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Partitions, q-Series and the Lusztig-Macdonald-Wall Conjectures

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1. Introduction

Recently Michael Hirschhorn (University of New South Wales) and G.E. Wall (University of Sydney) communicated the following two conjectures to me:

Let $\chi_n = \chi_n(a, b, q)$ with $\chi_{-1} = a$, $\chi_0 = b$ and for $n \geqq 0$

$$\chi_{2n+1} = \chi_{2n} + q^{2n+1} \chi_{2n-1}, \tag{1.1}$$

$$\chi_{2n+2} = \chi_{2n+1} + q^{n+1}(1+q^{n+1})(\chi_{2n+1} + (1-q^{2n+1})\chi_{2n-1}). \tag{1.2}$$

Let $\chi(a, b, q) = \lim_{n \to \infty} \chi_n(a, b, q)$.

Conjecture 1 (Theorem 1 below): $\chi(1, 1, q) = \dfrac{\sum_{i=-\infty}^{\infty} q^{i^2}}{\prod_{j=1}^{\infty} (1-q^j)}$.

Conjecture 2 (Theorem 2 below): $\chi(0, 1, q) = \dfrac{\sum_{i=0}^{\infty} q^{i(i+1)}}{\prod_{j=1}^{\infty} (1-q^j)}$.

Wall described the history of these conjectures thus: "[In [7] were] included generating functions for the numbers of conjugacy classes in the classical groups over finite fields. The power series $\chi(0, 1; t)$ and $\chi(1, 1; t)$ are what one needs to know in order to write down the generating functions for the symplectic and orthogonal groups over finite fields of characteristic 2 (see Theorem 3.7.3 on p. 59 [of [7]]) ...."

"Early in 1976, Professor George Lusztig (of the Mathematics Institute, University of Warwick, in England) wrote to me saying, 'I have very strong

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reasons (coming from representation theory) to believe that [Conjecture 2 holds]. I cannot prove this... . Independently, [Ian] Macdonald [Queen Mary College, London] has conjectured [Conjecture 2] by computing the first 30 terms of $\chi(0,1;\,T)$. I would also be interested in similar results for $\chi(1,1;\,T)$.' ... After receiving Lusztig's letter, I did some explicit calculations on $\chi(0,1;\,T)$ and $\chi(1,1;\,T)$; my calculations up to T^{25} confirmed Macdonald's for $\chi(0,1;\,T)$ and my calculations up to T^{16} on $\chi(1,1;\,T)$ strongly suggested [Conjecture 1]."

In the course of proving these conjectures it became necessary to prove an apparently new q-series identity:

Theorem 3.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}}{(q)_n(q)_m}=\frac{1+2q+2q^4+2q^9+\cdots}{1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-\cdots}$$
$$=\prod_{n=1}^{\infty}(1+q^n)(1+q^{2n-1})^2,$$

where $(q)_n=(1-q)(1-q^2)\dots(1-q^n)$, $(q)_0=1$.

Furthermore the identities in the conjectures (Theorems 1 and 2) imply two theorems on partition identities.

Theorem 4. Let $H_1(n)$ denote the number of partitions of n into red, green or yellow parts each congruent to 2 modulo 4 and only yellow parts may be repeated. Let $W_1(n)$ denote the number of partitions of n into red or green parts with the restrictions that: (i) only even parts may be green (any part may be red); (ii) consecutive integers (i.e., j and $j+1$) may not both appear as parts; (iii) each odd number appears twice or not at all; (iv) the same even part cannot appear in two different colors; (v) if an even part is green the next truly smaller part (if it exists) is at least 3 units smaller. Then $H_1(n)=W_1(n)$ for all n.

To illustrate Theorem 4, we consider partitions of 8 wherein plain integers are considered red, primed integers green, and double primed integers are yellow. The thirteen partitions enumerated by $H_1(8)$ are $6+2$, $6+2'$, $6+2''$, $6'+2$, $6'+2'$, $6'+2''$, $6''+2$, $6''+2'$, $6''+2''$, $2''+2''+2''+2''$, $2'+2''+2''+2''$, $2+2''+2''+2''$, $2+2'+2''+2''$. The thirteen partitions enumerated by $W_1(8)$ are 8, $8'$, $6+2$, $6+2'$, $6'+2$, $6'+2'$, $6+1+1$, $6'+1+1$, $4+4$, $4'+4'$, $4+2+2$, $4+2'+2'$, $3+3+1+1$.

