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Plane Partitions (I): The MacMahon Conjecture†

GEORGE E. ANDREWS‡

Mathematics Research Center
University of Wisconsin
Madison, Wisconsin

1. INTRODUCTION

In 1898 MacMahon [7] presented his first study of symmetric higher dimensional partitions. The most interesting part of his paper is a conjecture concerning the generating function for $M(j, s; n)$, the number of plane partitions $\sum n_{ik}$ of n with the added conditions that (i) $n_{ik} = n_{ki}$, (ii) $n_{ik} = 0$ if $i > s$, (iii) $n_{11} \leqslant j$. Thus $M(j, s; n)$ is the number of symmetric plane partitions with at most s rows and with each part at most j. MacMahon [7, p. 153] conjectures

$$\sum_{N \geqslant 0} M(j, s; N) q^N = \prod_{i=1}^{s} \left[\frac{(1 - q^{j+2i-1})}{(1 - q^{2i-1})} \prod_{h=i+1}^{s} \frac{(1 - q^{2(j+i+h-1)})}{(1 - q^{2(i+h-1)})} \right]. \quad (1.1)$$

He demonstrates the truth of this conjecture in a few instances and remarks: "The proof of this formula, the truth of which seems unquestionable, is much to be desired."

Subsequently, in his monumental treatise "Combinatory Analysis," MacMahon [8, pp. 262–271] again discusses this conjecture at length, and again he asserts that, "The result has not been rigorously established." At the conclusion of his discussion [8, p. 270–271] MacMahon carefully examines the symmetry in the product in (1.1) and states that, "This property of the enumerating function is of great beauty and mathematical elegance."

In the late 1960s, Gordon [3, 4] proved MacMahon's conjecture when $s = \infty$. Gordon [4, p. 158] observes that Sylvester's mapping of self-conjugate partitions into partitions with distinct odd parts may be directly extended to plane partitions to show that $M(j, s; n)$ is also the number of plane partitions of n with strict decrease along rows where each part is odd and at most $2s - 1$ and there are at most j rows.

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‡ Present address: Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania.

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Also in the late 1960s, Bender and Knuth [2] developed a very powerful combinatorial method for treating many problems in plane partitions. They also extended much of the work of Gordon [3, 4] and Gordon and Houten [5, 6], and they showed [2, p. 50] that if

$$g_j(q) = \sum_{N \geqslant 0} M(j, m; N) q^N, \tag{1.2}$$

then

$$g_{2n}(q) = \det(C_{i-j} + C_{i+j-1})_{n \times n}, \tag{1.3}$$

and

$$g_{2n+1}(q) = \left[\prod_{i=1}^{m} (1 + q^{2i-1})\right] \det(C_{i-j} - C_{i+j})_{n \times n}, \tag{1.4}$$

where

$$C_k = q^{k^2} \binom{2m}{m+k}_2, \tag{1.5}$$

and $\binom{N}{M}_r$ is the Gaussian polynomial (or q-binomial coefficient) defined by

$$\binom{N}{M}_r = \begin{cases} \dfrac{(1-q^{Nr})(1-q^{(N-1)r}) \cdots (1-q^{(N-M+1)r})}{(1-q^{Mr})(1-q^{(M-1)r}) \cdots (1-q^r)}, & 0 < M \leqslant N \\ 1, & M = 0 \\ 0, & M < 0, \quad M > N. \end{cases} \tag{1.6}$$

However, Bender and Knuth [2, p. 50] go on to state that, "We have not been able to simplify these determinants any further, even for the limiting case as $q \to 1$ . . . But the known results, and calculations for small j give overwhelming empirical evidence that the answer has a simple form."

The object of this paper is to prove MacMahon's conjecture. In Section 2, we prove preliminary lemmas that are simply results from basic hypergeometric series. In Sections 3 and 4, we prove the conjecture by transforming the determinants in (1.3) and (1.4) three times each. In each case the third transformation produces a lower triangular determinant, and MacMahon's conjecture follows immediately.

In a subsequent paper we hope to treat a second conjecture of this nature due to Bender and Knuth [2, p. 50]. Presumably, our methods are adequate to treat it also. In [10, p. 265], Stanley mentions that Gordon possesses (unpublished) a proof of this latter conjecture; however, the implication from Stanley's comments is that Gordon's methods differ substantially from ours.

2. Summation Lemmas

First we require two results that are merely extensions of the q-analog of the Chu–Vandermonde summation [1, p. 469, Theorem 4.2]:

Lemma 1. *For integers* $m \geqslant 0$, $i \geqslant 0$, $l \geqslant 1$,

$$\sum_{j=-\infty}^{\infty} \binom{2m}{m+i-j}_2 q^{(i-j)^2} \binom{2l-1}{l-j}_2 \left(q^{(j-l)(j-l-2m)} + q^{(j+l-1)(j+l+2m-1)}\right) = \binom{2m+2l-1}{m+i+l-1}_2 q^{(i-l)^2}\left(1 + q^{(2i-1)(2l-1)}\right).$$

Proof. We split the left sum into two parts:

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \binom{2m}{m+i-j}_2 q^{(i-j)^2} \binom{2l-1}{l-j}_2 q^{(j-l)(j-l-2m)} &= \sum_{j=0}^{2l-1} \binom{2m}{m+i-l+j}_2 \binom{2l-1}{j}_2 q^{(i-l+1)^2+j(j+2m)} \\ &= q^{(i-l)^2} \sum_{j=0}^{2l-1} \binom{2m}{m+i-l+j}_2 \binom{2l-1}{j}_2 q^{2j(j+m+i-l)} \\ &= q^{(i-l)^2} \binom{2m+2l-1}{m+i+l-1}_2, \end{aligned} \tag{2.1}$$

by the Chu–Vandermonde summation [1, p. 469, Theorem 4.2]. Next

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \binom{2m}{m+i-j}_2 q^{(i-j)^2} \binom{2l-1}{l-j}_2 q^{(j+l-1)(j+l+2m-1)} &= \sum_{j=0}^{2l+1} \binom{2m}{m+i-j+l-1}_2 q^{(i-j-l-1)^2} \binom{2l-1}{2l-1-j}_2 q^{j(j+2m)} \\ &= q^{(i+l-1)^2} \sum_{j=0}^{2l-1} \binom{2m}{m-i-l+1+j}_2 \binom{2l-1}{j}_2 q^{2j(j+m-i-l+1)} \\ &= q^{(i+l-1)^2} \binom{2m+2l-1}{m+i+l-1}_2, \end{aligned} \tag{2.2}$$

again by the Chu–Vandermonde summation [1, p. 469, Theorem 4.2].

If we now add together identities (2.1) and (2.2) we obtain Lemma 1. ∎

Lemma 2. *For integers* $m \geqslant 0$, $i \geqslant 0$, $l \geqslant 0$,

$$\sum_{j=-\infty}^{\infty} \binom{2m}{m+i-j}_2 q^{(i-j)^2} \binom{2l}{l+j}_2 \left(q^{(j-l)(j-l-2m)} - q^{(j+l)(j+l+2m)}\right) = \binom{2m+2l}{m+i+l}_2 \left(q^{(i-l)^2} - q^{(i+l)^2}\right).$$

Proof. We split the left sum into two parts:

$$\sum_{j=-\infty}^{\infty} \binom{2m}{m+i-j}_2 q^{(i-j)^2} \binom{2l}{l+j}_2 q^{(j-l)(j-l-2m)}$$
$$= \sum_{j=0}^{2l} \binom{2m}{m-i-l+j}_2 q^{(i-l+j)^2} \binom{2l}{2l-j}_2 q^{j(j+2m)}$$
$$= q^{(i-l)^2} \sum_{j=0}^{2l} \binom{2m}{m+i-l-j}_2 \binom{2l}{j}_2 q^{2j(j+m+i-l)}$$
$$= q^{(i-l)^2} \binom{2m-2l}{m-i-l}_2, \tag{2.3}$$

by the Chu–Vandermonde summation [1, p. 469, Theorem 4.2]. Finally,

$$\sum \binom{2m}{m+i-j}_2 q^{(i-j)2} \binom{2l}{l+j}_2 q^{(j+l)(j+l+2m)}$$
$$= \sum_{j=0}^{2l} \binom{2m}{m+i-l-j}_2 q^{(i-j+l)^2} \binom{2l}{j}_2 q^{j(j+2m)}$$
$$= q^{(i+l)^2} \sum_{j=0}^{2l} \binom{2m}{m-i+l-j}_2 \binom{2l}{j}_2 q^{2j(j+m-i-l)}$$
$$= q^{(i+l)^2} \binom{2m+2l}{m+i+l}_2, \tag{2.4}$$

by the Chu–Vandermonde summation [1, p. 469, Theorem 4.2].

