AN INTRODUCTION TO RAMANUJAN'S "LOST" NOTEBOOK

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1. Introduction. In the spring of 1976, I visited the Trinity College Library at Cambridge University. Dr. Lucy Slater had suggested to me that there were materials deposited there from the estate of the late G. N. Watson which might contain some work on q-series. In one box of materials from Watson's estate I found a number of items written by the famous Indian mathematician S. Ramanujan (1887–1920). The most interesting item in this box was a manuscript of more than one hundred pages in Ramanujan's distinctive handwriting which contains over six hundred mathematical formulae listed one after the other without proof. *It is my contention that this manuscript, or notebook, was written during the last year of Ramanujan's life, after his return to India from England.* My evidence (given in Section 3) for this assertion is all indirect; in the words of Stephen Leacock, "It is what we call circumstantial evidence—the same thing that people are hanged for."

The fascinating story of Ramanujan's short life and brilliant career has been told several times, and the interested reader is referred to the accounts in Ramanujan's *Collected Papers* **[18]**, in *Ramanujan* by G. H. Hardy **[14]**, and in *Ramanujan the Man and the Mathematician* by S. R. Ranganathan **[20]**. There are three famous notebooks written by Ramanujan **[19]**. During the past 60 years, these have formed the basis for numerous papers by many mathematicians, including G. H. Hardy, G. N. Watson, L. J. Rogers, and many others. G. N. Watson **[26]** presented a nice survey of the notebooks in 1931, and B. Berndt **[10]** has written a new survey giving an up-to-date account of recent work related to the notebooks. Watson **[26**, p. 138] suggests that Ramanujan's work on the three famous notebooks concluded around 1913. A fully edited version of them was never completed, although G. N. Watson and B. M. Wilson initiated such a project. In 1957, the Tata Trust brought out a photostat edition of the three notebooks **[19]**.

We shall consider the origin of this "lost" notebook later. Before going further, let us state a few of the marvelous identities that appear in this work:

$$1+\sum_{n=1}^{\infty}\frac{q^j}{\prod_{j=1}^{n}(1+aq^j)\left(1+\frac{q^j}{a}\right)}$$

$$=(1+a)\sum_{n=0}^{\infty}a^{3n}q^{\frac{1}{2}n(3n+1)}(1-a^2q^{2n+1})-a\frac{\sum_{n=0}^{\infty}(-1)^n a^{2n}q^{n(n+1)/2}}{\prod_{j=1}^{\infty}(1+aq^j)\left(1+\frac{q^j}{a}\right)}; \tag{1.1}$$

$$\sum_{n=0}^{\infty}\frac{q^n}{(1+q)(1+q^3)\cdots(1+q^{2n+1})}=\sum_{n=0}^{\infty}(-1)^n q^{6n^2+4n}(1+q^{4n+2}); \tag{1.2}$$

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$$\left[\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{\vdots}}}}}\right]^3=\frac{\sum_{n=0}^{\infty}q^{5n^2+4n}\frac{1+q^{5n+2}}{1-q^{5n+2}}-\sum_{n=0}^{\infty}q^{5n^2+6n+1}\frac{1+q^{5n+3}}{1-q^{5n+3}}}{\sum_{n=0}^{\infty}q^{5n^2+2n}\frac{1+q^{5n+1}}{1-q^{5n+1}}-\sum_{n=0}^{\infty}q^{5n^2+8n+3}\frac{1+q^{5n+4}}{1-q^{5n+4}}};\tag{1.3}$$

$$\frac{G(aq,\lambda q)}{G(a,\lambda)}=\cfrac{1}{1+\cfrac{aq+\lambda q}{1+\cfrac{bq+\lambda q^2}{1+\cfrac{aq^2+\lambda q^3}{1+\cfrac{bq^2+\lambda q^4}{\vdots}}}}},\tag{1.4}$$

where

$$G(a,\lambda)=G(a,\lambda;b;q)=1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}(a+\lambda)(a+\lambda q)\cdots(a+\lambda q^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)(1+bq)(1+bq^2)\cdots(1+bq^n)};$$

$$\frac{1}{1-\alpha}+\sum_{n=1}^{\infty}\frac{\beta^n}{(1-\alpha x^n)(1-\alpha x^{n-1}y)(1-\alpha x^{n-2}y^2)\cdots(1-\alpha y^n)}$$

$$=\frac{1}{1-\beta}+\sum_{n=1}^{\infty}\frac{\alpha^n}{(1-\beta x^n)(1-\beta x^{n-1}y)(1-\beta x^{n-2}y^2)\cdots(1-\beta y^n)}.\tag{1.5}$$

To my knowledge none of these five identities appears in the literature. Identity (1.1) was rediscovered by N. J. Fine in the early 1950's; however, he never published his proof.

Identity (1.2) is a "false" theta series identity. Results like (1.2) were studied by L. J. Rogers [**22**]; however, this elegant result appears to have escaped him. As we shall see, identity (1.2) implies a partition identity like that deduced from Euler's Pentagonal Number Theorem [**5**, p. 10].

Identity (1.3) is related to the famous Rogers–Ramanujan continued fraction [**5**, p. 104]:

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\vdots}}}}=\prod_{n=0}^{\infty}\frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+2})(1-q^{5n+3})}$$

$$=\frac{\sum_{n=0}^{\infty}(-1)^nq^{(5n^2+3n)/2}-\sum_{n=0}^{\infty}(-1)^nq^{(5n^2+7n+2)/2}}{\sum_{n=0}^{\infty}(-1)^nq^{(5n^2+n)/2}-\sum_{n=0}^{\infty}(-1)^nq^{(5n^2+9n+4)/2}};\tag{1.6}$$

however, it lies somewhat deeper.

Identity (1.4) is a new Rogers–Ramanujan type continued fraction from which Ramanujan deduces five corollaries (our equations (7.10)–(7.14)) in which continued fractions are represented by infinite products. The special case $a=0$ was given in Chapter 16 of Ramanujan's older notebooks [**19**, Vol. 2, Eq. (13), p. 195]. In the older notebooks Ramanujan gives a hint for proving the continued fraction identity by giving explicit formulae for the convergents. The proof we shall give relies on some q-series transformations due to Rogers.

Identity (1.5) looks disarmingly simple (and is). Its analytic proof is quite mundane; however, there is an amusing combinatorial proof.

I have chosen these five identities to give some flavor of Ramanujan's achievements in this "lost" notebook. Since there were over 600 to choose from, these results cannot really be called an accurate sample, but merely a tantalizing introduction. I plan to write a series of papers in which I shall organize these formulae into somewhat sensible groupings. I shall prove as many of Ramanujan's formulae as I can, and the rest I shall present for the consideration of others.