Theorem 5. Let $H_2(n)$ denote the number of partitions of n into red, green or yellow parts with each red or green congruent to 4 modulo 8 and each yellow congruent to 2 modulo 4. Let $W_2(n)$ denote the number of those partitions enumerated by $W_1(n)$ with the added restrictions that: (vi) each part is larger than 1; (vii) 2 may not be green. Then $H_2(n)=W_2(n)$ for all n.

From examination of the partitions enumerated by $W_1(8)$, we see that $W_2(8)=7$. The seven partitions enumerated by $H_2(8)$ are $6''+2''$, $4+4$, $4+4'$, $4'+4'$, $4+2''+2''$, $4'+2''+2''$, $2''+2''+2''+2''$.

In Section 2 we shall prove the conjectures. Section 3 will be devoted to Theorems 3–5, and in the conclusion we shall mention open problems arising from this study.

2. Proof of the Conjectures

We start by considering a more general sequence than that considered by Wall. Namely, we let $\chi_n(z)=\chi_n(a,b,q;z)$ with $\chi_{-1}(z)=a$, $\chi_0(z)=b$ and for $n\geqq 0$

$$\chi_{2n+1}(z)=\chi_{2n}(z)+z^2q^{2n+1}\chi_{2n-1}(z), \tag{2.1}$$

$$\chi_{2n+2}(z)=\chi_{2n+1}(z)+zq^{n+1}(1+zq^{n+1})\cdot(\chi_{2n+1}(z)+(1-z^2q^{2n+1})\chi_{2n-1}(z)), \tag{2.2}$$

and by applying (2.1) to the right side of (2.2) we see that

$$\chi_{2n+2}(z)=\chi_{2n+1}(z)+zq^{n+1}(1+zq^{n+1})(\chi_{2n}(z)+\chi_{2n-1}(z)). \tag{2.3}$$

Note that the initial values $\chi_{-1}(z)=a$, $\chi_0(z)=b$, and the recurrences (2.1) and (2.3) uniquely define the $\chi_n(z)$. Furthermore we see that the coefficient of q^N in $\chi_M(z)$ is some polynomial in z that is unaltered for any $M>2N$. Consequently we see that $\chi_n(z)$ converges in the ring of formal power series in q whose coefficients are polynomial functions of z. Furthermore we see immediately that the χ_n sequence is dominated by the sequence $\gamma_{-1}=\gamma_0=A=\max(a,b)$ (assumed positive):

$$\gamma_{2n+1}(z)=(1+z^2q^{2n+1})\gamma_{2n}(z)$$
$$\gamma_{2n+2}(z)=(1+2zq^{n+1}+2z^2q^{2n+1})\gamma_{2n+1}(z).$$

Clearly

$$\lim_{n\to\infty}\gamma_n(z)=A\prod_{n=0}^{\infty}(1+z^2q^{2n+1})(1+2zq^{n+1}+2z^2q^{2n+1})$$

which is an absolutely convergent infinite product provided only that $|q|<1$ with uniform convergence inside $|q|\leqq 1-\varepsilon$, $|z|\leqq M$. Hence we see that $\chi(a,b,q;z)$ is analytic is z and q for fixed nonnegative a and b provided $|q|<1$.

Let us now represent $\chi_{2n+1}(1,1,z;q)$ by a $(2n+2)\times(2n+2)$ determinant:

$$\chi_{2n+1}(1,1,q;z)$$
$$=\begin{vmatrix} 1 & z^2q & zq+z^2q^2 & 0 & 0 & \cdots & & \\ -1 & 1 & zq+z^2q^2 & 0 & 0 & \cdots & & \\ 0 & -1 & 1 & z^2q^3 & zq^2+z^2q^4 & \cdots & & \\ 0 & 0 & -1 & 1 & zq^2+z^2q^4 & \cdots & & \\ 0 & 0 & 0 & -1 & 1 & \cdots & zq^n+z^2q^{2n} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & zq^2+z^2q^{2n} & 0 \\ & & & & & -1 & 1 & z^2q^{2n+1} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & -1 & 1 \end{vmatrix}, \tag{2.4}$$