If we now subtract identity (2.4) from identity (2.3), we obtain Lemma 2. ∎

Our next result is a disguised form of the q-analog of the Pfaff–Saalschutz summation [9, p. 97, Eq. (3.3.2.2)].

LEMMA 3. *For integers $n \geqslant k \geqslant 1$, $b = 0$ or 1, $i \geqslant 0$, $m \geqslant 0$,*

$$\sum_{j=1}^{n} \binom{2m+2j-b}{m-i+j-b}_1 \frac{(-1)^{j+k}(q^{2m+2j+1-b})_{2k-2j} q^{\frac{1}{2}j(j+1)-kj}}{(q^{m+j+k-b})_{k-j}(q^{m+j})_{k-j}} \binom{k-1}{j-1}_1$$
$$= \frac{(-1)^{k+1} q^{1-k}(q)_{2m+2k-b}(q)_m(q^{1-i})_{k-1}(q^{b+1-i-k})_{k-1}(q)_{m+k-b}}{(q)_{m+2k-b-1}(q)_{m-k-1}(q)_{m+i-b-1}(q)_{m+k-i}(q^{b-i-m-k})_{k-1}},$$

where $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$.

Proof.

$$\sum_{j=1}^{n} \binom{2m+2j-b}{m+i+j-b}_1 \frac{(-1)^{j+k}(q^{2m+2j+1-b})_{2k-2j} q^{\frac{1}{2}j(j+1)-kj}}{(q^{m+j+k-b})_{k-j}(q^{m+j})_{k-j}} \binom{k-1}{j-1}_1$$
$$= \sum_{j=1}^{n} \frac{(q)_{2m+2k-b}(-1)^{j+k} q^{\frac{1}{2}j(j+1)-kj}(q)_{m+j+k-b-1}(q)_{m+j-1}}{(q)_{m+i+j-b}(q)_{m-i+j}(q)_{m+2k-b-1}(q)_{m+k-1}} \binom{k-1}{j-1}_1$$

$$= \frac{(-1)^{k+1}(q)_{2m+2k-b}(q)_{m+k-b}(q)_m}{(q)_{m+2k-b-1}(q)_{m+k-1}(q)_{m+i-b+1}(q)_{m-i+1}} \times \sum_{j=0}^{k-1} \frac{(q^{-k+1})_j(q^{m+k-b+1})_j(q^{m-1})_j q^j}{(q)_j(q^{m+i-b-2})_j(q^{m-i+2})_j}$$

$$= \frac{(-1)^{k+1}q^{1-k}(q)_{2m+2k-b}(q)_{m+k-b}(q)_m(q^{1-i})_{k-1}(q^{b+1-i-k})_{k-1}}{(q)_{m+2k-b-1}(q)_{m+k-1}(q)_{m+i-b+1}(q)_{m+k-i}(q^{b-i-m-k})_{k-1}},$$

by the q-analog of the Pfaff–Saalschutz summation [9, p. 97, Eq. (3.3.2.2)]. We remark that the above is meaningful and valid provided $b \leqslant m + i + 1$; however the more restricted cases $b = 0$ or 1 are sufficient for us. ∎

Our next two lemmas treat q-identities that are really further formulas for sums of basic hypergeometric series. Since we were unable to find them or generalizations of them in the literature, we choose to prove them *de novo* using Dougall's method [10, pp. 55, 95].

LEMMA 4. *For integers $s \geqslant j \geqslant 1$, $i \geqslant 1$, $m \geqslant 1$, if*

$$a_i(r, s) = \frac{(-1)^{r+s}(q^2, q^2)_{m+r+s-1}(q^2, q^2)_{m+s-r}(q^{2i-2s}; q^2)_s(q^{2i}; q^2)_s}{(q^2; q^2)_{m+i+s-1}(q^2, q^2)_{m-i+s}(1 - q^{2i-2r})(1 - q^{2i+2r-2})} \times \frac{(1 + q^{2i-1})(1 - q^{4r-2})q^{(i-s)^2+(s-r)}}{(1 + q^{2r-1})(q^2; q^2)_{s-1}(q^{2r}; q^2)_s} \binom{s-1}{r-1}_2, \tag{2.5}$$

then

$$\sum_{r=1}^{s} a_i(r, s)\binom{2m+2j-1}{m+r+j-1}_2 q^{(r-j)^2}(1 + q^{(2r-1)(2j-1)}) = \binom{2m+2j-1}{m+i+j-1}_2 q^{(i-j)^2}(1 + q^{(2i-1)(2j-1)}), \tag{2.6}$$

and

$$\sum_{r=1}^{s} a_i(r, s)\left[\binom{2m+2s+1}{m+r+s}_2 q^{(r-s-1)^2} \frac{(1 + q^{(2r-1)(2s+1)})}{(1 + q^{(2m+2s+1)(2s+1)})} - \binom{2m+2s+1}{m+r+s}_2 q^{(r-1)^2-2ms-s^2} \frac{(1 + q^{2r-1})}{(1 + q^{2m+2s+1})}\right]$$

$$= \binom{2m+2s+1}{m+i+s}_2 q^{(i-s-1)^2} \frac{(1 + q^{(2i-1)(2s+1)})}{(1 + q^{(2m+2s+1)(2s+1)})} - \binom{2m+2s+1}{m+i+s}_2 q^{(i-1)^2-2ms-s^2} \frac{(1 + q^{2i-1})}{(1 + q^{2m+2s+1})}, \tag{2.7}$$

where

$$(A;q)_n = \prod_{h=0}^{\infty} \frac{(1-Aq^h)}{(1-Aq^{h+n})}$$

$(= (1-A)(1-Aq)\cdots(1-Aq^{n-1})$ *when n is a positive integer*).

Remark. The apparent singularity of $a_i(r,s)$ for $i=r$ or $-r+1$ is removable since factors $(1-q^{2i-2r})$ and $(1-q^{2i+2r-2})$ appear in the numerator of $a_i(r,s)$. Note also that $1/(q^2;q^2)_{-n}=0$ for any positive integer n.