2. The mathematical setting for the "lost" notebook. The vast majority of the formulae in the lost notebook (including the results we have chosen) concern q-series and related theta functions. For example, we meet again and again the series

$$\sum_{n=0}^{\infty}\frac{(1-a)(1-aq)\cdots(1-aq^{n-1})(1-b)(1-bq)\cdots(1-bq^{n-1})t^n}{(1-q)(1-q^2)\cdots(1-q^n)(1-c)(1-cq)\cdots(1-cq^{n-1})}. \tag{2.1}$$

This is called the q-analog of the famous hypergeometric series studied by Euler, Gauss, and others:

$$\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)t^n}{n!\ \ c(c+1)\cdots(c+n-1)}. \tag{2.2}$$

It is called the q-analog because the rising factorial $a(a+1)\cdots(a+n-1)$ of the hypergeometric series has been uniformly replaced by the rising q-factorial $(1-a)(1-aq)\cdots(1-aq^{n-1})$. There are many, many results known about the hypergeometric series (2.2) (see [**25**, Chap. 1] for a nice introduction). For example, when $b=c$ in (2.2) we have the famous binomial series theorem:

$$\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)t^n}{n!}=(1-t)^{-a},\qquad |t|<1. \tag{2.3}$$

In parallel we have the q-binomial series theorem if $b=c$ in (2.1):

$$\sum_{n=0}^{\infty}\frac{(1-a)(1-aq)\cdots(1-aq^{n-1})t^n}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=0}^{\infty}\frac{(1-atq^n)}{(1-tq^n)}. \tag{2.4}$$

There is also an integral representation of the hypergeometric function (2.2) [**25**, p. 20]:

$$\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)t^n}{n!\ \ c(c+1)\cdots(c+n-1)}$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-tx)^{-a}\,dx. \tag{2.5}$$

The investigations of q-series have shown that the natural q-analog of (2.5) is not an integral representation for the series in (2.1), but rather an identity (due to Heine) between two such series given in this paper as identity (4.1). L. J. Rogers [**21**] noted that (4.1) could be applied to itself, and as a result he easily deduced the new and significant identities (4.6) and (7.1). Heine [**15**, p. 325] originally discovered (7.1), but he proved it in a more complicated way. Actually, (6.1) is a q-analog of Euler's famous identity for the hypergeometric function [**24**, p. 10].

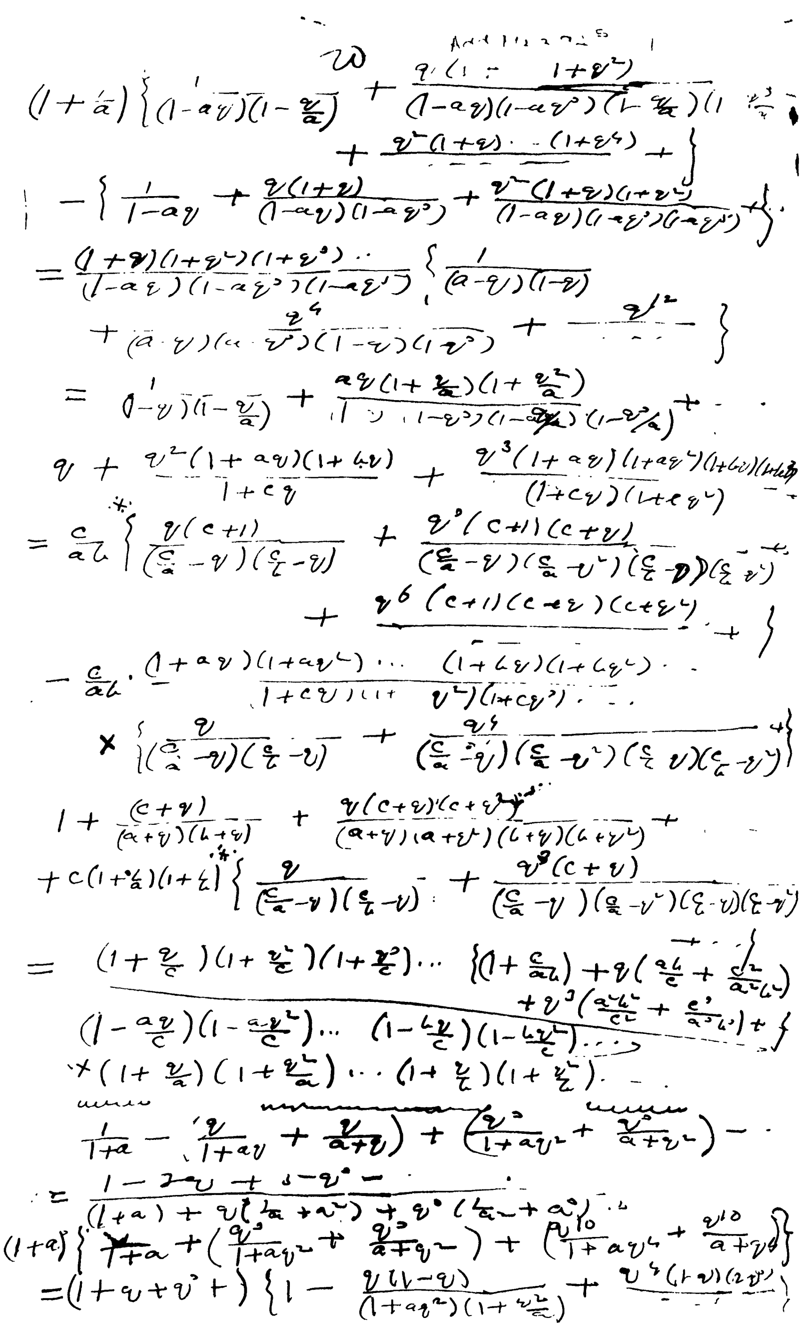

FIG. 1. Power series computations and comparisons from Ramanujan's "Lost" Notebook.

$$\sum_{n=0}^{\infty} \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)t^n}{n!\ \ c(c+1)\cdots(c+n-1)}$$
$$=(1-t)^{c-a-b}\sum_{n=0}^{\infty} \frac{(c-a)(c-a+1)\cdots(c-a+n-1)(c-b)(c-b+1)\cdots(c-b+n-1)t^n}{n!\ \ c(c+1)\cdots(c+n-1)}. \tag{2.6}$$

To understand fully Ramanujan's use of the terms "theta series," "false theta series," and "mock theta series," we quote from Ramanujan's last letter to Hardy [**27**, pp. 57–61] (also mentioned in Section 3 below). After the quotation we shall make a few comments to clarify further some of Ramanujan's ideas.