and for $\chi_{2n+2}(1,1,q;z)$ we have the $(2n+3)\times(2n+3)$ determinant

$$\chi_{2n+2}(1,1,q;z)$$
$$=\left|\begin{array}{cccccccc}
1 & z^2q & zq+z^2q^2 & 0 & 0 & \cdots & & \\
-1 & 1 & zq+z^2q^2 & 0 & 0 & \cdots & & \\
0 & -1 & 1 & z^2q^3 & zq^2+z^2q^4 & \cdots & & \\
0 & 0 & -1 & 1 & zq^2+z^2q^4 & \cdots & & \\
0 & 0 & 0 & -1 & 1 & \cdots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & 0 & 0 \\
 & & & & & & 0 & 0 \\
 & & & & & & z^2q^{2n+1} & zq^{n+1}z^2q^{2n+2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & & 1 & zq^{n+1}+z^2q^{2n+2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & & -1 & 1
\end{array}\right|. \tag{2.5}$$

It is not difficult to see that these representations are valid. By expansion along the final column we see that these determinants satisfy the recurrences (2.1) and (2.3). Furthermore if we set $n=-1$ in (2.4) and (2.5) we obtain the appropriate initial values $\chi_{-1}=\chi_0=1$.

Next we represent $\chi_{2n+2}(0,1,q;z)$ by $(2n+2)\times(2n+2)$ determinant

$$\chi_{2n+2}(0,1,q;z)$$
$$=\left|\begin{array}{ccccccccc}
1 & zq+z^2q^2 & 0 & 0 & \cdots & & & & \\
-1 & 1 & z^2q^3 & zq^2+z^2q^4 & \cdots & & & & \\
0 & -1 & 1 & zq^2+z^2q^4 & \cdots & & & & \\
0 & 0 & -1 & 1 & \cdots & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & & \\
 & & & & \cdots & -1 & 1 & z^2q^{2n+1} & zq^{n+1}+z^2q^{2n+2} \\
 & & & & \cdots & & -1 & 1 & zq^{n+1}+z^2q^{2n+2} \\
 & & & & \cdots & & & -1 & 1
\end{array}\right|, \tag{2.6}$$

and for $\chi_{2n+1}(0,1,q;z)$ we have the $(2n+1)\times(2n+1)$ determinant

$$\chi_{2n+1}(0,1,q;z)$$
$$=\left|\begin{array}{cccccccccc}
1 & zq+z^2q^2 & 0 & 0 & \cdots & & & & & \\
-1 & 1 & z^2q^3 & zq^2+z^2q^4 & \cdots & & & & & \\
0 & -1 & 1 & zq^2+z^2q^4 & \cdots & & & & & \\
0 & 0 & -1 & 1 & \cdots & & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & & & \\
 & & & & & & & zq^n+z^2q^{2n} & & \\
 & & & & \cdots & -1 & 1 & zq^n+z^2q^{2n} & & \\
 & & & & \cdots & & -1 & 1 & z^2q^{2n+1} & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 1 &
\end{array}\right|. \tag{2.7}$$

As with (2.4) and (2.5), the validity of (2.6) and (2.7) follows easily by expansion along the last column and the initial values for χ_0 and χ_1.

We may now expand all these determinants along the top row: from (2.4) we find that

$$\chi_{2n+1}(1,1,q;z)=\chi_{2n+1}(0,1,q;z)+z^2q\chi_{2n-1}(1,1,q;zq) \\ +(zq+z^2q^2)\chi_{2n-3}(0,1,q;zq); \tag{2.8}$$

from (2.6)

$$\chi_{2n+2}(0,1,q;z)=\chi_{2n}(1,1,q;zq)+(zq+z^2q^2)\chi_{2n}(0,1,q;zq). \tag{2.9}$$