Proof. If $i > m+s$ or $i \leqslant -m-s$, then (2.6) is trivial since both sides are identically zero. Multiply both sides of (2.6) by

$$q^{-(i-s)^2}(q^2;q^2)_{m+i+s-1}(q^2;q^2)_{m-i+s}(q^2;q^2)_{2m+2j-1}^{-1}(1+q^{2i-1})^{-1}.$$

Since we may assume $-m-s<i\leqslant m+s$, this quantity is neither infinite nor zero, and we have the equivalent identity:

$$\begin{aligned}\sum_{r=1}^{s} & (-1)^{r+s}(q^{2m+2r+2j};q^2)_{s-j}(q^{2m+2j-2r+2};q^2)_{s-j}q^{(s-r)+(r-j)^2} \\ & \times \frac{(1+q^{(2r-1)(2j-1)})(q^{2i-2s};q^2)_s(q^{2i};q^2)_s(1-q^{4r-2})}{(1+q^{2r-1})(1-q^{2i-2r})(1-q^{2i+2r-2})(q^2;q^2)_{s-1}(q^{2r};q^2)_s}\binom{s-1}{r-1}_s \\ & = q^{(2i-j-s)(s-j)}\frac{(1+q^{(2i-1)(2j-1)})}{(1+q^{2i-1})}(q^{2m+2i-2j};q^2)_{s-j}(q^{2m+2j-2i+2};q^2)_{s-j}. \end{aligned} \tag{2.8}$$

Now identity (2.8) can be viewed as a polynomial identity in q^{2i}, where the left polynomial has degree at most $2s-2$ and the right polynomial also has degree at most $(s-j)+(s-j)+(2j-2)=2s-2$. If we can prove that the two sides of (2.8) are identical for at least $2s-1$ values of q^{2i}, then identity (2.8) must be valid for all i.

First let $i=t$ with $1\leqslant t\leqslant s$. Then each term on the left of (2.8) vanishes with the exception of $r=t$; so in this case the left side of (2.8) is just

$$(q^{2m+2t-2j};q^2)_{s-j}(q^{2m+2j-2t-2};q^2)_{s-j}q^{-(s-t)^2+(t-j)^2}\frac{(1+q^{(2t-1)(2j-1)})}{(1+q^{2t-1})},$$

and this is the right side of (2.8) when $i=t$.

Next let $i=-t$ with $0\leqslant t\leqslant s-1$. Then again each term on the left of (2.8) vanishes with the exception of $r=t+1$; so in this case the left side of

(2.8) is just

$$(q^{2m+2t+2j+2};q^2)_{s-j}(q^{2m+2j-2t};q^2)_{s-j}\frac{1+q^{(2t+1)(2j-1)}}{1+q^{2t+1}}q^{(j-t)^2-(s+t)^2-4t-2j+2},$$

and this is precisely the right side of (2.8) when $i=-t$. Consequently (2.8) is valid in general and thus (2.6) is valid in general.

We now pass to the proof of (2.7). If $i>m+s+1$ or $i<-m-s$, then (2.7) is trivial since both sides are identically zero. Multiply both sides of (2.7) by

$$q^{2i-(i-s)^2}(q^2;q^2)_{m+i+s}(q^2;q^2)_{m-i+s+1}(q^2;q^2)^{-1}_{2m+2s+1}(1+q^{2i-1})^{-1}.$$

Since we may assume $-m-s\leqslant i\leqslant m+s+1$, this quantity is neither infinite nor zero, and we have the equivalent identity:

$$\begin{aligned}
&q^{2i}(1-q^{2m+2i+2s})(1-q^{2m-2i+2s+2})\\
&\quad\times\sum_{r=1}^{s}\frac{(-1)^{r+s}q^{(s-r)+(r-s-1)^2}(q^{2i-2s};q^2)_s(q^{2i},q^2)_s}{(1-q^{2m+2r+2s})(1-q^{2m+2s-2r+2})}\\
&\quad\times\frac{(1-q^{4r-2})\binom{s-1}{r-1}_2}{(1+q^{2r-1})(1-q^{2i-2r})(1-q^{2i+2r-2})(q^2;q^2)_{s-1}(q^{2r};q^2)_s}\\
&\quad\times\left[\frac{(1+q^{(2r-1)(2s+1)}}{(1+q^{(2m+2s+1)(2s+1)})}-\frac{q^{-2s-2rs-2ms-2s^2}(1+q^{2r-1})}{(1+q^{2m+2s+1})}\right]\\
&=\frac{q^{2s+1}}{(1+q^{(2m+2s+1)(2s+1)})}\\
&\quad\times\left[\frac{(1+q^{(2i-1)(2s+1)})}{(1+q^{2i-1})}-q^{2s(i-m-s-1)}\frac{(1+q^{(2m+2s+1)(2s+1)})}{(1+q^{2m+2s+1})}\right]. \qquad (2.9)
\end{aligned}$$

As before, identity (2.9) can be viewed as a polynomial identity in q^{2i}, where the left polynomial has degree at most $2s$ and the right polynomial has degree at most $2s$. We must now show that identity (2.9) is valid for $2s+1$ values of i in order to establish it for all i. First of all we note that if $i=m+s+1$ then both sides of (2.9) are identically zero.

Next let $i=t$ with $1\leqslant t\leqslant s$. Then each term on the left of (2.9) vanishes with the exception of $r=t$; so in this case the left side of (2.9) is just

$$\frac{q^{2s+1}}{(1+q^{(2m+2s+1)(2s+1)})}\left[\frac{(1+q^{(2t-1)(2s+1)})}{(1+q^{2t-1})}-\frac{q^{2s(t-m-s-1)}(1+q^{(2m+2s+1)(2s+1)})}{(1+q^{2m+2s+1})}\right]$$

and this is precisely the right side of (2.9) when $i=t$.

Now let $i = -t$ with $0 \leqslant t \leqslant s-1$. As before each term on the left of (2.9) vanishes with the exception of $r = t+1$; hence for $i = -t$ the left side of (2.9) is just

$$\frac{q^{2s+1}}{(1+q^{(2m+2s+1)(2s+1)})} \times \left[\frac{q^{-2s(2t+1)}(1+q^{(2t+1)(2s+1)})}{(1+q^{2t+1})} - q^{-2s(t+m+s+1)}\frac{(1+q^{(2m+2s+1)(2s+1)})}{(1+q^{2m+2s+1})}\right],$$

and this is the right side of (2.9) when $i = -t$.

Hence both sides of (2.9) are identical for $i = m+s+1$ and $-s+1 \leqslant i \leqslant s$; hence (2.9) is valid for all i. Thus (2.7) is established. ∎

LEMMA 5. *For integers $s \geqslant j \geqslant 1$, $i \geqslant 1$, $m \geqslant 1$, if*

$$b_i(r,s) = \frac{(-1)^{r+s}(q^2;q^2)_{m+r-s}(q^2;q^2)_{m+s-r}(q^{2i-2s};q^2)_s(q^{2i+2};q^2)_s}{(q^2;q^2)_{m+i+s}(q^2;q^2)_{m-i+s}(1-q^{2i-2r})(1-q^{2i+2r})} \times \frac{(1-q^{4i})q^{(i-s)^2+(s-r)}}{(q^2;q^2)_{s-1}(q^{2r+2};q^2)_s}\binom{s-1}{r-1}_2, \tag{2.10}$$

then

$$\sum_{r=1}^{s} b_i(r,s)\binom{2m+2j}{m+r+j}_2 q^{(r-j)^2}(1-q^{4rj}) = \binom{2m+2j}{m+i+j}_2 q^{(i-j)^2}(1-q^{4ij}), \tag{2.11}$$

and

$$\begin{aligned}\sum_{r=1}^{s} b_i(r,s)&\left\{\binom{2m+2s+2}{m+r+s+1}_2 q^{(r-s-1)^2}\frac{(1-q^{4r(s+1)})}{(1-q^{4(s+1)(s+m+1)})}\right.\\ &\left.-\binom{2m+2s+2}{m+r+s+1}_2\frac{q^{(r-1)^2-2ms-s^2}(1-q^{4r})}{(1-q^{4m+4s+4})}\right\}\\ &=\binom{2m+2s+2}{m+i+s+1}_2\frac{q^{(i-s-1)^2}(1-q^{4i(s+1)})}{(1-q^{4(s+1)(s+m+1)})}\\ &\quad-\binom{2m+2s+2}{m+i+s+1}_2\frac{q^{(i-1)^2-2ms-s^2}(1-q^{4i})}{(1-q^{4m+4s+4})}.\end{aligned} \tag{2.12}$$

Remark. The apparent singularity of $b_i(r,s)$ for $i = r$ or $-r$ is removable since factors $(1-q^{2i-2r})$ and $(1-q^{2i+2r})$ also appear in the numerator of $b_i(r,s)$.