If we consider a ϑ-function in the transformed Eulerian form, e.g.,

$$1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}+\frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\cdots, \tag{A}$$

$$1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\cdots, \tag{B}$$

and determine the nature of the singularities at the points

$$q=1, q^2=1, q^3=1, q^4=1, q^5=1, \ldots,$$

we know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance, when $q=e^{-t}$ and $t \to 0$,

$$(\mathrm{A})=\sqrt{\left(\frac{t}{2\pi}\right)}\exp\left(\frac{\pi^2}{6t}-\frac{t}{24}\right)+O(1),$$
$$(\mathrm{B})=\sqrt{\left(\frac{2}{5-\sqrt{5}}\right)}\exp\left(\frac{\pi^2}{15t}-\frac{t}{60}\right)+O(1),$$

and similar results at other singularities.

If we take a number of functions like (A) and (B), it is only in a limited number of cases the terms close as above; but in the majority of cases they never close as above. For instance, when $q=e^{-t}$ and $t \to 0$,

$$1+\frac{q}{(1-q)^2}+\frac{q^3}{(1-q)^2(1-q^2)^2}+\frac{q^6}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\cdots$$
$$=\sqrt{\left(\frac{t}{2\pi\sqrt{5}}\right)}\exp\left[\frac{\pi^2}{5t}+a_1t+a_2t^2+\cdots+O(a_kt^k)\right], \tag{C}$$

where $a_1=1/8\sqrt{5}$, and so on. The function (C) is a simple example of a function behaving in an unclosed form at the singularities.

Now a very interesting question arises. Is the converse of the statements concerning the forms (A) and (B) true? That is to say: Suppose there is a function in the Eulerian form and suppose that all or an infinity of points are exponential singularities, and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is: Is the function taken the sum of two functions one of which is an ordinary ϑ-function and the other a (trivial) function which is $O(1)$ at all the points $e^{2m\pi i/n}$? The answer is it is not necessarily so. When it is not so, I call the function a Mock ϑ-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function. Also I have shown that if it is necessarily so then it leads to the following assertion: viz., it is possible to construct two power series in x, namely, $\Sigma a_n x^n$ and $\Sigma b_n x^n$, both of which have essential singularities on the unit circle, are convergent when $|x|<1$, and tend to finite limits at every point $x=e^{2r\pi i/s}$, and at the same time the limit of $\Sigma a_n x^n$ at the point $x=e^{2r\pi i/s}$ is equal to the limit of $\Sigma b_n x^n$ at the point $x=e^{-2r\pi i/s}$.

This assertion seems to be untrue. Anyhow, we shall go to the examples and see how far our assertions are true. I have proved that, if

$$f(q)=1+\frac{q}{(1+q)^2}+\frac{q^4}{(1+q)^2(1+q^2)^2}+\cdots,$$

then

$$f(q)+(1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)=O(1)$$

at all the points $q=-1, q^3=-1, q^5=-1, q^7=-1, \ldots$; and at the same time

$$f(q)-(1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)=O(1)$$

at all the points $q^2=-1, q^4=-1, q^6=-1, \ldots$. Also, obviously, $f(q)=O(1)$ at all the points $q=1, q^3=1, q^5=1, \ldots$. And so $f(q)$ is a Mock ϑ-function.

When $q=-e^{-t}$ and $t\to 0$,

$$f(q)+\sqrt{\left(\frac{\pi}{t}\right)}\exp\left(\frac{\pi^2}{24t}-\frac{t}{24}\right)\to 4.$$

The coefficient of q^n in $f(q)$ is

$$(-1)^{n-1}\frac{\exp\left\{\pi\sqrt{\left(\frac{1}{6}n-\frac{1}{144}\right)}\right\}}{2\sqrt{\left(n-\frac{1}{24}\right)}}+O\left[\frac{\exp\left\{\frac{1}{2}\pi\sqrt{\left(\frac{1}{6}n-\frac{1}{144}\right)}\right\}}{\sqrt{\left(n-\frac{1}{24}\right)}}\right].$$

It is inconceivable that a single ϑ-function could be found to cut out the singularities of $f(q)$.

When Ramanujan refers to a ϑ-function, he apparently means sums, products, and quotients of series of the form

$$\sum_{n=-\infty}^{\infty}(-1)^{\epsilon n}q^{an^2+bn},$$

where $\epsilon=0$ or 1. Note that the full assertions connected with his series (A) and (B) are:

$$1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}+\cdots=\frac{1}{\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}} \tag{A$'$}$$

(an identity of Euler) [5, Chaps. 1 and 2], and

$$1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\cdots=\frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{n(5n-1)/2}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}} \tag{B$'$}$$

(the first Rogers–Ramanujan identity [5, Chap. 7]). The mock theta function $f(q)$ described by Ramanujan is also discussed in Section 3.

Theta functions have had considerable impact on various branches of mathematics, extending from mathematical physics to number theory. R. Bellman [9] has given a charming introduction to the many facets of theta functions. The mock theta functions are related to problems in additive number theory (see [8] and [7]).

Finally there are the "false theta functions." These are simply theta series with the "wrong" signs. For example, both

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+4n}=1-q^2-q^{10}+q^{16}+q^{32}-\cdots$$

and

$$\sum_{n=-\infty}^{\infty} q^{6n^2+4n}=1+q^2+q^{10}+q^{16}+q^{32}+\cdots$$

are theta series; however, the series on the right-hand side of (1.2) is not, in that

$$\sum_{n=0}^{\infty}(-1)^n q^{6n^2+4n}(1+q^{4n+2})=1+q^2-q^{10}-q^{16}+q^{32}+\cdots.$$

These "false" theta series do not seem to possess the analytic interest that Ramanujan describes for the ϑ-series and mock ϑ-series; however, they do crop up in some elegant identities such as (1.2).

There are other results from the general theory of q-series that we shall require in our treatment of the "lost" notebook. However, the ideas covered in this section constitute the fundamental tools in our treatment.

3. The origin of the "lost" notebook. There is no introduction or covering letter with this manuscript. Indeed there are only a few words scattered here and there throughout the manuscript. Concerning this notebook, Miss Rosemary Graham, Manuscript Cataloguer of the Trinity College Library, says: "... the notebook and other material was discovered among Watson's papers by Dr. J. M. Whittaker, who wrote the obituary of Professor Watson for the Royal Society. He passed the papers to Professor R. A. Rankin of Glasgow University, who, in December 1968, offered them to Trinity College so that they might join the other Ramanujan manuscripts already given to us by Professor Rankin on behalf of Professor Watson's widow."

Ramanujan's wife gives the following description of Ramanujan's last year (April 1919–April 1920) before his death: "He returned from England only to die, as the saying goes. He lived for less than a year. Throughout this period I lived with him without break. He was only skin and bones. He often complained of severe pain. In spite of it he was always busy doing his Mathematics. That evidently helped him to forget the pain. I used to gather the sheets of paper which he filled up. I would also give the slate whenever he asked for it." (Quoted from [**20**, p. 91].)

