ON THE THEOREMS OF WATSON AND DRAGONETTE FOR RAMANUJAN'S MOCK THETA FUNCTIONS.

By George E. Andrews.[1]

I. Introduction. The mock theta functions were first studied by Ramanujan; G. H. Hardy in [9; p. 354] defines such functions as follows

> . . . a "mock θ-function" is a function defined by a q-series convergent for $|q|<1$, for which we can calculate asymptotic formulae, when q tends to a "rational point" $e^{2r\pi i/s}$, of the same degree of precision as those furnished, for the ordinary θ-functions, by the theory of linear transformation.

The mock theta functions we shall study are the third order functions given below:

$$f(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}$$

$$\phi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})}$$

$$\psi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}$$

$$\omega(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2}$$

$$\nu(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(1+q)(1+q^3)\cdots(1+q^{2n+1})}$$

There are two other third order functions $\chi(q)$ and $\rho(q)$ [12; p. 62]. They satisfy the following equations [12; p. 63]

$$\chi(q)=\tfrac{1}{4}f(q)+\tfrac{3}{4}\theta_4{}^2(0,q^3)\prod_{r=1}^{\infty}(1-q^r)^{-1}$$

$$\rho(q)=-\tfrac{1}{2}\omega(q)+\tfrac{3}{2}[\tfrac{1}{2}q^{-3/8}\theta_2(0,q^{3/2})]^2\prod_{r=1}^{\infty}(1-q^{2r})^{-1}.$$

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We shall not study them; however, their behavior can be derived from these equations once the behavior of f and ω is known.

In the paper [2], Dragonette derived general linear transformation formulae for the third order mock theta functions and used her results to obtain very precise asymptotic formulae for the power series coefficients of these functions. G. N. Watson in [12] had already studied the third order mock theta functions under the specific transformations; $S\colon \tau \to \tau + 1$ and $T\colon \tau \to 1/\tau$. Since these transformations generate the modular group, Dragonette was able to derive the general modular transformation formulae by the process of iteration. Her method, however, left certain roots of unity unevaluated and also gave only estimates for the "error" terms arising. In this paper, we shall generalize Watson's method [12; pp. 73-76] and shall thus obtain precise linear transformation formulae with all roots of unity and "error" terms evaluated. This will carry us through Sections II and III. In Section IV, we shall show that all sums of roots of unity that are to be considered are actually generalized Kloosterman sums and therefore subject to the results of Salié [10] and Lehmer [6]. We shall show, in fact, that the exponential sums appearing in the asymptotic expansions of Dragonette [2; Theorems II and IV] are actually special cases of the exponential sum $A_k(n)$ associated with the partition function (cf. [6]). In Section V, we shall apply our results to the problem of improving the error terms in the asymptotic expansion theorems of Dragonette. For example, Dragonette has proved the following [2; p. 478]

THEOREM. *The coefficient of* q^n *in the series* $f(q) = \sum_{n=0}^{\infty} A(n) q^n$ *is*

$$A(n) = \sum_{0 < k \leq n^{\frac12}} \frac{\lambda(k) \exp\{\pi(n - 1/24)^{\frac12}/k\sqrt{6}\}}{k^{\frac12}(n - 1/24)^{\frac14}} + O(n^{\frac12} \operatorname{og} n)$$

where

$$\lambda(1) = \tfrac12(-1)^{n-1}$$

and

$$\lambda(k) = \tfrac12 \sum_{\substack{h \bmod 2k \\ \mathrm{g.c.d.}(h,2k)=1}} \exp(-\pi i h n/k)\, \epsilon_{h,k}{}^{(f)} \qquad (k = 2, 3, \cdots, [n^{\frac12}]).$$

We shall improve Dragonette's result to the following

THEOREM 5.1. *The coefficient of* q^n *in the series* $f(q) = \sum_{n=0}^{\infty} A(n)$ *is*

$$A(n) = \sum_{k=1}^{[n^{\frac12}]} \frac{\lambda(k) \exp\{\pi(n - 1/24)^{\frac12}/k\sqrt{6}\}}{k^{\frac12}(n - 1/24)^{\frac14}} + O(n^{\epsilon})$$

where

$$\lambda(k) = \begin{cases} \frac{1}{2}(-1)^{\frac{1}{2}(k+1)}A_{2k}(n) & k \text{ odd} \\ \frac{1}{2}(-1)^{\frac{1}{2}k}A_{2k}(n-\frac{1}{2}k) & k \text{ even,} \end{cases}$$

and ϵ *is positive and arbitrarily small.* $A_k(n)$ *is the exponential sum appearing in the Hardy-Ramanujan formula for* $p(n)$ [9; p. 284].

It seems likely that a closer study of the asymptotic expansion for $A(n)$ would show that the above series would diverge if extended to (since $\lambda(k) = O(k^{\frac{1}{2}+\epsilon})$ and probably $|\lambda(k)| > c \cdot k^{\frac{1}{2}}$ for some $c > 0$ and an infinite number of k). If, however, e^x is replaced by $2\sinh x$ in the above expansion, the problem becomes deeper. I would conjecture that the new series is not absolutely convergent (for probably $|\lambda(p)| > c \cdot p^{\frac{1}{2}}$ for some $c > 0$ and a positive proportion of all primes p), but the extreme accuracy of Dragonette's numerical results [2; p. 494] makes plausible the conjecture that the new series is conditionally convergent and that it converges to $A(n)$.

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II. The modular transformation formulae of f and ω. Our first two theorems of this section improve the results in Dragonette's Theorem I [2; p. 477]. The next two theorems improve Dragonette's results for ω in Theorem III [2; p. 499, line (7.7)]. We complete this section by simplifying the expressions for the roots of unity we have obtained.

Throughout the course of this paper we shall endeavor to keep our notation consistent with that of Dragonette. The following well-known theorem together with its notation and conventions will be used over and over.

If $P(q) = \prod_{r=1}^{\infty} (1-q^r)^{-1}$, then

$$\begin{aligned} P(\exp\{2\pi i(h+iz)/k\}) &= \omega_{h,k} z^{\frac{1}{2}} \exp\{\pi(z^{-1}-z)/12k\} \\ &\quad \cdot P(\exp\{2\pi i(h'+iz^{-1})/k\}) \end{aligned} \tag{2.0.1}$$

where $\operatorname{Re} z > 0$ with the principal branch of $z^{\frac{1}{2}}$ taken, $hh' \equiv -1 \pmod{k}$, and $\omega_{h,k}$ a $24k$-th root of unity [9; pp. 290-291]. We remark that $P(q)$ is essentially the modular form $\eta(\tau)$, and (2.0.1) is the transformation formula for $\eta(\tau)$ in thin disguise.

