^2, q^2)}$$

(by Lemma 1 with $k = 2$, $a = c = 0$, $t = \tau$, $b = \beta$)

$$= \sum_{n=0}^{\infty} \frac{\prod_n(-\beta q, q^2)\prod_n(-\beta, q^2)\tau^n}{\prod_n(-q^2, q^2)}$$

$$= \frac{\prod_\infty(-\beta, q^2)\prod_\infty(-\beta q\tau, q^2)}{\prod_\infty(-\tau, q^2)} \sum_{m=0}^{\infty} \frac{\prod_m(-\tau, q^2)\beta^m}{\prod_m(-q^2, q^2)\prod_m(-\beta q\tau, q^2)}$$

(replacing q by q^2 in Lemma 1, then setting $k = 1$, $a = \beta q$, $b = \beta$, $c = 0$, $t = \tau$). Simplifying this last expression, we obtain (I6).

References

1. G. E. Andrews, *On basic hypergeometric series, mock theta functions, and partitions* I, Quarterly Journal of Mathematics, Oxford Second Series, vol. 17(1966), pp. 64–80.
2. G. E. Andrews, *On basic hypergeometric series, mock theta functions, and partitions* II (to appear).
3. F. C. Auluck, *On some new types of partitions associated with generalized Ferrers graphs,* Proceedings of the Cambridge Philosophical Society, vol. 47(1951), pp. 679–686.
4. W. N. Bailey, *Generalized Hypergeometric Series,* Cambridge, 1935.
5. L. Carlitz, *Some q-identities related to the theta functions,* Bollettino della Unione Matematica Italiana, (3), vol. 17(1962), pp. 172–178.
6. L. Carlitz, *Advanced problem* 5196, American Mathematical Monthly, vol. 71(1964), pp. 440–441.
7. N. J. Fine, *Some new results on partitions,* Proceedings of the National Academy of Sciences, vol. 34(1948), pp. 616–618.
8. E. Heine, *Handbuch der Kugelfunctionen,* Vol. I, Berlin, 1878.
9. L. J. Rogers, *On a threefold symmetry in the elements of Heine's series,* Proceedings of the London Mathematical Society, vol. 24(1893), pp. 171–179.

10. L. J. ROGERS, *Second memoir on the expansion of certain infinite products*, Proceedings of the London Mathematical Society, vol. 25(1894), pp. 318–343.
11. D. B. SEARS, *On the transformation theory of basic hypergeometric functions*, Proceedings of the London Mathematical Society (2), vol. 53(1951), pp. 158–180.
12. D. B. SEARS, *Transformations of basic hypergeometric functions of any order*, Proceedings of the London Mathematical Society (2), vol. 53(1951), pp. 181–191.
13. L. J. SLATER, *An integral of hypergeometric type*, Proceedings of the Cambridge Philosophical Society, vol. 48(1952), pp. 578–582.
14. G. N. WATSON, *The final problem: an account of the mock theta functions*, Journal of the London Mathematical Society, vol. 11(1936), pp. 55–80.
15. G. N. WATSON, *The mock theta functions* II, Proceedings of the London Mathematical Society (2), vol. 42(1937), pp. 272–304.
16. G. N. WATSON, *A note on Lerch's functions*, Quarterly Journal of Mathematics, Oxford Series, vol. 8(1937), pp. 43–47.

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