Of this intense mathematical activity, we have up to now only known of the mock theta functions. These functions were described in Ramanujan's last letter to Hardy, written from the University of Madras and dated January 12, 1920 [**18**, p. xxxi]: "I am extremely sorry for not writing you a single letter up to now... I discovered very interesting functions recently which I call 'Mock' ϑ-functions. Unlike the 'False' ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples." Besides the material quoted in Section 2, Ramanujan also defines four third-order mock theta functions, ten fifth-order functions and three seventh-order functions. He also includes three identities satisfied by the third-order functions and five identities satisfied by his first five fifth-order functions. He states that the other five fifth-order functions also satisfy similar identities.

Subsequent authors ([**27**], [**28**], [**12**], [**1**], [**2**], [**3**], [**7**]) have studied the mock theta functions extensively. All of these papers study the work that Ramanujan described in his last letter. Now the "lost" notebook contains all of the formulae for the third- and fifth-order mock theta functions given in Ramanujan's last letter. Furthermore, it contains the five identities for the second family of fifth-order functions that were only mentioned but not stated in the letter. The "lost" notebook also contains one-parameter generalizations of the third-order identities that were rediscovered independently in 1965 [**2**].

It appears that either Watson did not possess the "lost" notebook in the late 1930's when he worked on mock theta functions or he had filed it away and forgotten it. In any event, Watson [**27**, p. 61] says that he believes Ramanujan did not possess transformation formulae for the third-order mock theta functions such as

$$f(q)\prod_{n=1}^{\infty}(1-q^n)=1+4\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}, \tag{3.1}$$

where $f(q)$ is one of Ramanujan's third-order mock theta functions given by

$$f(q)=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}. \tag{3.2}$$

However in the "lost" notebook we find

$$\prod_{n=1}^{\infty}(1-abq^n)\left(1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+aq)(1+aq^2)\cdots(1+aq^n)(1+bq)(1+bq^2)\cdots(1+bq^n)}\right)$$
$$=(1+a)(1+b)\sum_{n=0}^{\infty}\frac{(-1)^n a^n b^n q^{n(3n+1)/2}(1-abq^{2n})}{(1+aq^n)(1+bq^n)(1-ab)}\prod_{j=1}^{n}\frac{(1-abq^{j-1})}{(1-q^j)}, \tag{3.3}$$

which reduces to (3.1) for $a\to 1$, $b\to 1$. Indeed the master identity used by Watson to construct all of his mock theta function transformations is obtained directly from (3.3) by the substitutions $a\to e^{i\theta}$, $b\to e^{-i\theta}$.

Watson rightly suggests the importance of (3.1) by pointing out that he was only able to prove that $f(q)$ really was a mock theta function by utilizing (3.1). In a second paper [**28**], Watson studied the fifth-order mock theta functions and proved the five identities associated with each family. While Watson did succeed in proving that the fifth-order mock theta functions do indeed behave like theta functions [**28**, §6], he was unable to prove that they are not themselves theta functions. This was because he did not know a result comparable to (3.1) for the fifth-order functions. However, the "lost" notebook contains several such results. For example, $\phi_0(q)$ defined by

$$\phi_0(q)=1+\sum_{n=1}^{\infty}q^{n^2}(1+q)(1+q^3)\cdots(1+q^{2n-1}) \tag{3.4}$$

is one of the fifth-order functions, and in the "lost" notebook we find a result equivalent to

$$\phi_0(-q)=\prod_{n=0}^{\infty}\frac{(1-q^{5n+5})(1+q^{5n+2})(1+q^{5n+3})}{(1-q^{10n+2})(1-q^{10n+8})}$$
$$+1-\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(1-q)(1-q^6)\cdots(1-q^{5n+1})(1-q^4)(1-q^9)\cdots(1-q^{5n-1})}. \tag{3.5}$$

Hence by (3.3) with $q\to q^5$ and then $a\to -q$, $b\to -q^{-1}$, we see that

$$\phi_0(-q)=\prod_{n=0}^{\infty}\frac{(1-q^{5n+5})(1+q^{5n+2})(1+q^{5n+3})}{(1-q^{10n+2})(1-q^{10n+8})}$$
$$+1-\prod_{n=0}^{\infty}(1-q^{5n+5})^{-1}\left\{\frac{1}{1-q}+(1-q^{-1})\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(15n+5)/2}(1+q^{5n})}{(1-q^{5n+1})(1-q^{5n-1})}\right\}. \tag{3.6}$$

Now Watson was obviously totally unaware of (3.5) or he surely would have found (3.6). If he had found (3.6), the methods he used to prove that $f(q)$ is a mock theta function should have led him to a proof that the fifth-order functions are indeed true mock theta functions.

In a later paper I hope to examine all of the formulae in the "lost" notebook for the mock theta-functions. For now, I hope I have made the case for my assertion that this notebook was composed during the last year of Ramanujan's life, when, by his own words, he discovered the mock theta functions. I should add that while only a fraction (perhaps 5%) of the notebook is on the mock theta functions themselves, much of the rest of it involves related q-series expansions of theta functions and false theta functions. Thus it is not unreasonable to assume that the entire notebook was composed during Ramanujan's last year, especially since the results on the mock theta functions are scattered through it. Finally, the fact that its existence was never mentioned by anyone for over 55 years leads me to call it "lost." B. Birch [**11**] has found some other notes of Ramanujan's in the library of the Oxford Mathematical Institute; however, these notes comprise only 33 pages and, for the most part, apparently treat different formulae from those found in the "lost" notebook.

4. Proof of identity (1.1). In order to prove (1.1), we shall prove the following stronger result, which is also found in the "lost" notebook:

$$1+\sum_{n=1}^{\infty}\frac{q^n}{(1+aq)(1+aq^2)\cdots(1+aq^n)(1+bq)(1+bq^2)\cdots(1+bq^n)}$$

$$=(1+a^{-1})\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}b^n a^{-n}}{(1+bq)(1+bq^2)\cdots(1+bq^n)}\right)-\frac{a^{-1}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}b^n a^{-n}}{\prod_{j=1}^{\infty}(1+aq^j)(1+bq^j)}. \tag{4.1}$$

To obtain (1.1) from (4.1), first interchange a and b in (4.1) and then set $b=a^{-1}$; identity (1.1) follows once we recall the identity of L. J. Rogers [**22**, p. 335, Eq. (3)]:

$$1+\sum_{n=1}^{\infty}\frac{(-1)^n y^{2n} q^{n(n+1)/2}}{(1-yq)(1-yq^2)\cdots(1-yq^n)}=\sum_{n=0}^{\infty}(-1)^n y^{3n} q^{n(3n+1)/2}(1-y^2q^{2n+1}). \tag{4.2}$$