Proof. If $i > m + s$ or $i < -m - s$ or $i = 0$, then (2.11) is trivial since both sides are identically zero. Multiply both sides of (2.11) by

$$q^{-(i-s)^2}(q^2;q^2)_{m+i+s}(q^2;q^2)_{m-i+s}(q^2;q^2)^{-1}_{2m+2j}(1-q^{4i})^{-1}.$$

Since we may assume $-m - s \leqslant i \leqslant m + s$ and $i \neq 0$, this quantity is neither infinite nor zero, and we have the equivalent identity:

$$\begin{aligned}\sum_{r=1}^{s} (-1)^{r+s}(q^{2m+2r+2j+2};q^2)_{s-j}(q^{2m+2j-2r+2};q^2)_{s-j}q^{(s-r)+(r-j)^2} \\ \times \frac{(1-q^{4rj})(q^{2i-2s};q^2)_s(q^{2i+2};q^2)_s}{(1-q^{2i-2r})(1-q^{2i+2r})(q^2;q^2)_{s-1}(q^{2r+2};q^2)_s}\binom{s-1}{r-1}_2 \\ = q^{(2i-j-s)(s-j)}\frac{(1-q^{4ij})}{(1-q^{4i})}(q^{2m-2i-2j+2};q^2)_{s-j}(q^{2m+2j-2i+2};q^2)_{s-j}.\end{aligned} \tag{2.13}$$

Identity (2.13) may be viewed as a polynomial identity in q^{2i}, where the left polynomial has degree at most $2s - 2$ and the right polynomial also has degree at most $2s - 2$. If we can prove that the two sides of (2.13) are identical for at least $2s - 1$ values of q^{2i}, then (2.13) is valid for all i.

First let $i = t$ with $1 \leqslant t \leqslant s$. Then each term on the left of (2.13) vanishes with the exception of $r = t$; so in this case the left side of (2.13) is just

$$(q^{2m+2t+2j+2};q^2)_{s-j}(q^{2m+2j-2t+2};q^2)_{s-j}q^{(2t-j-s)(s-j)}\frac{(1-q^{4tj})}{(1-q^{4t})},$$

and this is the right side of (2.13) when $i = t$.

Now let $i = -t$ with $1 \leqslant t \leqslant s$. In this case each term on the left of (2.13) vanishes with the exception of $r = t$; hence for $i = -t$ the left side of (2.13) is just

$$q^{-(2t+j+s)(s-j)}\frac{(1-q^{-4tj})}{(1-q^{-4t})}(q^{2m-2t+2j-2};q^2)_{s-j}(q^{2m+2j+2t+2};q^2)_{s-j},$$

and this is the right side of (2.13) when $i = -t$. Consequently (2.13) is valid for all i, and so (2.11) is also.

To conclude the proof of Lemma 5 we treat identity (2.12). If $i > m + s + 1$ or $i < -m - s - 1$, or $i = 0$, then (2.12) is trivial since both sides are identically zero. Multiply both sides of (2.12) by

$$q^{2i-(i-s)^2}(q^2;q^2)_{m+i+s+1}(q^2;q^2)_{m-i+s+1}(q^2;q^2)^{-1}_{2m+2s+2}(1-q^{4i})^{-1}.$$

Since we may assume $-m - s - 1 \leqslant i \leqslant m + s + 1$ and $i \neq 0$, this quantity is neither infinite nor zero and we have the equivalent identity:

$$
\begin{aligned}
q^{2i}&(1-q^{2m+2i+2s+2})(1-q^{2m-2i+2s+2})\\
&\times\sum_{r=1}^{s}\frac{(-1)^{r+s}q^{(s-r)}}{(1-q^{2m+2r+2s+2})(1-q^{2m-2r+2s+2})}\\
&\times\frac{(q^{2i-2s};q^2)_s(q^{2i+2};q^2)_s\binom{s-1}{r-1}_2}{(1-q^{2i-2r})(1-q^{2i+2r})(q^2;q^2)_{s-1}(q^{2r+2};q^2)_s}\\
&\times\left[\frac{q^{(r-s-1)^2}(1-q^{4r(s+1)})}{(1-q^{4(s+1)(s+m+1)})}-\frac{q^{(r-1)^2-2ms-s^2}(1-q^{4r})}{(1-q^{4m+4s+4})}\right]\\
&=\frac{q^{2s+1}}{(1-q^{4(s+1)(s+m+1)})}\left[\frac{(1-q^{4i(s+1)})}{(1-q^{4i})}-\frac{q^{2s(i-m-s-1)}(1-q^{4(s-1)(s+m+1)})}{(1-q^{4m+4s+4})}\right].
\end{aligned}
\tag{2.14}
$$

Identity (2.14) can be viewed as a polynomial identity in q^{2i}, where each side has degree at most $2s$. We must now show that identity (2.14) is valid for $2s+1$ values of i in order to establish it for all i.

If $i=m+s+1$, then both sides of (2.14) are identically zero.

If $i=t$ with $1\leqslant t\leqslant s$, then each term on the left side of (2.14) vanishes with the exception of $r=t$; hence in this case the left side of (2.14) is just

$$\frac{q^{2s+1}}{(1-q^{4(s+1)(s+m+1)})}\left[\frac{(1-q^{4t(s+1)})}{(1-q^{4t})}-\frac{q^{2s(t-m-s-1)}(1-q^{4(s+1)(s-m+1)})}{(1-q^{4m+4s+4})}\right],$$

and this is, in fact, the right side of (2.14) when $i=t$.

Finally, let $i=-t$ with $1\leqslant t\leqslant s$. As before, each term on the left of (2.14) vanishes with the exception of $r=t$: so for $i=-t$ the left side of (2.14) reduces to

$$\frac{q^{2s+1}}{(1-q^{4(s+1)(s+m+1)})}\left[\frac{(1-q^{-4t(s+1)})}{(1-q^{-4t})}-\frac{q^{-2s(t+m+s+1)}(1-q^{4(s-1)(s+m+1)})}{(1-q^{4m+4s+4})}\right],$$

and this is the right side of (2.14) when $i=-t$.

Thus identity (2.14) is valid for $i=m+s+1$, $1\leqslant i\leqslant s$, and $-s\leqslant i\leqslant -1$; this implies that (2.14) is valid for all i. Identity (2.12) now follows. ∎

We now have all the necessary summations. These will be used in the next section for matrix multiplication lemmas.

3. Determinants and Matrices

To accomplish our goal, we must prove six lemmas that involve the following six matrices:

$$\alpha_n = \left(q^{(i-j)^2}\binom{2m}{m+i-j}_2 + q^{(i+j-1)^2}\binom{2m}{m-i-j-1}_2\right)_{n\times n}, \tag{3.1}$$

$$\alpha_n' = \left(q^{(i-j)^2}\binom{2m}{m+i-j}_2 - q^{(i+j)^2}\binom{2m}{m+i+j}_2\right)_{n\times n}, \tag{3.2}$$

$$\beta_n = \left[\binom{2m+2j-1}{m+i+j-1}_2 \frac{q^{(i-j)^2}(1+q^{(2i-1)(2j-1)})}{(1+q^{(2m+2j-1)(2j-1)})}\right]_{n\times n}, \tag{3.3}$$

$$\beta_n' = \left[\binom{2m+2j}{m+i+j}_2 \frac{q^{(i-j)^2}(1-q^{4ij})}{(1-q^{4j(m+j)})}\right]_{n\times n}, \tag{3.4}$$

$$\gamma_n = \left[\binom{2m+2j-1}{m+i+j-1}_2 \frac{q^{(i-1)^2-2m(j-1)-(j-1)^2}(1+q^{2i-1})}{(1+q^{2m+2j-1})}\right]_{n\times n}, \tag{3.5}$$

$$\gamma_n' = \left[\binom{2m+2j}{m+i+j}_2 \frac{q^{(i-1)^2-2m(j-1)-(j-1)^2}(1-q^{4i})}{(1-q^{4m+4j})}\right]_{n\times n}. \tag{3.6}$$

LEMMA 6. *Let*

$$\delta_n = \left[\binom{2j-1}{j-i}_2 \frac{(q^{(i-j)(i-j-2m)} + q^{(i+j-1)(i+j+2m-1)})}{(1+q^{(2j-1)(2j+2m-1)})}\right]_{n\times n}.$$

Then $\det \delta_n = 1$, *and* $\alpha_n \cdot \delta_n = \beta_n$.