THEOREM 2.1. *Let* $\text{g.c.d.}(h,k) = 1$, $hh' + kk' = -1$, h *be even, and* $\operatorname{Re} z > 0$ *with the principal branch of* $z^{\frac{1}{2}}$ *taken. Then*

$$f(\exp\{\pi i(h+iz)/k\}) = \eta_{h,k}{}^{(f)} 2^{\frac{3}{2}} z^{-\frac{1}{2}} \exp(-\pi(4z^{-1}+z/8)/3k)$$
$$\cdot\,\omega(\exp\{2\pi i(h'+iz^{-1})/k\}) + E_1{}^{(f)}(h,k;z)$$

where

$$\eta_{h,k}{}^{(f)} = \exp\{\tfrac{1}{2}\pi i(2h'+k'-3-3h'k'+3h'/k)\}\omega_{\frac{1}{2}h,k},$$

and

$$E_1{}^{(f)}(h,k;z) = 2^{\frac{1}{2}} k^{-1} \exp(-\pi z/24k) \sum_{\nu \bmod k} \delta_1{}^{(f)}(\nu;\tfrac{1}{2}h;2h';k) z^{\frac{1}{2}} J_1{}^{(f)}(\nu;k;z)$$

with

$$\delta_1{}^{(f)}(\nu;h;h';k) = (-1)^{\nu}\omega_{h,k}\exp\{\pi i h'(-3\nu^2+\nu)/k\}$$

and

$$J_1{}^{(f)}(\nu;k;z) = \int_{-\infty}^{\infty} \frac{\exp(-3\pi z x^2/2k)}{\cosh(\pi i(\nu-\frac{1}{6})/k - \pi z x/2k)}\,dx.$$

Finally if $1 \leqq \nu \leqq k$, *we have*

$$|\, z^{\frac{1}{2}} J_1{}^{(f)}(\nu;k;z) \,| \leqq 3^{-\frac{1}{2}} 2^{\frac{3}{2}} N^{\frac{1}{2}} k \,|\, k - 2(\nu - \tfrac{1}{6}) \,|^{-1}$$

provided $z = kN^{-2} - ik\phi$ *with* $|\,\phi\,| \leqq k^{-1}N^{-1}$ *and* $1 \leqq k \leqq N$.

THEOREM 2.2. *Let* g. c. d. $(h,k)=1$, $hh''+2kk''=-1$, h *be odd, and* $\operatorname{Re} z > 0$ *with the principal branch of* $z^{\frac{1}{2}}$ *taken. Then*

$$f(\exp\{\pi i(h+iz)/k\})$$
$$= \epsilon_{h,k}{}^{(f)} z^{-\frac{1}{2}} \exp(-\pi(z+z^{-1})/24k) f(\exp\{\pi i(h''+iz^{-1})/k\}) + E_2{}^{(f)}(h,k;z)$$

where

$$\epsilon_{h,k}{}^{(f)} = (-1)^{k+\frac{1}{2}(hh''-1)} \exp(\pi i h''/2 + 3\pi i h h''^2 k/2)\,\omega_{h,2k},$$

and

$$E_2{}^{(f)}(h,k;z) = E_1{}^{(f)}(2h,2k;2z)$$
$$= \exp(-\pi z/24k) k^{-1} \sum_{\nu \bmod k} \delta_1{}^{(f)}(\nu;h;h'';2k) z^{\frac{1}{2}} J_1{}^{(f)}(\nu;2k;2z)$$

with $\delta_1{}^{(f)}(\nu;h;h'';2k)$ *and* $J_1{}^{(f)}(\nu;2k;2z)$ *as in Theorem* 2.1.

THEOREM 2.3. *Let* g. c. d. $(h,k)=1$, $hh'+kk'=-1$, h *be even, and* $\operatorname{Re} z > 0$ *with the principal branch of* $z^{\frac{1}{2}}$ *taken. Then*

$$\omega(\exp\{\pi i(h+iz)/k\}) = \tfrac{1}{2} z^{-\frac{1}{2}} \epsilon_{h,k}{}^{(\omega)} \exp(2\pi z/3k + \pi z^{-1}/12k)$$
$$\cdot f(\exp\{2\pi i(h'+iz^{-1})/k\}) + E_1{}^{(\omega)}(h,k;z)$$

where

$$\epsilon_{h,k}{}^{(\omega)} = (-1)^{\frac{1}{2}(k-1)} \exp(3\pi ihk/4 - 3\pi ih/4k)\,\omega_{h,k}$$

and

$$E_1{}^{(\omega)}(h,k;z) = k^{-1}\exp(2\pi z/3k) \sum_{\nu \bmod k} \delta_1{}^{(\omega)}(\epsilon;\nu;h;h';k) z^{\frac{1}{2}} J_1{}^{(\omega)}(\nu;k;z)$$

with

$$\delta_1{}^{(\omega)}(\epsilon;\nu;h;h';k) = i\epsilon_{h,k}{}^{(\omega)} \exp(-3\pi ih'\nu^2/k + \pi ih'/k)$$

and

$$J_1{}^{(\omega)}(\nu;k;z) = \int_{-\infty}^{\infty} \frac{\exp(-3\pi zx^2/k)}{\tanh\{\pi i(\nu - \frac{1}{6})/k - \pi zx/k\}}\,dx.$$

Finally if $-\frac{1}{2}(k+1) \leqq \nu \leqq \frac{1}{2}(k-1)$, *we have*

$$|\, z^{\frac{1}{2}} J_1{}^{(\omega)}(\nu;k;z)\,| \leqq 2^{-\frac{1}{2}}3^{-\frac{1}{2}} \exp(\pi/6) N^{\frac{1}{2}} k \,|\, \nu - \tfrac{1}{6}\,|^{-1}$$

provided $z = kN^{-2} - ik\phi$ *with* $|\,\phi\,| \leqq (kN)^{-1}$ *and* $1 \leqq k \leqq N$.