To simplify working with these series, we introduce the following standard notation:

$$(a)_n=(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1}). \tag{4.3}$$

The symbol $(a)_n$ is defined for all real n by the relation

$$(a)_n=(a;q)_n=\prod_{m=0}^{\infty}\frac{(1-aq^m)}{(1-aq^{m+n})},\qquad |q|<1.$$

Finally, we define

$$(a)_\infty=(a;q)_\infty=\prod_{m=0}^{\infty}(1-aq^m),\qquad |q|<1. \tag{4.4}$$

To prove (4.1) let us recall an identity of Euler [**5**, p. 19]:

$$\sum_{m=0}^{\infty}\frac{A^m q^{m(m+1)/2}}{(q)_m}=(-Aq)_\infty, \tag{4.5}$$

and an identity of L. J. Rogers [**21**, p. 171]:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n\tau^n}{(q)_n(\gamma)_n}=\frac{(\gamma/\beta)_\infty(\beta\tau)_\infty}{(\gamma)_\infty(\tau)_\infty}\sum_{n=0}^{\infty}\frac{(\beta)_m(\alpha\beta\tau/\gamma)_m\gamma^m\beta^{-m}}{(q)_m(\beta\tau)_m}. \tag{4.6}$$

Now multiply both sides of (3.1) by $(-bq)_\infty$ and apply (3.5) with $A=bq^n$:

$$\sum_{n=0}^{\infty}\frac{q^n}{(-aq)_n}\sum_{m=0}^{\infty}\frac{q^{mn+m(m+1)/2}b^m}{(q)_m}$$
$$=(1+a^{-1})\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}b^n a^{-n}\sum_{n=0}^{\infty}\frac{q^{mn+m(m+1)/2}b^m}{(q)_m}-\frac{a^{-1}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}b^n a^{-n}}{(-aq)_\infty}. \tag{4.7}$$

If we compare coefficients of b^N on both sides of (4.7) and multiply the resulting identity by $q^{-N(N+1)/2}$, we find that (4.1) is equivalent to

$$\frac{1}{(q)_N}\sum_{n=0}^{\infty}\frac{q^{(N+1)n}}{(-aq)_n}=a^{-N}(1+a^{-1})\sum_{m=0}^{N}\frac{(-1)^{N+m}a^m}{(q)_m}-\frac{(-1)^N a^{-N-1}}{(-aq)_\infty}. \tag{4.8}$$

To prove (4.8) we set $\beta=q,\alpha=0,\gamma=-aq$, and $\tau=q^{N+1}$ in (4.6); hence after dividing by $(q)_N$ we find

$$\frac{1}{(q)_N}\sum_{n=0}^{\infty}\frac{q^{(N+1)n}}{(-aq)_n}=\frac{1}{(q)_N}\frac{(q^{N+2})_\infty(1+a)}{(q^{N+1})_\infty}\sum_{n=0}^{\infty}\frac{(-a)^n}{(q^{N+2})_n}$$
$$=(1+a)(-a)^{-N-1}\sum_{n=0}^{\infty}\frac{(-a)^{n+N+1}}{(q)_{n+N+1}}$$
$$=(1+a)(-a)^{-N-1}\left\{\sum_{n=0}^{\infty}\frac{(-a)^n}{(q)_n}-\sum_{n=0}^{N}\frac{(-a)^n}{(q)_n}\right\}$$
$$=(1+a)(-a)^{-N-1}\left\{\frac{1}{(-a)_\infty}-\sum_{n=0}^{N}\frac{(-a)^n}{(q)_n}\right\}$$
$$=\frac{-(-1)^N a^{-N-1}}{(-aq)_\infty}+a^{-N}(1+a^{-1})\sum_{n=0}^{N}\frac{(-1)^{n+N}a^n}{(q)_n}, \tag{4.9}$$

which is (4.8). Thus (4.1) is established. Note that in the penultimate step in (4.9) we used a second identity of Euler [5, p. 19]:

$$\sum_{n=0}^{\infty}\frac{A^n}{(q)_n}=\frac{1}{(A)_\infty}. \tag{4.10}$$

There exist generalizations of (4.1), and we shall discuss these later in this series of papers.

5. Proof of identity (1.2). To prove this result we shall require (4.2) and (4.6) as well as the fundamental transformation of E. Heine [15, p. 306]:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n\tau^n}{(q)_n(\gamma)_n}=\frac{(\beta)_\infty(\alpha\tau)_\infty}{(\gamma)_\infty(\tau)_\infty}\sum_{m=0}^{\infty}\frac{(\gamma/\beta)_m(\tau)_m\beta^m}{(q)_m(\alpha\tau)_m}. \tag{5.1}$$

To begin we replace q by q^2 in (5.1) and then set $\alpha=0,\beta=q^2,\gamma=q^3$ and $\tau=-q$; after dividing both sides of the resulting identity by $(1-q)$ we find that

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(q;q^2)_{n+1}}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty(-q;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(q;q^2)_m(-q;q^2)_m q^{2m}}{(q^2;q^2)_m}$$

$$=(q^2;q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{2m}}{(q^2;q^2)_m} \frac{1}{(q^{4m+2};q^4)_\infty}$$

$$=(q^2;q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{2m}}{(q^2;q^2)_m} \sum_{n=0}^{\infty} \frac{q^{n(4m+2)}}{(q^4;q^4)_n} \quad \text{(by (4.10))}$$

$$=(q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^4;q^4)_n} \frac{1}{(q^{4n+2};q^2)_\infty} \quad \text{(by (4.10))}$$

$$=\sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n} q^{2n}}{(q^4;q^4)_n}$$

$$=\sum_{n=0}^{\infty} (q^2;q^4)_n q^{2n}. \tag{5.2}$$

Now in (4.6) replace q by q^4, then set $\beta=q^4, \alpha=q^2, \tau=q^2$, and let $c\to 0$; this yields

$$\sum_{n=0}^{\infty} (q^2;q^4)_n q^{2n} = \sum_{m=0}^{\infty} \frac{(-1)^m q^{2m^2+2m}}{(q^2;q^4)_{m+1}}$$

$$=-\sum_{m=1}^{\infty} \frac{(-1)^m q^{2m^2-2m}}{(q^2;q^4)_m}$$

$$=1-\sum_{n=0}^{\infty} (-1)^n q^{6n^2-4n}(1-q^{8n}) \quad \text{by (4.2)}$$

$$=\sum_{n=0}^{\infty} (-1)^n q^{6n+4n}(1+q^{4n+2}). \tag{5.3}$$

Now if we equate the first member of (5.2) with the final member of (5.3), we deduce (1.2) upon replacing q by $-q$.