Proof. We note immediately that δ_n is upper triangular with ones on the main diagonal. Hence $\det \delta_n = 1$.

The (i, j)th entry in $\alpha_n \cdot \delta_n$ is

$$\sum_{k=1}^{n} \left(q^{(i-k)^2}\binom{2m}{m+i-k}_2 + q^{(i+k-1)^2}\binom{2m}{m+i+k-1}_2\right)$$
$$\times \left[\binom{2j-1}{j-k}_2 \frac{(q^{(k-j)(k-j-2m)} + q^{(k+j-1)(k+j+2m-1)})}{(1+q^{(2j-1)(2j+2m-1)})}\right]$$
$$= \sum_{k=-\infty}^{\infty} \binom{2m}{m+i-k}_2 q^{(i-k)^2}\binom{2j-1}{j-k}_2 \frac{(q^{(k-j)(k-j-2m)} + q^{(k+j-1)(k+j-2m-1)})}{(1+q^{(2j-1)(2j+2m-1)})}$$
$$= \binom{2m+2j-1}{m+i+j-1}_2 \frac{q^{(i-j)^2}(1+q^{(2i-1)(2j-1)})}{(1+q^{(2j-1)(2j+2m-1)})},$$

by Lemma 1. Since this last expression is precisely the (i, j)th entry of β_n, Lemma 6 is established. ∎

LEMMA 7. *Let*

$$\delta_n' = \left(\binom{2j}{j+i}_2 \frac{(q^{(i-j)(i-j-2m)} - q^{(i+j)(i+j+2m)})}{(1-q^{4j(j-m)})}\right)_{n\times n}.$$

Then $\det \delta_n' = 1$, *and* $\alpha_n' \cdot \delta_n' = \beta_n'$.

Proof. The matrix δ_n' is upper triangular with ones on the main diagonal. Hence $\det \delta_n' = 1$.

The (i, j)th entry in $\alpha_n' \cdot \delta_n'$ is

$$\begin{aligned}
&\sum_{k=1}^{n} \left(q^{(i-k)^2}\binom{2m}{m+i-k}_2 - q^{(i+k)^2}\binom{2m}{m-i+k}_2\right) \\
&\qquad \times \left(\binom{2j}{j+k}_2 \frac{(q^{(k-j)(k-j-2m)} - q^{(k+j)(k+j+2m)})}{(1-q^{4j(j+m)})}\right) \\
&= \sum_{k=-\infty}^{\infty} \binom{2m}{m+i-k} q^{(i-k)^2}\binom{2j}{j+k}_2 \frac{(q^{(k-j)(k-j-2m)} - q^{(k+j)(k-j+2m)})}{(1-q^{4j(j+m)})} \\
&= \binom{2m+2j}{m+i+j}_2 \frac{q^{(i-j)^2}(1-q^{4ij})}{(1-q^{4j(j+m)})},
\end{aligned}$$

by Lemma 2. Since this last expression is the (i, j)th entry of β_n', we have established Lemma 7. ▌

LEMMA 8.

$$\det \gamma_n = \prod_{i=1}^{m}\left[\frac{(1-q^{2n+2i-1})}{(1-q^{2i-1})} \prod_{h=i+1}^{m} \frac{(1-q^{2(2n-i+h-1)})}{(1-q^{2(i+h-1)})}\right].$$

Proof.

$$\begin{aligned}
\det \gamma_n &= \det\left[\binom{2m+2j-1}{m+i+j-1}_2 \frac{q^{(i-1)^2-2m(j-1)-(j-1)^2}(1+q^{2i-1})}{(1+q^{2m+2j-1})}\right]_{n\times n} \\
&= \frac{q^{-mn(n-1)}(-q;q^2)_n}{(-q^{2m+1};q^2)_n} \det\left[\binom{2m+2j-1}{m+i+j-1}_2\right]_{n\times n}.
\end{aligned}$$

Define

$$\epsilon_n = \left(\frac{(-1)^{i+j}(q^{4m+4i};q^2)_{2j-2i}q^{i^2+i-2ij}}{(q^{2m+2i+2j-2};q^2)_{j-i}(q^{2m+2i};q^2)_{j-1}}\binom{j-1}{i-1}_2\right)_{n\times n}.$$

Then ϵ_n is upper triangular, and its (j, j)th entry is q^{-j^2+j}. Therefore $\det \epsilon_n = \prod_{j=1}^{n} q^{j-j^2}$. If we let

$$\gamma_n^* = \left[\binom{2m+2j-1}{m+i+j-1}_2\right]_{n\times n}$$

then the (i, j)th entry of $\gamma_n^* \cdot \epsilon_n$ is (by Lemma 3 with $b = 1$ and q replaced by q^2):

$$\frac{(-1)^{j+1}q^{2-2j}(q^2;q^2)_{2m+2j-1}(q^2;q^2)_m(q^{2-2i};q^2)_{j-1}(q^{4-2i-2j};q^2)_{j-1}}{(q^2;q^2)_{m+2j-2}(q^2;q^2)_{m+i}(q^2;q^2)_{m+j-i}(q^{2-2i-2m-2j};q^2)_{j-1}}.$$

Now since this last expression is zero for $j > i$, we see that $\gamma_n^* \cdot \epsilon_n$ is lower triangular. Consequently,

$$\det \gamma_n = \frac{q^{-mn(n-1)}(-q;q^2)_n(\det \gamma_n^*)}{(-q^{2m+1};q^2)_n}$$

$$= \frac{q^{-mn(n-1)}(-q;q^2)_n}{(-q^{2m+1};q^2)_n(\det \epsilon_n)} \prod_{j=1}^{n}$$

$$\times \frac{(-1)^{j+1}q^{2-2j}(q^2;q^2)_{2m+2j-1}(q^{2-2j};q^2)_{j-1}(q^{4-4j};q^2)_{j-1}}{(q^2;q^2)_{m+2j-2}(q^2;q^2)_{m+j}(q^{2-2m-4j};q^2)_{j-1}}$$

$$= \frac{(-q;q^2)_n}{(-q^{2m+1};q^2)_n} \prod_{j=1}^{n} \frac{(q^2;q^2)_{2m+2j-1}(q^2;q^2)_{j-1}(q^{2j};q^2)_{j-1}}{(q^2;q^2)_{m+2j+2}(q^2;q^2)_{m+j}(q^{2m+2j+2};q^2)_{j-1}}$$

$$= \frac{(-q;q^2)_n}{(-q^{2m+1};q^2)_n} \prod_{j=1}^{n} \frac{(q^2;q^2)_{2m+2j-1}(q^2;q^2)_{2j-2}}{(q^2;q^2)_{m+2j-2}(q^2;q^2)_{m+2j-1}}$$

$$= \frac{(q^{2n+1};q^2)_m}{(q;q^2)_m} \prod_{j=1}^{n} \frac{(q^2;q^2)_{2m+2j-2}(q^2;q^2)_{2j-1}}{(q^2;q^2)_{m+2j-2}(q^2;q^2)_{m+2j-1}}$$