Now we let $n\to\infty$ in each of Equations (2.8) and (2.9); this yields

$$\chi(1,1,q;z)=\chi(0,1,q;z)+z^2q\chi(1,1,q;zq)+(zq+z^2q^2)\chi(0,1,q;zq); \tag{2.10}$$

$$\chi(0,1,q;z)=\chi(1,1,q;zq)+(zq+z^2q^2)\chi(0,1,q;zq). \tag{2.11}$$

We may eliminate $(zq+z^2q^2)\chi(0,1,q;zq)$ from these equations and deduce that

$$\chi(0,1,q;z)=\tfrac{1}{2}(\chi(1,1,q;z)+(1-z^2q)\chi(1,1,q;zq)). \tag{2.12}$$

Now we apply (2.12) to (2.11) to eliminate both $\chi(0,1,q;z)$ and $\chi(0,1,q;zq)$; after simplification we obtain

$$\chi(1,1,q;z)=(1+zq+z^2q+z^2q^2)\chi(1,1,q;zq) \\ +(zq+z^2q^2)(1-z^2q^3)\chi(1,1,q;zq^2). \tag{2.13}$$

We now observe that the functional equation (2.13), the initial conditions $\chi(1,1,q;0)=\chi(1,1,0;z)=1$ and the requirement that $\chi(1,1,q;z)$ be analytic in z and q for $|q|<1$ together uniquely define $\chi(1,1,q;z)$. This is easily seen by consideration of the double series expansion

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}z^mq^n=\chi(1,1,q;z). \tag{2.14}$$

The initial conditions show that $a_{00}=1$ and $a_{m,0}=a_{0,m}=0$ for $m>0$. Finally if we substitute the double series expansion (2.14) into (2.13) and then compare coefficients of q^nz^m, we find a recurrence for the a_{mn} which uniquely define them given the above initial conditions.

Next we consider

$$X_1(z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}z^{n+m}}{(q)_n(q)_m}, \tag{2.15}$$

where $(A)_n=(A;q)_n=(1-A)(1-Aq)\dots(1-Aq^{n-1}),(A)_0=1$, and

$$X_2(z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}z^{n+m}(q^n+q^m-q^{n+m})}{(q)_n(q)_m}. \tag{2.16}$$

We note immediately that

$$
\begin{aligned}
&X_1(z)-z^2qX_1(zq)\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}z^{n+m}}{(q)_n(q)_m}-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm+1}z^{n+m+2}q^{n+m}}{(q)_n(q)_m}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}z^{n+m}}{(q)_n(q)_m}\\
&\quad-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{(n-1)^2+(m-1)^2-(n-1)(m-1)+1}z^{n+m}q^{n+m-2}}{(q)_n(q)_m}\cdot(1-q^n)(1-q^m)
\end{aligned}
\tag{2.17}
$$

(we have shifted $n\to n-1$, $m\to m-1$ in the second double sum)

$$
=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}z^{n+m}(1-(1-q^n)(1-q^m))}{(q)_n(q)_m}=X_2(z).
$$

Next

$$
X_2(z)+X_1(zq)=\sum_{n=0}\sum_{m=0}\frac{q^{n^2+m^2-nm}z^{n+m}(q^n+q^m)}{(q)_n(q)_m}. \tag{2.18}
$$

Also

$$
\begin{aligned}
&X_2(z)-X_1(zq)\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}z^{n+m}((q^n-q^{n+m})+(q^m-q^{n+m}))}{(q)_n(q)_m}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm+n}z^{n+m}(1-q^m)}{(q)_n(q)_m}+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm+m}z^{n+m}}{(q)_n(q)_m}(1-q^n)\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2+2m+1-nm}z^{n+m+1}}{(q)_n(q)_m}+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+2n+1+m^2-nm}z^{n+m+1}}{(q)_m(q)_n}\\
&=zq\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}(q^{2m}+q^{2n})z^{n+m}}{(q)_n(q)_m}.
\end{aligned}
\tag{2.19}
$$

Consequently by (2.18) and (2.19)

$$
\begin{aligned}
&(X_2(z)-X_1(zq))-zq(X_2(zq)+X_1(zq^2))\\
&=zq\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}(q^{2m}+q^{2n})z^{n+m}}{(q)_m(q)_n}\\
&\quad-zq\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm+n+m}z^{n+m}(q^m+q^n)}{(q)_m(q)_n}\\
&=zq\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}q^{2m}(1-q^n)z^{n+m}}{(q)_m(q)_n}\\
&\quad+zq\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}q^{2n}(1-q^m)z^{n+m}}{(q)_m(q)_n}
\end{aligned}
$$

$$=zq\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{(n+1)^2+m^2-(n+1)m+2m}z^{n+m+1}}{(q)_n(q)_m}$$
$$+zq\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+(m+1)^2-n(m+1)+2n}z^{n+m+1}}{(q)_m(q)_n}$$
$$=z^2q^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}(zq)^{n+m}(q^n+q^m)}{(q)_n(q)_m}$$
$$=z^2q^2(X_2(zq)+X_1(zq^2)). \tag{2.20}$$