The following partition theorem is easily deduced from (1.2); it would be nice to have a combinatorial proof of this result:

THEOREM. *Let $R_i(n)$ denote the number of partitions of n into odd parts wherein the largest part is congruent to i* (mod 4) *and appears an odd number of times while all other parts appear an even number of times. Then*

$$R_1(n)-R_3(n)=\begin{cases}(-1)^j & \text{if } n=12j^2+8j+1 \quad \text{or} \quad 12j^2+16j+5\\ 0 & \text{otherwise.}\end{cases} \tag{5.4}$$

Proof. We note that

$$1+\sum_{n=1}^{\infty} (R_1(n)-R_3(n))q^n = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{(1-q^2)(1-q^6)\cdots(1-q^{4n+2})}. \tag{5.5}$$

since

$$\frac{1}{(1-q^{1+1})(1-q^{3+3})\cdots(1-q^{(2n+1)+(2n+1)})} \tag{5.6}$$

is the generating function for partitions with odd parts each $\leqslant 2n+1$ and each appearing an even number of times. Our theorem now follows if we replace q by $-q^2$ in (1.2) and multiply the resulting identity by q.

For example, when $n=5$ the partitions enumerated by $R_1(5)$ are 5 and $1+1+1+1+1$ while that enumerated by $R_3(5)$ is $3+1+1$; hence $R_1(5)-R_3(5)=2-1=1$ as predicted. When $n=11$, $R_1(11)$ enumerates $9+1+1, 5+3+3, 5+1+1+1+1+1+1$ and $1+1+1+1+1+1+1+1+1+1+1$ while $R_3(11)$ enumerates $11, 7+1+1+1+1, 3+3+3+1+1, 3+1+1+1+1+1+1+1+1$; hence $R_1(11)-R_3(11)=4-4=0$ as predicted.

Later in this series we shall present further partition theorems implied by Ramanujan's work in this "lost" notebook.

6. Proof of identity (1.3). This continued fraction identity depends both on the well-known Rogers–Ramanujan continued fraction (1.6) and on what has become known as Ramanujan's ${}_1\psi_1$ summation:

$$
\begin{aligned}
{}_1\psi_1\left[\begin{matrix} a; q, t \\ b \end{matrix}\right] &= \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} t^n \\
&= \frac{(b/a)_\infty (az)_\infty (q/az)_\infty (q)_\infty}{(q/a)_\infty (b/az)_\infty (b)_\infty (z)_\infty}. \qquad (6.1)
\end{aligned}
$$

Proofs of (6.1) can be found in [4], [6], [7].

We start on (1.3) by proving an elementary series identity: for $i=1, 2, 3$, or 4

$$
\sum_{n=0}^{\infty} q^{5n^2+2in} \frac{1+q^{5n+i}}{1-q^{5n+i}} = \sum_{n=0}^{\infty} \frac{q^{in}}{1-q^{5n+i}}. \qquad (6.2)
$$

This identity holds because

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{in}}{1-q^{5n+i}} &= \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} q^{in+5nm+mi} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} + \sum_{m=n+1}^{\infty} \right) q^{in+5nm+im} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q^{i(n+m)+5(n+m)n+im} \\
&\quad + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q^{in+5n(m+n+1)+i(m+n+1)} \\
&= \sum_{m=0}^{\infty} \frac{q^{5m^2+2im}}{1-q^{5m+i}} + \sum_{n=0}^{\infty} \frac{q^{5n^2+5n+2in+i}}{1-q^{5n+i}} \\
&= \sum_{n=0}^{\infty} q^{5n^2+2in} \frac{1+q^{5n+i}}{1-q^{5n+i}}. \qquad (6.3)
\end{aligned}
$$

Next we see that by (6.1) for $i=1, 2, 3$, or 4

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{q^{in}}{1-q^{5n+i}} &= \frac{1}{1-q^i} {}_1\psi_1\left[\begin{matrix} q^i; q^5, q^i \\ q^{5+i} \end{matrix}\right] \\
&= \frac{(q^5;q^5)_\infty^2 (q^{2i};q^5)_\infty (q^{5-2i};q^5)_\infty}{(a^{5-i};q^5)_\infty^2 (q^i;q^5)_\infty^2}, \qquad (6.4)
\end{aligned}
$$

Hence

$$\frac{\sum_{n=0}^{\infty} q^{5n^2+4n}\frac{1+q^{5n+2}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} q^{5n^2+6n+1}\frac{1+q^{5n+3}}{1-q^{5n+3}}}{\sum_{n=0}^{\infty} q^{5n^2+2n}\frac{1+q^{5n+1}}{1-q^{5n+1}} - \sum_{n=0}^{\infty} q^{5n^2+8n+3}\frac{1+q^{5n+4}}{1-q^{5n+4}}}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{q^{2n}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1-q^{5n+3}}}{\sum_{n=0}^{\infty} \frac{q^{n}}{1-q^{5n+1}} - \sum_{n=0}^{\infty} \frac{q^{4n+3}}{1-q^{5n+4}}} \quad \text{(by (6.3))}$$

$$= \frac{\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}}}{\sum_{n=-\infty}^{\infty} \frac{q^{n}}{1-q^{5n+1}}}$$

$$= \frac{(q^5;q^5)_\infty^2(q^4;q^5)_\infty(q;q^5)_\infty}{(q^3;q^5)_\infty^2(q^2;q^5)_\infty^2} \cdot \frac{(q^4;q^5)_\infty^2(q;q^5)_\infty^2}{(q^5;q^5)_\infty^2(q^2;q^5)_\infty(q^3;q^5)_\infty}$$

$$= \left(\frac{(q^4;q^5)_\infty(q;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\right)^3$$

$$= \left[\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\ddots}}}}\right]^3 \quad \text{(by (1.6)).}$$

Also in the "lost" notebook are more than a dozen identities deduced from special cases of (6.4) combined with (1.6).

7. Proof of identity (1.4). Instead of determining the convergents of this continued fraction, we deduce this identity from a family of relations. To obtain this family, we require a further identity of E. Heine [**15**, p. 325] (see also L. J. Rogers [**21**, p. 171]):

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n\tau^n}{(q)_n(\gamma)_n} = \frac{(\alpha\beta\tau/\gamma)_\infty}{(\tau)_\infty} \sum_{n=0}^{\infty} \frac{(\gamma/\alpha)_n(\gamma/\beta)_n}{(q)_n(\gamma)_n}\left(\frac{\alpha\beta\tau}{\gamma}\right)^n. \tag{7.1}$$

Next we define for nonnegative integral i:

$$F_i(a,\lambda;b;q) = \sum_{n=0}^{\infty} \frac{q^{(n+2i+1)n/2}(-\lambda a^{-1}q^i)_n a^n}{(q)_n(-bq)_{n+i}}, \tag{7.2}$$

$$H_i(a,\lambda;b;q) = \sum_{n=0}^{\infty} \frac{q^{(n+2i+3)n/2}(-\lambda a^{-1}q^i)_n a^n}{(q)_n(-bq)_{n+i}}. \tag{7.3}$$