$$= \frac{(q^{2n+1};q^2)_m \prod_{j=1}^{n} (q^{4j-2};q^2)_{2m}}{(q;q^2)_m \prod_{j=1}^{2n} (q^{2j};q^2)_m}$$

$$= \frac{(q^{2n+1};q^2)_m \prod_{j=1}^{2n} (q^{2j};q^2)_{2m}}{(q;q^2)_m (\prod_{j=1}^{2n} (q^{2j};q^2)_m)(\prod_{j=1}^{n} (q^{4j};q^2)_{2m})}$$

$$= \frac{(q^{2n+1};q^2)_m \prod_{j=1}^{2n} (q^{2j+2m};q^2)_m}{(q;q^2)_m \prod_{j=1}^{n} (q^{4j};q^2)_{2m}}$$

$$= \frac{(q^{2n+1};q^2)_m \prod_{j=1}^{2n} (q^{2j+2m};q^2)_m}{(q;q^2)_m \prod_{j=1}^{n} (q^{4j};q^4)_m(q^{4j+2};q^4)_m}$$

$$= \frac{(q^{2n+1};q^2)_m}{(q;q^2)_m} \prod_{j=1}^{2n-1} \prod_{i=1}^{m} \frac{(1-q^{2m+2j+2i})}{(1-q^{2j+4i})}$$

$$= \frac{(q^{2n+1};q^2)_m}{(q;q^2)_m} \prod_{i=1}^{m} \frac{(q^{2m+2i};q^2)_{2n}}{(q^{4i};q^2)_{2n}}$$

$$= \frac{(q^{2n+1};q^2)_m}{(q;q^2)_m} \prod_{i=1}^{m} \frac{(q^2;q^2)_{2n+i+m-1}(q^2;q^2)_{2i-1}}{(q^2;q^2)_{m+i-1}(q^2;q^2)_{2n+2i-1}}$$

$$= \frac{(q^{2n+1};q^2)_m}{(q;q^2)_m} \prod_{i=1}^{m} \prod_{j=i+1}^{m} \frac{(1-q^{2(2n+i+j-1)})}{(1-q^{2i+2j-2})}$$

$$= \prod_{i=1}^{m} \left\{ \frac{(1-q^{2n+2i-1})}{(1-q^{2i-1})} \prod_{j=i+1}^{m} \frac{(1-q^{2(2n+i+j-1)})}{(1-q^{2(i+j-1)})} \right\}. \quad \blacksquare$$

LEMMA 9.

$$(-q;q^2)_m \det \gamma_n' = \prod_{i=1}^{m}\left[\frac{(1-q^{2n+2i})}{(1-q^{2i-1})}\prod_{h=i+1}^{m}\frac{(1-q^{2(2n+i+h)})}{(1-q^{2(i+h-1)})}\right].$$

Proof.

$$\det \gamma_n' = \det\left[\binom{2m+2j}{m+i+j}_2 \frac{q^{(i-1)^2-2m(j-1)-(j-1)^2}(1-q^{4i})}{(1-q^{4m+4j})}\right]_{n\times n}$$

$$= \frac{q^{-mn(n-1)}(q^4;q^4)_n}{(q^{4m+4};q^4)_n}\det\left[\binom{2m+2j}{m+i+j}_2\right]_{n\times n}.$$

Define

$$\epsilon_n' = \left(\frac{(-1)^{i+j}(q^{4m+4i+2};q^2)_{2j-2i}q^{i^2+i-2ij}}{(q^{2m+2i+2j};q^2)_{j-i}(q^{2m-2i};q^2)_{j-i}}\binom{j-1}{i-1}_2\right)_{n\times n}.$$

Then ϵ_n' is upper triangular, and its (j,j)th entry is q^{-j^2+j}. Therefore $\det \epsilon_n' = \prod_{j=1}^{n} q^{j-j^2}$. If we let

$$\gamma_n^{\#} = \left(\binom{2m+2j}{m+i+j}_2\right)_{n\times n},$$

then the (i,j)th entry of $\gamma_n^{\#}\cdot\epsilon_n'$ is (by Lemma 3 with $b=0$ and q replaced by q^2):

$$\frac{(-1)^{j+1}q^{2-2j}(q^2;q^2)_{2m+2j}(q^2;q^2)_m(q^{2-2i};q^2)_{j-1}(q^{2-2i-2j};q^2)_{j-1}(1-q^{2m+2j})}{(q^2;q^2)_{m+2j-1}(q^2;q^2)_{m+i+1}(q^2;q^2)_{m+j-i}(q^{-2i-2m-2j};q^2)_{j-1}}.$$

This last expression is zero for $j>i$, so we see that $\gamma_n^{\#}\cdot\epsilon_n'$ is lower triangular. Consequently,

$$(-q;q^2)_m \det \gamma_n' = \frac{(-q;q^2)_m q^{-mn(n-1)}(q^4;q^4)_n(\det \gamma_n^{\#})}{(q^{4m+4};q^4)_n}$$

$$= \frac{(-q;q^2)_m q^{-mn(n-1)}(q^4;q^4)_n}{(q^{4m+4};q^4)_n(\det \epsilon_n')}$$

$$\times \prod_{j=1}^{n}\frac{(-1)^{j+1}q^{2-2j}(q^2;q^2)_{2m+2j}(q^{2-2j};q^2)_{j-1}(q^{2-4j};q^2)_{j-1}(1-q^{2m+2j})}{(q^2;q^2)_{m+2j-1}(q^2;q^2)_{m+j+1}(q^{-2m-4j};q^2)_{j-1}}$$

$$= \frac{(-q;q^2)_m(q^4;q^4)_n}{(-q^{2m+2};q^2)_n}\prod_{j=1}^{n}\frac{(q^2;q^2)_{2m+2j}(q^2;q^2)_{j-1}(q^{2j+2};q^2)_{j-1}}{(q^2;q^2)_{m+2j-1}(q^2;q^2)_{m+j+1}(q^{2m+2j+4};q^2)_{j-1}}$$

$$= \frac{(-q;q^2)_m(-q^2;q^2)_n}{(-q^{2m+2};q^2)_n}\prod_{j=1}^{n}\frac{(q^2;q^2)_{2m+2j}(q^2;q^2)_{2j-1}}{(q^2;q^2)_{m+2j-1}(q^2;q^2)_{m+2j}}$$

$$= \frac{(-q;q^2)_m(-q^2;q^2)_n\prod_{j=1}^{n}(q^{4j+2};q^2)_{2m}}{(-q^{2m+2};q^2)_n\prod_{j=1}^{2n}(q^{2j+2};q^2)_m}$$

$$= \frac{(-q;q^2)_m(-q^2;q^2)_n \prod_{j=1}^{2n} (q^{2j+2};q^2)_{2m}}{(-q^{2m+2};q^2)_n (\prod_{j=1}^{2n} (q^{2j+2};q^2)_m)(\prod_{j=1}^{n} (q^{4j};q^2)_{2m})}$$

$$= \frac{(-q;q^2)_m(-q^2;q^2)_n \prod_{j=1}^{2n} (q^{2j+2m+2};q^2)_m}{(-q^{2m+2};q^2)_n \prod_{j=1}^{n} (q^{4j};q^2)_{2m}}$$

$$= \frac{(-q;q^2)_m(-q^2;q^2)_n \prod_{j=1}^{2n} (q^{2j+2m+2};q^2)_m}{(-q^{2m+2};q^2)_n \prod_{j=1}^{n} (q^{4j};q^4)_m(q^{4j+2};q^4)_m}$$

$$= \frac{(-q;q^2)_m(-q^2;q^2)_n}{(-q^{2m+2};q^2)_n} \prod_{j=0}^{2n-1} \prod_{i=1}^{m} \frac{(1-q^{2m+2j+2i+2})}{(1-q^{2j+4i})}$$