THEOREM 2.4. *Let* g.c.d.$(h,k) = 1$, $hh'' + 2kk'' = -1$, *h be odd, and* $\mathrm{Re}\, z > 0$ *with the principal branch of* $z^{\frac{1}{2}}$ *taken. Then*

$$\omega(\exp\{\pi i(h+iz)/k\}) = \eta_{h,k}{}^{(\omega)} z^{-\frac{1}{2}} \exp(2\pi z/3k - 2\pi z^{-1}/3k)$$
$$\cdot\, \omega(\exp\{\pi i(h'' + iz^{-1})/k\}) + E_2{}^{(\omega)}(h,k;z)$$

where

$$\eta_{h,k}{}^{(\omega)} = i(-1)^{\frac{1}{2}(h''+1)} \exp\{-3\pi ih''k''/2 - 3\pi i(h''-h)/4k\}\,\omega_{h,k}$$

and

$$E_2{}^{(\omega)}(h,k;z) = k^{-1}\exp(2\pi z/3k) \sum_{\nu \bmod k} (-1)^{\nu} \exp(-3\pi ih''/4k)$$
$$\cdot\, \delta_1{}^{(\omega)}(\eta;\mu;h;h'';k) z^{\frac{1}{2}} J_1{}^{(\omega)}(\mu;k;z)$$

with $\delta_1{}^{(\omega)}(\eta;\mu;h;h'';k)$ and $J_1{}^{(\omega)}(\mu;k;z)$ *as in Theorem* 2.3 *and* $\mu = \nu + \frac{1}{2}$ *written for simplicity of notation.*

We now prove these four theorems. The proof of Theorem 2.1 is given in detail; since the proofs of all four theorems are similar, we omit many of the details in the remaining theorems. Also, it should be noted that Theorem 2.3 may be derived directly from Theorem 2.1; however, such a derivation would be complicated and would not likely shorten our exposition.

Proof of Theorem 2.1. We shall freely utilize the results in [12]. We let

$$F(q) = f(q)/P(q)$$

$$= f(q) \prod_{r=1}^{\infty} (1 - q^r)$$

(2.1.1)

$$= 1 + 4 \sum_{n=1}^{\infty} (-1)^n q^{\frac{1}{2}n(3n+1)} (1+q^n)^{-1} \quad [12; \text{ p. } 64]$$

$$= 1 + 4 \sum_{n=1}^{\infty} (-1)^n q^{3n^2/2} (q^{-\frac{1}{2}n} + q^{\frac{1}{2}n})^{-1}.$$

If $q = \exp\{\pi i(h+iz)/k\}$, then

$$F(q) = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n \exp\{3n^2\pi i(h+iz)/2k\}}{\cosh(n\pi i(h+iz)/2k)}$$

$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^n \exp(3\pi i h n^2/2k - 3\pi z n^2/2k)}{\cosh(\pi i h n/2k - \pi z n/2k)}, \text{ letting } n = mk + \nu$$

(2.1.3)

$$= \sum_{\nu=0}^{k-1} \sum_{m=-\infty}^{\infty} \frac{(-1)^{mk+\nu} \exp\{3\pi i h(m^2k^2 + 2mk\nu + \nu^2)/2k - 3\pi z(mk+\nu)^2/2k\}}{\cosh\{\pi i h m/2 + \pi i h \nu/2k - \pi z(mk+\nu)/2k\}}$$

$$= \sum_{\nu=0}^{k-1} (-1)^\nu \exp(3\pi i h \nu^2/2k) \sum_{m=-\infty}^{\infty} \frac{(-1)^m \exp\{-3\pi z(mk+\nu)^2/2k\}}{\cosh\{\pi i h \nu/2k - \pi z(mk+\nu)/2k\}};$$

by the Poisson summation formula [1; pp. 7-9] we obtain

$$F(q) = \sum_{\nu=0}^{k-1} (-1)^\nu \exp(3\pi i h \nu^2/2k) \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\{-3\pi z(kx+\nu)^2/2k + \pi i x + 2\pi i n x\}}{\cosh\{\pi i h \nu/2k - \pi z(kx+\nu)/2k\}} \, dx;$$

making the change of variable $kx + \nu \to x$, we obtain

$$F(q) = k^{-1} \sum_{\nu=0}^{k-1} (-1)^\nu \exp(3\pi i h \nu^2/2k) \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\{(2n+1)\pi i(x-\nu)/k - 3\pi z x^2/2k\}}{\cosh(\pi i h \nu/2k - \pi z x/2k)} \, dx.$$

Notice now that if ν is replaced by $\nu \pm k$ the change introduced is

$$(-1)^{\pm k} \exp\{3\pi i h(\pm 2k + k^2)/2k\} (-1)^{(2n+1)-\frac{1}{2}h} = 1.$$

Hence we have

$$F(q) = k^{-1} \sum_{\nu \bmod k} (-1)^\nu \exp(3\pi i h \nu^2/2k) \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\{(2n+1)\pi i(x-\nu)/k - 3\pi z x^2/2k\}}{\cosh(\pi i h \nu/2k -)zx/2k)} \, dx.$$

Now

$$\sum_{n=-\infty}^{\infty} = \sum_{n=0}^{\infty} + \sum_{n=-1}^{-\infty} = \sum_{n=0}^{\infty} + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\{-(2n-1)\pi i(x-\nu)/k - 3\pi z x^2/2k\}}{\cosh(\pi i h \nu/2k -)zx/2k)} \, dx;$$

sending $x \to -x$ and $n \to n+1$ we derive

$$\sum_{n=-\infty}^{\infty} = \sum_{n=0}^{\infty} + \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\{(2n+1)\pi i(x+\nu)/k - 3\pi z x^2/2k\}}{\cosh(-\pi i h \nu/2k - \pi z x/2k)} \, dx.$$

Replacing ν by $-\nu$ in the second sum we see that

$$(2.1.4)\quad F(q)=2k^{-1}\sum_{\nu \bmod k}(-1)^{\nu}\exp(3\pi ih\nu^{2}/2k)$$
$$\cdot\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\{(2n+1)\pi i(x-\nu)/k-3\pi zx^{2}/2k\}}{\cosh(\pi ih\nu/2k-\pi zx/2k)}\,dx.$$

We now define $C_k(x)=\dfrac{\cosh 3x}{\cosh\,(x/k)}$.