We now define

$$X_0(z)=\tfrac{1}{2}(X_1(z)+X_1(zq)(1-z^2q)), \tag{2.21}$$

and we are prepared to establish the most important identifies of this section. Namely

Lemma 1.

$$X_1(z)=\chi(1,1,q;z), \tag{2.22}$$

$$X_0(z)=\chi(0,1,q;z). \tag{2.23}$$

Proof. We note that each of the four functions defined in (2.22) and (2.23) is analytic in z and q for $|q|<1$. Furthermore each of these four functions reduces to 1 when either z or q is set equal to 0.

Let us substitute the expression for $X_2(z)$ from (2.17) into (2.20) in order to eliminate $X_2(z)$ and $X_2(zq)$; after simplification we see that

$$X_1(z)=(1+zq+z^2q+z^2q^2)X_1(zq)+(zq+z^2q^2)(1-z^2q^3)X_1(zq^2). \tag{2.24}$$

We recall our remark following equation (2.13) (which is merely (2.24) with $\chi(1,1,q;z)$ replacing $X_1(z)$); since (i) $X_1(z)$ is analytic in z and q for $|q|<1$, (ii) $X_1(z)$ equals 1 for $z=0$ or $q=0$, and (iii) $X_1(z)$ satisfies (2.13), we deduce that

$$X_1(z)=\chi(1,1,q;z). \tag{2.25}$$

Finally comparing (2.21) with (2.12) and bearing (2.25) in mind, we see that

$$X_0(z)=\chi(0,1,q;z). \tag{2.26}$$

This establishes Lemma 1.

To obtain Wall's conjectures we require one identity from the theory of basic hypergeometric functions:

$$\sum_{n=0}^{\infty}\frac{q^{n^2-n}c^nx^n}{(q)_n(c)_n}=\frac{1}{(c)_\infty}\sum_{n=0}^{\infty}\frac{(x)_n(-1)^nc^nq^{n(n-1)/2}}{(q)_n}, \tag{2.27}$$

where $(A)_n=(1-A)(1-Aq)\dots(1-Aq^{n-1})$ and $(A)=\lim_{n\to\infty}(A)_n$. This result follows from the second iterate of Heine's fundamental transformation; in identity (I2) of [1; p. 576] replace τ by $\frac{cx}{\alpha\beta}$, replace γ by c and let α and $\beta\to\infty$.

Theorem 1. $\chi(1,1,q)=\dfrac{\sum_{i=-\infty}^{\infty} q^{i^2}}{(q)_\infty}$.

Proof. $\chi(1,1,q)=\chi(1,1,q;1)$ (by definition)

$$=X_1(1) \quad \text{(by Lemma 1)}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}}{(q)_n(q)_m} \quad \text{(by (2.15))}$$

$$=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n^2}+2\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\frac{q^{n^2+m^2-nm}}{(q)_n(q)_m}$$

$$=\frac{1}{(q)_\infty}+2\sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty}\frac{q^{n^2+m^2-nm}}{(q)_n(q)_m} \quad \text{(by (2.27) with } c=q,\ x=1)$$

$$=\frac{1}{(q)_\infty}+2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{(n+m+1)^2+m^2-(n+m+1)m}}{(q)_{n+m+1}(q)_m}$$

$$=\frac{1}{(q)_\infty}+2\sum_{n=0}^{\infty}q^{(n+1)^2}\sum_{m=0}^{\infty}\frac{q^{m^2+m(n+1)}}{(q)_m(q)_{m+n+1}}$$

$$=\frac{1}{(q)_\infty}+2\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}}{(q)_\infty} \quad \text{(by (2.27) with } c=q^{n+2}, x=1)$$

$$=\frac{\sum_{i=-\infty}^{\infty} q^{i^2}}{(q)_\infty},$$

as desired.

Theorem 2. $\chi(0,1,q)=\dfrac{\sum_{i=0}^{\infty} q^{i(i+1)}}{(q)_\infty}$.