$$= \frac{(-q;q^2)_m(-q^2;q^2)_n}{(-q^{2m+2};q^2)_n} \prod_{i=1}^{m} \frac{(q^{2m+2i+2};q^2)_{2n}}{(q^{4i};q^2)_{2n}}$$

$$= \frac{(-q;q^2)_m(-q^2;q^2)_n}{(-q^{2m+2};q^2)_n} \prod_{i=1}^{m} \frac{(q^2;q^2)_{2n+m+i}(q^2;q^2)_{2i-1}}{(q^2;q^2)_{2n+2i-1}(q^2;q^2)_{m+i}}$$

$$= \frac{(-q;q^2)_m(-q^2;q^2)_n}{(-q^{2m+2};q^2)_n} \prod_{i=1}^{m} \left[\frac{(1-q^{4n+4i})}{(1-q^{2m+2i})} \prod_{j=i+1}^{m} \frac{(1-q^{4n+2i+2j})}{(1-q^{2(i+j-1)})}\right]$$

$$= \frac{(-q;q^2)_m(-q^2;q^2)_n(q^{4n+4};q^4)_m}{(-q^{2m+2};q^2)_n(q^{2m+2};q^2)_m} \prod_{i=1}^{m} \prod_{j=i+1}^{m} \frac{(1-q^{2(2n+i+j)})}{(1-q^{2(i+j-1)})}$$

$$= \frac{(-q;q^2)_m(q^4;q^4)_{m+n}(-q^2;q^2)_m(q^2;q^2)_m}{(q^2;q^2)_n(-q^2;q^2)_{m+n}(q^2;q^2)_{2m}} \prod_{i=1}^{m} \prod_{j=i+1}^{m} \frac{(1-q^{2(2n+i+j)})}{(1-q^{2(i+j-1)})}$$

$$= \frac{(-q;q^2)_m(q^2;q^2)_{m+n}(q^4;q^4)_m}{(q^2;q^2)_n(q^2;q^4)_m(q^4;q^4)_m} \prod_{i=1}^{m} \prod_{j=i+1}^{m} \frac{(1-q^{2(2n+i+j)})}{(1-q^{2(i+j-1)})}$$

$$= \frac{(q^{2n+2};q^2)_m}{(q;q^2)_m} \prod_{i=1}^{m} \prod_{j=i+1}^{m} \frac{(1-q^{2(2n+i+j)})}{(1-q^{2(i+j-1)})}$$

$$= \prod_{i=1}^{m} \left[\frac{(1-q^{2n+2i})}{(1-q^{2i-1})} \prod_{j=i+1}^{m} \frac{(1-q^{2(2n+i+j)})}{(1-q^{2(i+j-1)})}\right]. \quad \blacksquare$$

LEMMA 10. *There exist c_{ij} such that $c_{ij} = 0$ if $i > j$, $c_{ii} = 1$, and for all n*

$$\beta_n \cdot (c_{ij})_{n \times n} = \gamma_n.$$

Proof. We proceed by induction on n. In the case $n = 1$, the result is trivial since β_1 and γ_1 are the 1×1 matrix whose single entry is

$$\binom{2m+1}{m+1}_2 \frac{(1+q)}{(1+q^{2m+1})}.$$

Now we assume that the c_{ij} have been found for $i < n$ and $j < n$. We choose $c_{nj} = 0$ for $j < n$, $c_{nn} = 1$, and for $1 \leqslant h < n$ we choose c_{hn} so that the following system of equations is valid:

$$\sum_{j=1}^{n} \binom{2m+2j-1}{m+i+j-1}_2 \frac{q^{(i-j)^2}(1+q^{(2i-1)(2j-1)})}{(1+q^{(2m+2j-1)(2j-1)})} c_{jn} = \binom{2m+2n-1}{m+i+n-1}_2 \frac{q^{(i-1)^2-2m(n-1)-(n-1)^2}(1+q^{2i-1})}{(1+q^{2m+2n-1})}, \tag{3.7}$$

where $1 \leqslant i \leqslant n-1$. Now $c_{nn} = 1$, so that (3.7) is a system of $(n-1)$ equations in $(n-1)$ unknowns. The determinant of the system is $\det \beta_{n-1}$, and by the induction hypothesis

$$\det \beta_{n-1} = \frac{\det \gamma_{n-1}}{\det(c_{ij})_{n-1 \times n-1}} = \det \gamma_{n-1} \neq 0$$

by Lemma 8. Therefore the c_{jn} exist and are unique.

I claim now that (3.7) is valid for every $i \geqslant 1$. This is because by Lemma 4,

$$\begin{aligned}
&\sum_{j=1}^{n} \binom{2m+2j-1}{m+i+j-1}_2 \frac{q^{(i-j)^2}(1+q^{(2i-1)(2j-1)})}{(1+q^{(2m+2j-1)(2j-1)})} c_{jn} \\
&\quad - \binom{2m+2n-1}{m+i+n-1}_2 \frac{q^{(i-1)^2-2m(n-1)-(n-1)^2}(1+q^{2i-1})}{(1-q^{2m+2n-1})} \\
&= \sum_{j=1}^{n-1} \sum_{r=1}^{n-1} a_i(r, n-1) \binom{2m+2j-1}{m+r+j-1}_2 \frac{q^{(r-j)^2}(1+q^{(2r-1)(2j-1)})}{(1+q^{(2m+2j-1)(2j-1)})} c_{jn} \\
&\quad + \sum_{r=1}^{n-1} a_i(r, n-1) \left[\binom{2m+2n-1}{m+r+n-1}_2 \frac{q^{(r-n)^2}(1+q^{(2r-1)(2n-1)})}{(1+q^{(2m+2n-1)(2n-1)})} \right. \\
&\quad \left. - \binom{2m+2n-1}{m+r+n-1}_2 \frac{q^{(r-1)^2-2m(n-1)-(n-1)^2}(1+q^{2r-1})}{(1+q^{2m+2n-1})} \right] \\
&= \sum_{r=1}^{n-1} a_i(r, n-1) \left[\sum_{j=1}^{n} \binom{2m+2j-1}{m+r+j-1}_2 \frac{q^{(r-j)^2}(1+q^{(2r-1)(2j-1)})}{(1+q^{(2m+2j-1)(2j-1)})} c_{jn} \right. \\
&\quad \left. - \binom{2m+2n-1}{m+r+n-1}_2 \frac{q^{(r-1)^2-2m(n-1)-(n-1)^2}(1+q^{2r-1})}{(1+q^{2m+2n-1})} \right] \\
&= 0,
\end{aligned}$$

since the internal expressions are zero by (3.7) with $1 \leqslant r \leqslant n-1$. Therefore in multiplying β_n on the right by $(c_{ij})_{n \times n}$ we see that the (i, j)th entry is the (i, j)th entry in γ_n if $j < n$ and $i < n$ by the induction hypothesis and if $j = n$ by (3.7). When $i = n$ and $j = j_0 < n$, we see that our above argument establishes (3.7) for all i with n replaced by j_0, and so again the resulting