Immediately from (7.2) and (7.3) we obtain our first relation:

$$F_i(a,\lambda;b;q)-H_i(a,\lambda;b;q) = \sum_{n=0}^{\infty}\frac{q^{(n+2i+1)n/2}(-\lambda a^{-1}q^i)_n a^n}{(q)_n(-bq)_{n+i}}(1-q^n)$$

$$=\sum_{n=0}^{\infty}\frac{q^{(n+2i+2)(n+1)/2}(-\lambda a^{-1}q^i)_{n+1}a^{n+1}}{(q)_n(-bq)_{n+i+1}}$$

$$=(aq^{i+1}+\lambda q^{2i+1})\sum_{n=0}^{\infty}\frac{q^{(n+2(i+1)+1)n/2}(-\lambda a^{-1}q^{i+1})_n a^n}{(q)_n(-bq)_{n+i+1}}$$

$$=(aq^{i+1}+\lambda q^{2i+1})F_{i+1}(a,\lambda;b;q). \tag{7.4}$$

Next we transform $F_i(a,\lambda;b;q)$ and $H_i(a,\lambda;b;q)$ utilizing (6.1). First in (7.1) we set

$$\gamma=-bq^{i+1},\quad \tau=-\frac{q^{i+2}a}{\alpha},\quad \beta=-\gamma a^{-1}q^i$$

and let $\alpha\to\infty$:

$$H_i(a,\lambda;b;q)=(-\lambda b^{-1}q^{i+1})_\infty\sum_{n=0}^{\infty}\frac{(ba\lambda^{-1})_n(-\lambda b^{-1}q^{i+1})^n}{(q)_n(-bq)_{n+i}}; \tag{7.5}$$

next we set

$$\gamma=-bq^{i+1},\tau=-\frac{q^{i+1}a}{\alpha},\beta=-\lambda a^{-1}q^i$$

and let $\alpha\to\infty$:

$$F_i(a,\lambda;b;q)=(-\lambda b^{-1}q^i)_\infty\sum_{n=0}^{\infty}\frac{(ba\lambda^{-1})_n(-\lambda b^{-1}q^i)^n}{(q)_n(-bq)_{n+i}}. \tag{7.6}$$

From (7.5) and (7.6) we obtain our second relation:
$H_i(a,\lambda;b;q)-F_{i+1}(a,\lambda;b;q)$

$$=(-\lambda b^{-1}q^{i+1})_\infty\sum_{n=0}^{\infty}\frac{(ba\lambda^{-1})_n(-\lambda b^{-1}q^{i+1})^n}{(q)_n(-bq)_{n+i+1}}(1+bq^{n+i+1}-1)$$

$$=(bq^{i+1}+\lambda q^{2i+2})(-\lambda b^{-1}q^{i+2})_\infty\sum_{n=0}^{\infty}\frac{(ba\lambda^{-1})_n(-\lambda b^{-1}q^{i+2})^n}{(q)_n(-bq)_{n+i+1}}$$

$$=(bq^{i+1}+\lambda q^{2i+2})H_{i+1}(a,\lambda;b;q). \tag{7.7}$$

Finally we note that from the definition of $G(a,\lambda)$ given after (1.4), we have

$$G(a,\lambda)=F_0(a,\lambda;b;q), \tag{7.8}$$

and

$$G(aq,\lambda q)=H_0(a,\lambda;b;q). \tag{7.9}$$

We now deduce (1.4) from iterated application of (7.7) and (7.4):

$$\frac{G(aq,\lambda q)}{G(a,\lambda)}=\cfrac{1}{\cfrac{F_0(a,\lambda;b;q)}{H_0(a,\lambda;b;q)}}\quad \text{(by (7.8) and (7.9))}$$

$$=\cfrac{1}{1+\cfrac{aq+\lambda q}{\cfrac{H_0(a,\lambda;b;q)}{F_1(a,\lambda;b;q)}}} \quad \text{(by (7.4))}$$

$$=\cfrac{1}{1+\cfrac{aq+\lambda q}{1+\cfrac{bq+\lambda q^2}{\cfrac{F_1(a,\lambda;b;q)}{H_1(a,\lambda;b;q)}}}} \quad \text{(by (7.7))}$$

$$=\cfrac{1}{1+\cfrac{aq+\lambda q}{1+\cfrac{bq+\lambda q^2}{1+\cfrac{aq^2+\lambda q^3}{\cfrac{H_1(a,\lambda;b;q)}{F_2(a,\lambda;b;q)}}}}} \quad \text{(by (7.4))}$$

$$\vdots \quad .$$

Convergence of the continued fraction is assured by the fact that both H_i/F_{i+1} and F_i/H_i are analytic in $|q|<1$ with each of the form $1+O(q^{i+1})$.

Having proved (1.4), let us now examine the corollaries discovered by Ramanujan:

$$\prod_{n=0}^{\infty}\frac{(1-q^{2n+1})}{(1-q^{4n+2})^2}=\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2+q}{1+\cfrac{q^3}{1+\cfrac{q^4+q^2}{1+\vdots}}}}} \quad . \tag{7.10}$$

Identity (7.10) follows if we set $a=0$, $b=1$, $\lambda=1$ in (1.4) and recall (4.5), which implies

$$G(0,\lambda)=\sum_{n=0}^{\infty}\frac{q^{n^2}\lambda^n}{(q^2;q^2)_n}$$
$$=(-\lambda q;q^2)_\infty,$$

so

$$\frac{G(0,q)}{G(0,1)}=\frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty}=\frac{1}{(q^2;q^4)_\infty(-q;q^2)_\infty} \quad \text{(by [5, p. 5, Eq. (1.2.5)])} \quad =\frac{(q;q^2)_\infty}{(q^2;q^4)_\infty^2}.$$

Next we have an identity originally discovered by Eisenstein [13, p. 36]:

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}=\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2-q}{1+\cfrac{q^3}{1+\cfrac{q^4-q^2}{\vdots}}}}} \quad . \tag{7.11}$$

This result follows by our setting $a=0$, $\lambda=1$, $b=-1$ in (1.4) and noting

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}\tau^n}{(q)_n(\gamma)_n} = \frac{1}{(\gamma)_\infty} \sum_{m=0}^{\infty} \frac{(\tau/\gamma)_m}{(q)_m}(-1)^m \gamma^m q^{m(m-1)/2},$$

which follows from (3.6) if we replace τ by $\tau\alpha^{-1}\beta^{-1}$ and let α and β tend to infinity. Hence

$$\frac{G(0,q;-1;q)}{G(0,1;-1;q)} = \frac{\displaystyle\sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q)_m^2}}{\displaystyle\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m^2}} = \frac{\displaystyle\frac{1}{(q)_\infty} \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)/2}}{1/(q)_\infty} = \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)/2}.$$

$$\prod_{n=0}^{\infty} \frac{(1-q^{6n+1})(1-q^{6n+5})}{(1-q^{6n+3})^2} = \cfrac{1}{1+\cfrac{q+q^2}{1+\cfrac{q^2+q^4}{1+\cfrac{q^3+q^6}{1+\vdots}}}} \, . \tag{7.12}$$