Clearly $C_k(x)$ is an entire function of x if k is an odd integer. Hence if

$$\mathcal{F}_n(x)=\frac{\exp\{(2n+1)\pi i(x-\nu)/k-3\pi zx^{2}/2k\}}{\cosh(\pi ih\nu/2k-\pi zx/2k)}$$
$$=\frac{\exp\{(2n+1)\pi i(x-\nu)/k-3\pi zx^{2}/2k\}}{\cosh\{3(\pi ih\nu-\pi zx)/2\}}C_k(\{\pi ih\nu-\pi zx\}/2)$$
$$=\frac{(-1)^{\frac{1}{2}h\nu}\exp\{(2n+1)\pi i(x-\nu)/k-3\pi zx^{2}/2k\}}{\cosh(3\pi zx/2)}C_k(\{\pi ih\nu-\pi zx\}/2)$$

then $\mathcal{F}_n(x)$ has (at most) simple poles at

$$x_m=(2m+1)i/3z,\ m \text{ any integer.}$$

The residue at x_m is

$$\lambda_{n,m}=2(3\pi z)^{-1}(-1)^{\frac{1}{2}h\nu-m}\exp\{(2n+1)\pi i(x_m-\nu)/k$$
$$-3\pi zx_m{}^{2}/2k\}C_k(\{\pi ih\nu-\pi zx_m\}/2).$$

Now by Cauchy's theorem,

$$\left\{\int_{-\infty}^{\infty}-P\int_{-\infty+(2n+1)i/3z}^{\infty+(2n+1)i/3z}\right\}\mathcal{F}_n(x)\,dx=2\pi i(\lambda_{n,0}+\lambda_{n,1}+\cdots+\tfrac{1}{2}\lambda_{n,n}).$$

Hence

$$F(q)=2k^{-1}\sum_{\nu \bmod k}(-1)^{\nu}\exp(3\pi ih\nu^{2}/2k)2\pi i\sum_{n=0}^{\infty}(\lambda_{n,0}+\lambda_{n,1}+\cdots+\tfrac{1}{2}\lambda_{n,n})$$
$$+2k^{-1}\sum_{\nu \bmod k}(-1)^{\nu}\exp(3\pi ih\nu^{2}/2k)\sum_{n=0}^{\infty}P\int_{-\infty+x_n}^{\infty+x_n}\mathcal{F}_n(x)\,dx$$
$$=\Sigma_1+\Sigma_2.$$

It will turn out that Σ_1 will yield the main term in the transformation formula involving $\omega(q)$, and Σ_2 will yield the "error" term.

In Σ_1, we have

$$\sum_{n=0}^{\infty}\{\lambda_{n,0}+\lambda_{n,1}+\cdots+\tfrac{1}{2}\lambda_{n,n}\}=\sum_{n=0}^{\infty}\{\tfrac{1}{2}\lambda_{m,m}+\lambda_{m+1,m}+\lambda_{m+2,m}+\cdots\},$$

by rearrangement of the first series,

$$=\sum_{n=0}^{\infty}\lambda_{m,m}\{\tfrac{1}{2}+\sum_{l=1}^{\infty}\exp\{2\pi i(x_m-\nu)l/k\}$$

since $\lambda_{m+l,m}=\exp\{2\pi i(x_m-\nu)l/k\}\cdot\lambda_{m,m}$

(2.1.6)

$$=\tfrac{1}{2}\sum_{n=0}^{\infty}\lambda_{m,m}\frac{1+\exp\{2\pi i(x_m-\nu)/k\}}{1-\exp\{2\pi i(x_m-\nu)/k\}}.$$

We now take a closer look at $\lambda_{m,m}$. Since $\lambda_{m,m}$ is a residue of $\mathfrak{F}_m(x)$, we note that $\lambda_{m,m}\neq 0$ if and only if $\cosh(\pi ih\nu/2k-\pi zx_m/2k)=0$, i.e. $\cos\{\tfrac{1}{2}\pi(h\nu/k-(2m+1)/3k\}=0$. Thus we must have $3h\nu-(2m+1)\equiv 0 \pmod{3k}$.

If $2m+1\not\equiv 0 \pmod 3$, then the above congruence is impossible. Hence $2m+1=6j+3$. Therefore,

$$h\nu-(2j+1)\equiv 0 \pmod k, \text{ or}$$

$$\nu\equiv -h'(2j+1) \pmod k$$

provided $hh'\equiv -1 \pmod k$.

Thus when considering $\sum_{\nu \bmod k}$ we see that the only non-zero term is the term $\nu\equiv -h'(2j+1) \pmod k$.

We now evaluate $\lambda_{m,m}$ when $m=3j+1$ and $\nu\equiv -h'(2j+1) \pmod k$.

$$\lambda_{m,m}=\lim_{x\to x_m}(x-x_m)\mathfrak{F}_m(x)$$

$$=\frac{\exp\{(2m+1)\pi i(x_m-\nu)/k-3\pi zx_m{}^2/2k\}}{-\tfrac{1}{2}\pi zk^{-1}\sinh(\pi ih\nu/2k-\pi zx_m/2k)}$$

$$=\frac{\exp\{3(2j+1)\pi i[(2j+1)iz^{-1}+h'(2j+1)]/k+3\pi(2j+1)^2/2kz\}}{-\tfrac{1}{2}\pi zk^{-1}\sinh\{-\pi ihh'(2j+1)/2k-\pi i(2j+1)/2k\}}$$

(2.1.7)

$$=\frac{-2k\cdot\exp\{3(2j+1)^2\pi i(h'+iz^{-1})/k+3\pi(2j+1)^2/2kz\}}{i\pi z\sin\{\pi(2j+1)kk'/2k\}}$$

$$=-2k(\pi iz)^{-1}(-1)^{k'j+\frac{1}{2}(k'-1)}\exp\{3(2j+1)^2\pi i(h'+iz^{-1})/k$$
$$+3\pi(2j+1)^2/2kz\}.$$