(i, j)th entry is the (i, j_0)th entry of γ_n. Hence

$$\beta_n \cdot (c_{ij})_{n \times n} = \gamma_n. \quad \blacksquare$$

LEMMA 11. *There exist d_{ij} such that $d_{ij} = 0$ if $i > j$, $d_{ii} = 1$, and for all n*

$$\beta_n' \cdot (d_{ij})_{n \times n} = \gamma_n'.$$

Proof. We proceed by induction on n. In the case $n = 1$, the result is trivial since β_1' and γ_1' are the 1×1 matrix whose single entry is

$$\binom{2m+2}{m+2}_2 \frac{(1 - q^4)}{(1 - q^{4m+4})}.$$

Now we assume that the d_{ij} have been found for $i < n$ and $j < n$. We choose $d_{nj} = 0$ for $j < n$, $d_{nn} = 1$, and for $1 \leqslant h < n$, we choose d_{nn} so that the following system of equations is valid:

$$\sum_{j=1}^{n} \binom{2m+2j}{m+i+j}_2 \frac{q^{(i-j)^2}(1 - q^{4ij})}{(1 - q^{4j(m+j)})} d_{jn} = \binom{2m+2n}{m+i+n}_2 \frac{q^{(i-1)^2 - 2m(n-1) - (n-1)^2}(1 - q^{4i})}{(1 - q^{4m-4n})}, \tag{3.8}$$

where $1 \leqslant i \leqslant n - 1$. Since $d_{nn} = 1$, (3.8) is a system of $(n - 1)$ equations in $(n - 1)$ unknowns. The determinant of the system is $\det \beta'_{n-1}$, and by the induction hypothesis $\det \beta'_{n-1} = \det \gamma'_{n-1} / \det(d_{ij})_{n-1 \times n-1} = \det \gamma'_{n-1} \neq 0$ by Lemma 9. Therefore the d_{jn} exist and are unique. I claim now that (3.8) is valid for every $i \geqslant 1$. This is because by Lemma 5,

$$\begin{aligned}
&\sum_{j=1}^{n} \binom{2m+2j}{m+i+j}_2 \frac{q^{(i-j)^2}(1 - q^{4ij})}{(1 - q^{4j(m+j)})} d_{jn} - \binom{2m+2n}{m+i+n}_2 \frac{q^{(i-1)^2 - 2m(n-1) - (n-1)^2}(1 - q^{4i})}{\cdot(1 - q^{4m+4n})} \\
&= \sum_{j=1}^{n-1} \sum_{r=1}^{n-1} b_i(r, n-1) \binom{2m+2j}{m+r+j}_2 \frac{q^{(r-j)^2}(1 - q^{4rj})}{(1 - q^{4j(m+j)})} d_{jn} \\
&\quad + \sum_{r=1}^{n-1} b_i(r, n-1) \left[\binom{2m+2n}{m+r+n}_2 \frac{q^{(r-n)^2}(1 - q^{4rn})}{(1 - q^{4n(m+n)})} \right. \\
&\quad \left. - \binom{2m+2n}{m+r+n}_2 \frac{q^{(r-1)^2 - 2m(n-1) - (n-1)^2}(1 - q^{4r})}{(1 - q^{4m+4n})} \right] \\
&= \sum_{r=1}^{n-1} b_i(r, n-1) \left[\sum_{j=1}^{n} \binom{2m+2j}{m+r+j}_2 \frac{q^{(r-j)^2}(1 - q^{4rj})}{(1 - q^{4j(m+j)})} d_{jn} \right. \\
&\quad \left. - \binom{2m+2n}{m+r+n}_2 \frac{q^{(r-1)^2 - 2m(n-1) - (n-1)^2}(1 - q^{4r})}{(1 - q^{4m+4n})} \right] \\
&= 0,
\end{aligned}$$

since the internal expressions are zero by (3.8) with $1 \leqslant r \leqslant n-1$. Therefore in multiplying β_n' on the right by $(d_{ij})_{n\times n}$ we see that the (i,j)th entry is the (i,j)th entry in γ_n' if $j<n$ and $i<n$ by the induction hypothesis and if $j=n$ by (3.8). When $i=n$ and $j=j_0<n$, we see that our above argument establishes (3.8) for all i with n replaced by j_0, and so in any event the resulting (i,j_0)th entry is the (i,j_0)th entry of γ_n'. Therefore

$$\beta_n' \cdot (d_{ij})_{n\times n} = \gamma_n'. \quad \blacksquare$$

4. MacMahon's Conjecture

Theorem 1 (MacMahon's conjecture).

$$\sum_{N\geqslant 0} M(n,m;N)q^N = \prod_{i=1}^{m}\left[\frac{(1-q^{n+2i-1})}{(1-q^{2i-1})}\prod_{h=i+1}^{m}\frac{(1-q^{2(n+i+h-1)})}{(1-q^{2(i+h-1)})}\right]. \tag{4.1}$$

Proof. We must treat separately the cases n even and n odd. First

$$\begin{aligned}\sum_{N\geqslant 0} &M(2n,m;N)q^N\\ &= g_{2n}(q) && \text{(by (1.2))}\\ &= \det \alpha_n && \text{(by (1.3) and (3.1))}\\ &= \det \beta_n && \text{(by Lemma 6)}\\ &= \det \gamma_n && \text{(by Lemma 10)}\\ &= \prod_{i=1}^{m}\left[\frac{(1-q^{2n+2i-1})}{(1-q^{2i-1})}\prod_{h=i+1}^{m}\frac{(1-q^{2(2n+i+h-1)})}{(1-q^{2(i+h-1)})}\right] && \text{(by Lemma 8).}\end{aligned}$$

Hence (4.1) is valid when n is even.

When $n=1$, the right side of (4.1) is

$$\begin{aligned}\prod_{i=1}^{m}\frac{(1-q^{2i})}{(1-q^{2i-1})}\cdot\frac{(1-q^{2i+2m})}{(1-q^{4i})} &= \frac{(q^2;q^2)_m(q^{2m+2};q^2)_m}{(q;q^2)_m(q^4;q^4)_m}\\ &= \frac{(q^2;q^2)_{2m}}{(q;q^2)_m(q^4;q^4)_m}\\ &= \frac{(q^2;q^4)_m(q^4;q^4)_m}{(q;q^2)_m(q^4;q^4)_m}\\ &= (-q;q^2)_m.\end{aligned}$$

On the other hand $M(1,m;N)$ is clearly just the number of linear partitions with distinct odd parts each at most $2m - 1$. Hence

$$\sum_{N \geq 0} M(1, m; N)q^N = (1 + q)(1 + q^3) \cdots (1 + q^{2m-1}) = (-q; q^2)_m.$$

Thus (4.1) is valid when $n = 1$.

$$\begin{aligned}
&\sum_{N \geq 0} M(2n + 1, m; N)q^N \\
&\quad = g_{2n+1}(q) && \text{(by (1.2))} \\
&\quad = (-q; q^2)_m \det \alpha_n' && \text{(by (1.4) and (3.2))} \\
&\quad = (-q; q^2)_m \det \beta_n' && \text{(by Lemma 7)} \\
&\quad = (-q; q^2)_m \det \gamma_n' && \text{(by Lemma 11)} \\
&\quad = \prod_{i=1}^{m} \left\{ \frac{(1 - q^{2n+2i})}{(1 - q^{2i-1})} \prod_{h=i+1}^{m} \frac{(1 - q^{2(2n+i+h)})}{(1 - q^{2(i+h-1)})} \right\} && \text{(by Lemma 9).}
\end{aligned}$$

Hence (4.1) is valid when n is odd and larger than 1. Therefore (4.1) is valid for all n. ∎

5. Conclusion

First of all we remark that the Bender–Knuth conjecture [2, p. 50] is so clearly of the same type as MacMahon's conjecture that it ought to be provable by our methods.

We also mention that the case $q = 1$ of MacMahon's conjecture can be treated much more easily in that Lemmas 4, 5, 10, and 11 are unnecessary.

Note added in proof. Ian Macdonald of Queen Mary College has independently obtained a proof of MacMahon's conjecture from group representation theory. The interrelationship of Lemmas 4 and 5 with reciprocal polynomials and basic hypergeometric series is explored in my paper Implications of the MacMahon conjecture, *in* "Combinatoire et représentation du groupe symetrique" (D. Foata, ed.), Lecture Notes in Mathematics, No. 579, Springer-Verlag, Berlin and New York pp. 287–296, 1977. The relationship between MacMahon's conjecture and the Bender–Knuth conjecture has been established in my paper Plane partitions (II): The equivalence of the Bender–Knuth and MacMahon conjectures, *Pac. J. Math* **72** (1977), 283–291.

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