Proof. $\chi(0,1,q)=\chi(0,1,q;1)$ (by definition)

$$=X_0(1) \quad \text{(by Lemma 1)}$$

$$=\tfrac{1}{2}(X_1(1)+X_1(q)(1-q)) \quad \text{(by (2.12)).}$$

$$=\frac{\sum_{i=-\infty}^{\infty} q^{i^2}}{2(q)_\infty}+\frac{(1-q)}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm+n+m}}{(q)_n(q)_m} \quad \text{(by Theorem 1 and (2.15))}$$

$$=\frac{\sum_{i=-\infty}^{\infty} q^{i^2}}{2(q)_\infty}+\frac{(1-q)}{2}\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q)_n^2}+(1-q)\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\frac{q^{n^2+m^2-nm+n+m}}{(q)_n(q)_m}$$

$$=\frac{\sum_{i=-\infty}^{\infty} q^{i^2}}{2(q)_\infty}+\frac{(1-q)}{2}\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q)_n^2}$$
$$+(1-q)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{(n+m+1)^2+m^2-(n+m+1)m+n+2m+1}}{(q)_{n+m+1}(q)_m}$$
$$=\frac{\sum_{i=-\infty}^{\infty} q^{i^2}}{2(q)_\infty}+\frac{(1-q)}{2}\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q)_n^2}+(1-q)\sum_{n=0}^{\infty}q^{(n+1)^2+(n+1)}\sum_{m=0}^{\infty}\frac{q^{m^2+(n+1)m+2m}}{(q)_m(q)_{n+m+1}}$$
$$=\frac{\sum_{i=-\infty}^{\infty} q^{i^2}}{2(q)_\infty}+\frac{1}{2(q)_\infty}\sum_{n=0}(1-q^{n+1})(-1)^n q^{n(n+1)/2} \qquad \text{(by (2.27))}$$
$$+\sum_{n=0}^{\infty}q^{(n+1)^2+(n+1)}\frac{1}{(q)_\infty}\sum_{m=0}^{\infty}(1-q^{m+1})(-1)^m q^{(n+1)m+m(m+1)/2}$$
$$=\frac{\sum_{i=1}^{\infty} q^{i^2}+\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}+\sum_{n=1}^{\infty}q^{n^2+n}\sum_{m=0}^{\infty}(1-q^{m+1})(-1)^m q^{nm+m(m+1)/2}}{(q)_\infty}$$
$$=\frac{1}{(q)_\infty}\Bigg\{\sum_{i=1}^{\infty} q^{i^2}+\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}$$
$$+\sum_{n=2}^{\infty}\sum_{m=-2}^{\infty}q^{n^2+n+nm+m(m+1)/2+m+1}(-1)^m$$
$$-\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q^{n^2+n+nm+m(m+1)/2+m+1}(-1)^m\Bigg\}$$
$$=\frac{1}{(q)_\infty}\Bigg\{\sum_{i=1}^{\infty} q^{i^2}+\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}$$
$$+\sum_{n=2}^{\infty}q^{n^2-n}-\sum_{n=2}^{\infty}q^{n^2}-\sum_{m=0}^{\infty}(-1)^m q^{(m+3)(m+2)/2}\Bigg\}$$
$$=\frac{1}{(q)_\infty}\left\{q+1-q+\sum_{n=2}^{\infty}q^{n^2-n}\right\}=\frac{\sum_{i=0}^{\infty}q^{i(i+1)}}{(q)_\infty},$$

as desired.

3. Applications

The results in Section 2 may be applied to both the q-series and partition theorems stated in the introduction.

Proof of Theorem 3. By (2.15) and Theorem 1:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+m^2-nm}}{(q)_n(q)_m}=X_1(1)=\frac{1+2\sum_{i=1}^{\infty}q^{i^2}}{(q)_\infty}$$

$$=\frac{1+2q+2q^4+2q^9+\cdots}{1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-\cdots}$$

(by Euler's Pentagonal Number Theorem [3; p. 177])

$$=\prod_{n=0}^{\infty}\frac{(1-q^{2n+2})(1+q^{2n+1})^2}{(1+q^{n+1})}$$

(by Jacobi's Triple Product Identity [3; p. 170])

$$=\prod_{n=1}^{\infty}(1+q^n)(1+q^{2n-1})^2.$$

Now we treat the two partition theorems implied by Theorems 1 and 2.