Hence

$$\Sigma_1=2k^{-1}2\pi i\tfrac{1}{2}\sum_{j=0}^{\infty}(-1)^{-h'(2j+1)}\exp\{3\pi ih(h'(2j+1))^2\}$$
$$\cdot(-2k)(\pi iz)^{-1}(-1)^{k'j+\frac{1}{2}(k'-1)}$$

$$\cdot \exp\{3(2j+1)^2\pi i(h'+iz^{-1})/k+3\pi(2j+1)^2/2kz\}$$

$$\frac{1+\exp\{2\pi i[(2j+1)i/z+h'(2j+1)]/k\}}{1-\exp\{2\pi i[(2j+1)i/z+h'(2j+1)]/k\}}.$$

$$(2.1.8)\quad = 4z^{-1}(-1)^{h'-1+\frac{1}{2}(k'-1)}\sum_{j=0}^{\infty}(-1)^j\exp\{3\pi ih'(2j+1)^2(-1-kk')/2k\}$$

$$\cdot \exp\{3(2j+1)^2\pi i(h'+iz^{-1})/k+3\pi(2j+1)^2/2kz\}$$

$$\frac{1+\exp\{2\pi i[(2j+1)i/z+h'(2j+1)]/k\}}{1-\exp\{2\pi i[(2j+1)i/z+h'(2j+1)]/k\}}.$$

$$= 4z^{-1}(-1)^{h'-1-\frac{1}{2}(k'-1)}\exp\{-3\pi ih'k'/2+3\pi i(h'+iz^{-1})/2k\}$$

$$\cdot\sum_{j=0}^{\infty}(-1)^j\exp\{3j(j+1)2\pi i(h'+iz^{-1})/k\}$$

$$\frac{1+\exp\{2\pi i(2j+1)(h'+iz^{-1})/k\}}{1-\exp\{2\pi i(2j+1)(h'+iz^{-1})/k\}}.$$

$$= 4z^{-1}(-1)^{h'-1-\frac{1}{2}(k'-1)}\exp\{-3\pi ih'k'/2+3\pi ih'/2k-3\pi/2kz\}$$

$$\frac{\omega(\exp[2\pi i(h'+iz^{-1})/k])}{P(\exp[4\pi i(h'+iz^{-1})/k])}\quad [12;\text{ p. } 66].$$

We now evaluate Σ_2. We recall that

$$\Sigma_2 = 2k^{-1}\sum_{\nu \bmod k}(-1)^\nu\exp(3\pi ih\nu/2k)\sum_{n=0}^{\infty}P\int_{-\infty+x_n}^{\infty+x_n}\mathcal{J}_n(x)\,dx,$$

where $x=(2n+1)i/3z$.

Now

$$2\sum_{n=0}^{\infty}=\sum_{n=0}^{\infty}+\sum_{n=0}^{-\infty}P\int_{-\infty-(2n+1)i/3z}^{\infty+(2n+1)i/3z}\frac{\exp\{-(2n-1)\pi i(x-\nu)/k-3\pi zx^2/2k\}}{\cosh(\pi ih\nu/2k-\pi zx/2k)}dx$$

$$=\sum_{n=0}^{\infty}+\sum_{n=1}^{\infty-}P\int_{-\infty+(-2n+1)i/3z}^{\infty+(-2n+1)i/3z}\frac{\exp\{(2n+1)\pi i(x-\nu)/k-3\pi zx^2/2k\}}{\cosh(\pi ih\nu/2k-\pi zx/2k)}\,dx$$

upon replacing ν by $-\nu$ and sending $x\to -x$; thus

$$\Sigma_2 = k^{-1}\sum_{\nu \bmod k}(-1)^\nu\exp(3\pi ih\nu^2/2k)\sum_{n=-\infty}^{\infty}P\int_{-\infty+x_n}^{\infty+x_n}\mathcal{J}_n(x)\,dx.$$

Now

$$P\int_{-\infty+x_n}^{\infty+x_n}\mathcal{J}_n(x)\,dx = P\int_{-\infty}^{\infty}\mathcal{J}_n(x+x_n)\,dx,$$

and

$$\exp\{(2n+1)\pi i(x+x_n-\nu)/k-3\pi z(x+x_n)^2/2k\}$$

$$= \exp\{(2n+1)\pi ix/k - (2n+1)^2\pi/3kz - (2n+1)\pi i\nu/k$$
$$- 3\pi zx^2/2k - \pi x(2n+1)i/k\} \cdot \exp\{\pi(2n+1)^2/6kz\}$$
$$= \exp\{-(2n+1)\pi i\nu/k - (2n+1)^2\pi/6kz - 3\pi zx^2/2k\}$$

We return to Σ_2 letting $n = 3p + \delta$,

$$\Sigma_2 = k^{-1} \sum_{\nu \bmod k} (-1)^\nu \exp(3\pi ih\nu^2/2k) \sum_{\delta=-1}^{1} \sum_{p=-\infty}^{\infty} \exp\{-(2n+1)\pi i\nu/k - (2n+1)^2\pi/6kz\}$$

(2.1.9)

$$\cdot P\int_{-\infty}^{\infty} \frac{\exp(-3\pi zx^2/2k)}{\cosh\{\pi ih\nu/2k - \pi zx/2k - \pi ip/k - \pi i(2\delta+1)/6k\}} dx$$

If we replace $\frac{1}{2}h - p$ by ν, i.e. $\nu \to -2h'\nu - 2ph'$, then

$$\cosh\{\pi i(\tfrac{1}{2}h\nu - p)/k - \pi zx/2k - \pi i(2\delta+1)/6k\}$$
$$\to \cosh\{-\pi ip/k - \pi i(1+kk')(\nu+p)/k - \pi zx/2k - \pi i(2\delta+1)/6k\}$$
$$= (-1)^{\nu+p}\cosh\{\pi i\nu/k - \pi zx/2k - \pi i(2\delta+1)/6k\},$$

and

$$(-1)^\nu \exp\{3\pi ih\nu^2/2k - (2n+1)\pi i\nu/k\} \to \exp\{3\pi ih4h'^2(\nu+p)^2 + 2(6p+2\delta+1)\pi ih'(\nu+p)/k\}$$
$$= \exp\{-6\pi ih'(\nu^2+2\nu p+p^2)/k + 12p\pi ih'(\nu+p)/k + 2(2\delta+1)\pi ih'p/k + 2(2\delta+1)\pi ih'\nu/k\}$$
$$= \exp\{6\pi ih'p^2/k + (2\delta+1)\pi i2h'p/k - 6\pi ih'\nu^2/k + (2\delta+1)\pi i2h'\nu/k\}.$$

Thus in Σ_2 we now have $\sum_{p=-\infty}^{\infty}$ free of x and ν. We define

$$U_\delta(z) = \sum_{p=-\infty}^{\infty} (-1)^p \exp\{6\pi ih'p^2/k + (2\delta+1)\pi ih'p/k - (6p+2\delta+1)^2\pi/6kz\}$$

(2.1.10)

$$= \exp\{-(2\delta+1)^2\pi/6kz\} \sum_{p=-\infty}^{\infty} (-1)^p \exp\{2\pi i(h'+iz^{-1}) \cdot (3p^2+(2\delta+1)p)/k\}.$$

Letting $\tau' = (h'+iz^{-1})/k$, we have for $\delta = 1$

$$U_1(z) = \exp(-3\pi/2kz) \sum_{p=-\infty}^{\infty} (-1)^p \exp\{2\pi i\tau' 3p(p+1)\}.$$

In this series the p-th term and the $(-p-1)$-st term cancel each other, hence $U_1(z)=0$.