To obtain this result we replace q by q^2 in (1.4), then we set $a=q^{-1}$, $b=1$, $\lambda=1$. Then we observe

$$G(q^{-1},1;1;q^2) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty (q^3;q^3)_\infty (q^3;q^6)_\infty}{(q^2;q^2)_\infty},$$

a result given by Slater [24, p. 154, Eq. (25)], and finally

$$G(q,q^2;1;q^2) = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty (q^6;q^6)_\infty (q;q^6)_\infty (q^5;q^6)_\infty}{(q^2;q^2)_\infty}.$$

This last formula has apparently not been stated in the literature before; however, it is easily deduced if we set $y=-q^{1/2}$ and let $z\to\infty$ in Slater's identity E(4) [23, p. 469].

$$\prod_{n=0}^{\infty} \frac{(1-q^{8n+1})(1-q^{8n+7})}{(1-q^{8n+3})(1-q^{8n+5})} = \cfrac{1}{1+\cfrac{q^2+q}{1+\cfrac{q^4}{1+\cfrac{q^6+q^3}{\vdots}}}} \, . \tag{7.13}$$

Here we replace q by q^2 in (1.4); then we set $a=q^{-1}$, $b=0$, $\lambda=1$. Under these substitutions

$$\frac{G(q,q^2;0;q^2)}{G(q^{-1},1;0;q^2)}=\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n}}$$

$$=\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty},$$

where the last equation follows from two identities of Slater [**24**, p. 155, Eqs. (34) and (36)].

Besides the case $a=b=0,\lambda=1$ which yields (1.6), Ramanujan presented one other corollary which involves a false theta function.

$$\sum_{n=0}^{\infty}(-1)^n q^{3n^2+2n}(1+q^{2n+1})=\cfrac{1}{1+\cfrac{q^2-q}{1+\cfrac{q^4-q^2}{1+\cfrac{q^6-q^3}{\vdots}}}}. \tag{7.14}$$

To obtain this result we replace q by q^2 in (1.4); then we set $a=-q^{-1}$, $b=-1$, $\lambda=1$. Thus

$$\frac{G(-q,q^2;-1;q^2)}{G(-q^{-1},1;-1;q^2)}=\lim_{\tau\to\infty}\frac{\sum_{n=0}^{\infty}\frac{(q/\tau;q^2)_n(q;q^2)_n\tau^n q^{2n}}{(q^2;q^2)_n(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(q/\tau;q^2)_n(q;q^2)_n\tau^n}{(q^2;q^2)_n(q^2;q^2)_n}}$$

$$=\frac{\frac{(q^3;q^2)}{(q^2;q^2)}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}}{(q^3;q^2)_n}}{\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}} \quad \text{(by (4.6))}$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{q^{n^2+n}}{(q;q^2)_{n+1}}$$

$$=\sum_{n=0}^{\infty}(-1)^n q^{3n^2+2n}(1+q^{2n+1}),$$

where the last equation is due to L. J. Rogers [**22**, p. 333, Eq. (4)].

8. Proof of identity (1.5). This result is a good deal easier than the others we have considered. First we give an analytic proof. For this proof we require a special case of the q-analog of the binomial series (2.4) [**5**, p. 36, Eq. (3.37)]:

$$\frac{1}{(z)_{n+1}}=\sum_{m=0}^{\infty}\frac{(q)_{n+m}}{(q)_n(q)_m}z^m. \tag{8.1}$$

Thus the left-hand side of (1.5) is

$$\sum_{n=0}^{\infty} \frac{\beta^n}{(\alpha x^n)_{n+1}}$$

where $q=y/x$. Hence

$$\sum_{n=0}^{\infty} \frac{\beta^n}{(\alpha x^n)_{n+1}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \beta^n \alpha^m \frac{(q)_{n+m}}{(q)_n (q)_m} x^{nm},$$

and the symmetry in α and β is now obvious. Hence (1.5) is established.

Combinatorially we see that the coefficient of $\alpha^m \beta^n$ in

$$\sum_{n=0}^{\infty} \frac{\beta^n}{(1-\alpha x^n)(1-\alpha x^{n-1}y)\cdots(1-\alpha y^n)} \tag{8.2}$$

is a homogeneous polynomial in x and y of degree nm. In this polynomial the coefficient of $y^j x^{nm-j}$ is the number of partitions of j into at most m parts each at most n, say $j = \alpha_1 + \cdots + \alpha_r$. Then the exponent on x is

$$nm - j = (n-\alpha_1) + (n-\alpha_2) + \cdots + (n-\alpha_r) + \underbrace{n + \cdots + n}_{m-r \text{ terms}}$$

which is another partition of at most m parts each $\leqslant n$ and it is uniquely determined by $\alpha_1 + \cdots + \alpha_r$, n and m. Now if we conjugate all the partitions under consideration we merely interchange the roles of n and m. Hence (8.2) is symmetric in α and β.

9. Conclusion. This is the first of several papers that will treat Ramanujan's "lost" notebook. I hope that the results discussed here indicate that Ramanujan discovered many results of interest in the last year of his life. I wish to close by quoting the last two paragraphs of G. N. Watson's presidential address to the London Mathematical Society in 1935 [27, p. 80]. In this address Watson has just finished an extensive discussion of the mock theta functions based on the last letter of Ramanujan, which we quoted in Sections 2 and 3. Watson concludes as follows:

> The study of Ramanujan's work and of the problems to which it gives rise inevitably recalls to mind Lamé's remark that, when reading Hermite's papers on modular functions, "on a la chair de poule." I would express my own attitude with more prolixity by saying that such a formula as
>
> $$\int_0^{\infty} e^{-3\pi x^2} \frac{\sinh \pi x}{\sinh 3\pi x}\, dx = \frac{1}{e^{2\pi/3}\sqrt{3}} \sum_{n=0}^{\infty} \frac{e^{-2n(n+1)\pi}}{(1+e^{-\pi})^2(1+e^{-3\pi})^2 \cdots (1+e^{-(2n+1)\pi})^2}$$
>
> gives me a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Capelle Medicee and see before me the austere beauty of the four statues representing Day, Night, Evening, and Dawn which Michelangelo has set over the tombs of Guiliano de'Medici and Lorenzo de'Medici.
>
> Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine where
>
> "Pale, beyond porch and portal,
> Crowned with calm leaves, she stands
> Who gathers all things mortal
> With cold immortal hands."

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