Proof of Theorems 4 and 5. Let us replace q by q^2 and z by 1 in the defining recurrences (2.1) and (2.3). This yields

$$\chi_{2n+1}=\chi_{2n}+q^{(2n+1)+(2n+1)}\chi_{2n-1}, \tag{3.1}$$

$$\chi_{2n+2}=\chi_{2n+1}+(q^{2n+2}+q^{(2n+2)+(2n+2)})\chi_{2n} +(q^{2n+2}+q^{(2n+2)+(2n+2)})\chi_{2n-1}. \tag{3.2}$$

Now it is clear that $\chi_N(1,1,q;1)$ is the generating function for the partitions of the type enumerated by $W_1(n)$ (n arbitrary) subject to the restriction that each part is $\leqq N$. This made obvious by (3.1) and (3.2). For example we split the considered partitions with parts $\leqq 2n+1$ into two exhaustive and disjoint classes: (1) those in which $2n+1$ does not appear, (2) those in which $2n+1$ appears twice. Those in the first class are enumerated by χ_{2n} and those in the second class are enumerated by $q^{(2n+1)+(2n+1)}\chi_{2n-1}$. The same reasoning applies to Equation (3.2) once we observe that the third term on the right side takes care of the "green" appearances of even parts. Hence

$$\sum_{n=0}^{\infty}W_1(n)q^n=\lim_{N\to\infty}\chi_N(1,1,q^2;1)=\chi(1,1,q^2)$$

$$=\prod_{n=1}^{\infty}(1+q^{2n})(1+q^{4n-2})^2 \quad \text{(by Theorem 1)}$$

$$=\prod_{n=1}^{\infty}\frac{(1+q^{4n-2})^2}{(1-q^{4n-2})} \quad \text{(by Euler's identity [3; pp. 164–165])}$$

$$=\sum_{n=0}^{\infty}H_1(n)q^n. \tag{33}$$

Now comparing coefficients of q^n in the extremes of (3.3) we deduce that $W_1(n)=H_1(n)$ for all n.

The only alteration when we pass to Theorem 5 is that the change in initial conditions for $\chi_N(0,1,q;1)$ produces the restrictions on appearances of 1's and 2's. Hence

$$\sum_{n=0}^{\infty} W_2(n)q^n = \lim_{N\to\infty} \chi_N(0,1,q^2;1) = \chi(0,1,q^2)$$

$$= \frac{\prod_{n=1}^{\infty}(1-q^{4n})(1+q^{4n})^2}{(q^2;q^2)_\infty}$$

(by Theorem 2 and Jacobi's Triple Product Identity [3; p. 170])

$$= \prod_{n=1}^{\infty}(1+q^{2n})(1+q^{4n})^2$$

$$= \prod_{n=1}^{\infty}\frac{1}{(1-q^{4n-2})(1-q^{8n-4})^2}$$

(by Euler's identity [3; pp. 164–165])

$$= \sum_{n=0}^{\infty} H_2(n)q^n, \tag{3.4}$$

and comparing coefficients of q^n in the extremes of (3.4) we deduce that $W_2(n) = H_2(n)$ for all n.

4. Conclusion

The proof of the conjectures given in Section 3 appears to be unmotivated since the function $X_1(z)$ appeared from nowhere in (2.15). It was discovered from an empirical examination of the lower powers of z in $\chi(1,1,q;z)$. The technique of representing recurrent sequences by determinants and then passing to q-difference equations was utilized in [2] to treat Sylvester's generalization of Euler's theorem. Subsequently, M.D. Hirschhorn [5] has greatly simplified the treatment in [2], and V. Ramamani and K. Venkatachaliengar [6] have given a nice combinatorial treatment.

It seems surprising that an identity as elementary and as elegant as Theorem 3 has not been discovered previously. This naturally leads to the question: For what positive definite quadratic forms $Q(m,n)$ is

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{Q(m,n)}}{(q)_m(q)_n}$$

summable to an infinite product. The only nondiagonal forms I know of are $km^2 + kn^2 - (2k-1)mn$ (k positive integral) and n^2+2m^2+2nm [4; Eq. (1.8)]. Obviously this question can be generalized to k-fold series where a number of such results have recently been found; one such family is given in [4; Th. 1].

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