On the other hand,

$$
\begin{aligned}
U_0(z) &= \exp(-\pi/6kz) \sum_{p=-\infty}^{\infty} (-1)^p \exp\{2\pi i\tau' p(3p+1)\} \\
&= \exp(-\pi/6kz) \sum_{p=-\infty}^{\infty} (-1)^p \exp\{2\pi i\tau' p(3p-1)\} \\
&= U_{-1}(z) \qquad\qquad (2.1.11) \\
&= \frac{\exp(-\pi/6kz)}{P(\exp[4\pi i(h'+iz^{-1})/k])} \qquad [4;\ \text{p. } 284].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Sigma_2 = \frac{k^{-1}\exp(-\pi/6kz)}{P[4\pi i(h'+iz^{-1})/k]} \sum_{\delta=-1}^{0} \sum_{\nu \bmod k} (-1)^\nu \exp\{&-6\pi i h'\nu^2/k \\
&+ (2\delta+1)\pi i 2h'\nu/k\}
\end{aligned}
$$
$$
\cdot \int_{-\infty}^{\infty} \frac{\exp(-3\pi z x^2/2k)}{\cosh\{\pi i\nu/k - \pi z x/2k - \pi i(2\delta+1)/6k\}}\, dx.
$$

In the case $\delta=-1$, if we replace ν by $-\nu$ and send $x\to -x$ we get the case $\delta=0$, hence

$$
\begin{aligned}
\Sigma_2 &= \frac{k^{-1}2\cdot\exp(-\pi/6kz)}{P[4\pi i(h'+iz^{-1})/k]} \sum_{\nu \bmod k} (-1)^\nu \exp\{-6\pi i h'\nu^2/k + 2\pi i h'\nu/k\} \qquad (2.1.12) \\
&\quad \cdot \int_{-\infty}^{\infty} \frac{\exp(-3\pi z x^2/2k)}{\cosh\{\pi i(\nu-\frac{1}{6})/k - \pi z x/2k\}}\, dx.
\end{aligned}
$$

Since h is even, we may write (2.0.1) as

$$
\begin{aligned}
P(\exp\{2\pi i(\tfrac{1}{2}h + i\tfrac{1}{2}z)/k\}) = \omega_{\frac{1}{2}h,k} 2^{-\frac{1}{2}} z^{\frac{1}{2}} \exp(\pi/6kz& \\
-\pi z/24k) P(\exp\{4\pi i(h'+iz^{-1})/k\})&\,;
\end{aligned}
$$

thus since we have established

$$
\frac{f(\exp\{\pi i(h+iz)/k\})}{P(\exp\{\pi i(h+iz)/k\})} = \Sigma_1 + \Sigma_2,
$$

we have

$$
\begin{aligned}
&f(\exp[\pi i(h+iz)/k]) \\
&\quad = (-1)^{h'-1+\frac{1}{2}(k'-1)} \exp\{-3\pi i h'k'/2 + 3\pi i h'/2k\} \omega_{\frac{1}{2}h,k} 2^{\frac{3}{2}} z^{-\frac{1}{2}} \\
&\qquad \cdot \exp(-4\pi/3kz - \pi z/24k) \omega(\exp[2\pi i(h'+iz^{-1})/k])
\end{aligned}
$$

$$(2.1.13) \qquad + 2^{\frac12} z^{\frac12} k^{-1} \exp(-\pi z/24k)\,\omega_{\frac12 h,k} \sum_{\nu \bmod k} (-1)^\nu \exp(-6\pi i h'\nu^2/k - 2\pi i h'\nu/k) \cdot \int_{-\infty}^{\infty} \frac{\exp(-3\pi z x^2/2k)}{\cosh\{\pi i(\nu - \frac16)/k - \pi z x/2k\}}\, dx.$$

This is the required transformation formula; thus when h is even, the general linear transformation sends f into ω plus a correction $E_1^{(f)}(h,k;z)$.

We proceed to the final part of the theorem in which we estimate the integral involved in $E_1^{(f)}(h,k;z)$.

$$\begin{aligned} z^{\frac12} J_1^{(f)}(\nu;k;z) &= z^{\frac12} \int_{-\infty}^{\infty} \frac{\exp(-3\pi z x^2/2k)}{\cosh\{\pi i(\nu - \frac16)/k - \pi z k/2k\}}\, dx \\ &= 2k\pi^{-1} z^{-\frac12} \int_{-\infty}^{\infty} \frac{\exp(-6kt^2/\pi z)}{\cosh\{\pi i(\nu - \frac16)/k - t\}}\, dt \end{aligned}$$

by Cauchy's theorem. Now with $z = kN^{-2} - ik\phi$ and $1 \leqq \nu \leqq k$, we have

$$| z^{\frac12} J_1^{(f)}(\nu;k;z) | \leqq 2k\pi^{-1} | z |^{-\frac12} \int_{-\infty}^{\infty} \frac{\exp\{-6k(\operatorname{Re} z^{-1}) t^2/\pi\}}{| \cosh\{\pi i(\nu - \frac16)/k - t\} |}\, dt$$

$$\leqq 2k\pi^{-1} | z |^{-\frac12} | \cos\{\pi(\nu - \tfrac16)/k\} |^{-1} \int_{-\infty}^{\infty} \exp\{-6k(\operatorname{Re} z^{-1}) t^2/\pi\}\, dt$$

$$\leqq 2k\pi^{-1} | z |^{-\frac12} | 1 - 2k^{-1}(\nu - \tfrac16) |^{-1} \pi^{\frac12} \{12k(\operatorname{Re} z^{-1})\}^{-\frac12} \int_{-\infty}^{\infty} \exp(-u^2/2)\, du$$

$$\leqq k^{\frac32} (3\pi)^{-\frac12} | k - 2(\nu - \tfrac16) |^{-1} | z |^{-\frac12} (\operatorname{Re} z^{-1})^{-\frac12} (2\pi)^{\frac12}$$

$$= k^{\frac32} (\tfrac23)^{\frac12} | k - 2(\nu - \tfrac16) |^{-1} | z |^{-\frac12} \{(\operatorname{Re} z)/| z |^2\}^{-\frac12}.$$

Now $\operatorname{Re} z = kN^{-2}$, and

$$\begin{aligned} | z | &= (k^2 N^{-4} + k^2 \phi^2)^{\frac12} \\ &\leqq (k^2 N^{-4} + k^2 k^{-2} N^{-2})^{\frac12} \\ &= \{ (k^2 + N^2) N^{-4} \}^{\frac12} \\ &\leqq 2^{\frac12} N^{-1}. \end{aligned}$$

Hence

$$| z |^{\frac12} (\operatorname{Re} z)^{-\frac12} \leqq 2^{\frac14} N^{-\frac12} N k^{-\frac12} = 2^{\frac14} N^{\frac12} k^{-\frac12}.$$

Thus

$$| z^{\frac12} J_1^{(f)}(\nu;k;z) | \leqq 2^{\frac34} N^{\frac12} k 3^{-\frac12} | k - 2(\nu - \tfrac16) |^{-1}$$

This concludes the proof of Theorem 2.1.

Proof of Theorem 2.2. As before

$$F(q) = f(q)/P(q) \tag{2.2.1}$$

$$= 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(3n+1)}}{1+q^n} \tag{2.2.2}$$

$$= \sum (-1)^\nu \exp(3\pi i h\nu^2/2) \sum_{\nu=0}^{2k-1} \frac{(-1)^m \exp\{-3\pi z(2mk+\nu)^2/2k\}}{\cosh\{\pi i h\nu/2k - \pi z(2mk+\nu)/2k\}} \tag{2.2.3}$$

by the Poisson summation formula [1; pp. 7-9] we obtain

$$\begin{aligned} F(q) &= k^{-1} \sum_{\nu \bmod 2k} (-1)^\nu \\ &\cdot \exp(3\pi i h\nu^2/2k) \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\{(2n+1)\pi i(x-\nu)/2k - 3\pi z x^2/2k\}}{\cosh(\pi i h\nu/2k - \pi z x/2k)} dx. \end{aligned} \tag{2.2.4}$$

We now define

$$S_{2k}(x) = \frac{\sinh 3x}{\cosh(x/2k)}.$$

Clearly $S_{2k}(x)$ is an entire function of x since k is an integer. Hence if

$$\begin{aligned} \mathfrak{F}_n(x) &= \frac{\exp\{(2n+1)\pi i(x-\nu)/2k - 3\pi z x^2/2k\}}{\cosh(\pi i h\nu/2k - \pi z x/2k)} \\ &= \frac{-(-1)^\nu \exp\{(2n+1)\pi i(x-\nu)/2k - 3\pi z x^2/2k\} S_{2k}(\pi i h\nu - \pi z x)}{\sinh 3\pi z x}, \end{aligned}$$

then $\mathfrak{F}_n(x)$ has (at most) simple poles at

$$x_m = 2mi/6z = mi/3z.$$

The residue at x_m is

$$\begin{aligned} \lambda_{n,m} = -(-1)^{\nu+m}(3\pi z)^{-1}\exp\{(2n+1)\pi i(x_m-\nu)/2k \\ - 3\pi z x_m{}^2/2k\} S_{2k}(\pi i h\nu - \pi z x_m). \end{aligned}$$

Now by Cauchy's theorem,

$$\left\{\int_{-\infty}^{\infty} - \int_{-\infty+(2n+1)i/6z}^{\infty+(2n+1)i/6z}\right\} \mathfrak{F}_n(x)dx = 2\pi i\{\tfrac{1}{2}\lambda_{n,0} + \lambda_{n,1} + \cdots + \lambda_{n,n}\}.$$

Hence

$$\begin{aligned} F(q) &= k^{-1} \sum_{\nu \bmod 2k} (-1)^\nu \exp(3\pi i h\nu^2/2k) \sum_{n=0}^{\infty} 2\pi i\{\tfrac{1}{2}\lambda_{n,0} + \lambda_{n,1} + \cdots + \lambda_{n,n}\} \\ &\quad + k^{-1} \sum_{\nu \bmod 2k} (-1)^\nu \exp(3\pi i h\nu^2/2k) \sum_{n=0}^{\infty} \int_{-\infty+(2n+1)i/6z}^{\infty+(2n+1)i/6z} \mathfrak{F}_n(x)\,dx \\ &= \Sigma_1 + \Sigma_2. \end{aligned} \tag{2.2.5}$$

In this instance, Σ_1 will yield the main term in the transformation formula involving $f(q)$, and Σ_2 will again yield the "error" term.

In Σ_1, we have

$$(2.2.6)\quad \begin{aligned}&\sum_{n=0}^{\infty}\{\tfrac{1}{2}\lambda_{n,0}+\lambda_{n,1}+\cdots+\lambda_{n,n}\}\\ &\qquad=\tfrac{1}{2}\frac{\lambda_{0,0}}{1-\exp\{\pi i(x_m-\nu)/k\}}+\sum_{m=1}^{\infty}\frac{\lambda_{m,m}}{1-\exp\{\pi i(x_m-\nu)/k\}}.\end{aligned}$$

We thus obtain

$$(2.2.6a)\quad \Sigma_1=\pi ik^{-1}\sum_{\nu \bmod 2k}(-1)^{\nu}\exp(3\pi ih\nu^2/2k)\sum_{m=-\infty}^{\infty}\frac{\lambda_{m,m}}{1-\exp\{\pi i(x_m-\nu)/k\}}.$$

We now take a closer look at $\lambda_{m,m}$. Since $\lambda_{m,m}$ was a residue of $\mathfrak{F}_m(x)$, we note that $\lambda_{m,m}\neq 0$ if and only if $\cosh(\pi ih\nu/2k-\pi zx_m/2k)=0$, i.e. $\cos\{\pi(h\nu/k-m/3k)/2\}=0$. Thus we must have $3h\nu-m\equiv 3k \pmod{6k}$. If $m\not\equiv 0 \pmod 3$, then this last congruence is impossible. Hence $m=3j$.

$$h\nu-j\equiv k \pmod{2k}$$

or $$\nu\equiv -h''(k+j) \pmod{2k}$$

provided $hh''\equiv -1 \pmod{2k}$.

Thus when considering $\sum_{\nu \bmod 2k}$, we see that the only non-zero term is the term $\nu\equiv -h''(k+j) \pmod{2k}$.

We now evaluate $\lambda_{m,m}$ with $m=3j$ and $\nu\equiv -h''(k+j) \pmod{2k}$.

$$(2.2.7)\quad \begin{aligned}\lambda_{m,m}&=\lim_{x\to x_m}(x-x_m)\mathfrak{F}_m(x)\\ &=2k(\pi iz)^{-1}(-1)^{jk''+\frac{1}{2}(hh''-1)+j}\exp\{\pi ih''/2\\ &\qquad+(6j+1)\pi ij(h''+iz^{-1})/2k+3\pi j^2/2kz\}\end{aligned}$$

where $hh''+2kk''=-1$. Hence

$$(2.2.8)\quad \begin{aligned}\Sigma_1&=z^{-1}(-1)^{k+\frac{1}{2}(hh''-1)}\exp(\pi ih''/2\\ &\qquad+3\pi ihh''^2k/2)\frac{f(\exp\{\pi i(h''+iz^{-1})/k\})}{P(\exp\{\pi i(h''+iz^{-1})/k\})}\text{ by [12; p. 64].}\end{aligned}$$

We now evaluate Σ_2.

$$\begin{aligned}\Sigma_2&=k^{-1}\sum_{\nu \bmod 2k}(-1)^{\nu}\exp(3\pi ih\nu^2/2k)\sum_{n=0}^{\infty}\int_{-\infty+(2n+1)i/6z}^{\infty+(2n+1)i/6z}\mathfrak{F}_n(x)\,dx\\ &=\tfrac{1}{2}k^{-1}\sum_{\nu \bmod 2k}(-1)^{\nu}\exp(3\pi ih\nu^2/2k)\sum_{n=-\infty}^{\infty}\int_{-\infty+(2n+1)i/6z}^{\infty+(2n+1)i/6z}\mathfrak{F}_n(x)\,dx,\end{aligned}$$

as in Theorem 2.1.

Now

$$\int_{-\infty+(2n+1)i/6z}^{\infty+(2n+1)i/6z} \mathcal{J}_n(x)\,dx = \int_{\infty}^{-\infty} \mathcal{J}_n(x(2n+1)i/6z)\,dx,$$

and in Σ_2 letting $n = 3p + \delta$, we have

$$\Sigma_2 = \tfrac{1}{2}k^{-1} \sum_{h \bmod 2k} (-1)^{\nu}$$

$$(2.2.9) \qquad \cdot \exp(3\pi i h\nu^2/2k) \sum_{\delta=-1}^{1} \sum_{p=-\infty}^{\infty} \exp\{-(2n+1)\pi i\nu/2k\}$$

$$\cdot \exp\{-(2n+1)^2\pi/24kz\}$$

$$\int_{-\infty}^{\infty} \frac{\exp(-3\pi z x^2/2k)}{\cosh\{\pi i h\nu/2k - \pi z x/2k - \pi i(6p+2\delta+1)/12k\}} dx$$

If we replace $h\nu - p$ by ν i.e. $\nu \to -h''\nu - h''p$, then in Σ_2 we now have $\sum_{p=-\infty}^{\infty}$ free of x and ν. We define

$$(2.2.10) \qquad U_\delta(z) = \sum_{p=-\infty}^{\infty} (-1)^p \exp\{3\pi i h'' p^2/2k + (2\delta+1)\pi i h'' p/2k - (6p+2\delta+1)^2\pi/24kz\}$$

$$(2.2.11) \qquad = \begin{cases} 0 \text{ if } \delta = 1 \\ \exp(-\pi/24kz)\dfrac{1}{P(\exp\{\pi i(h''+iz^{-1})/k\})} \end{cases}$$

as in Theorem 2.1. Hence

$$\Sigma_2 = \frac{k^{-1}\exp(-\pi/24kz)}{P(\exp\{\pi i(h''+iz^{-1})/k\})} \sum_{\nu \bmod 2k} (-1)^{\nu}$$

$$(2.2.12) \qquad \cdot \exp(-3\pi i h''\nu^2/2k + \pi i h''\nu/2k)$$

$$\cdot \int_{-\infty}^{\infty} \frac{\exp(-3\pi z x/2k)}{\cosh(\pi i h\nu/2k - \pi z x/2k - \pi i/12k)} dx.$$

Since h is odd, we may write (2.0.1) as

$$P(\exp\{2\pi i(h+iz)/2k\}) = \omega_{h,2k} z^{\frac{1}{2}} \exp(\pi/24kz - \pi z/24k) P(\exp\{2\pi i(h''+iz^{-1})/2k\});$$

thus since we have established

$$\frac{f(\exp\{\pi i(h+iz)/k\})}{P(\exp\{\pi i(h+iz)/k\})} = \Sigma_1 + \Sigma_2,$$

we have

$$
\begin{aligned}
&f(\exp\{\pi i(h+iz)/k\}) \\
&\quad = (-1)^{k+\frac{1}{2}(hh''-1)}\exp(\pi ih''/2 + 3\pi ihh''^2k/2)\omega_{h,2k}z^{-\frac{1}{2}} \\
&\qquad \cdot \exp(\pi/24kz - \pi z/24k) f(\exp\{\pi i(h''+iz^{-1})/k\}) \\
&\quad + k^{-1}\omega_{h,2k}z^{\frac{1}{2}}\exp(-\pi z/24k)\sum_{\nu \bmod 2k}(-1)^{\nu} \\
&\qquad \cdot \exp(-3\pi ih''\nu^2/2k + \pi ih''\nu/2k) \\
&\qquad \cdot \int_{-\infty}^{\infty}\frac{\exp(-3\pi zx^2/2k)}{\cosh(\pi i\nu/2k - \pi zx/2k - \pi i/12k)}\,dx.
\end{aligned} \tag{2.2.13}
$$

Thus when h is odd, the general linear transformation sends f into f plus a correction $E_2^{(f)}(h,k;z)$; this completes Theorem 2.2.

Proof of Theorem 2.3. We let

$$\Omega(q) = \omega(q)/P(q^2) \tag{2.3.1}$$

$$= q^{-\frac{3}{4}}\sum_{n=0}^{\infty}(-1)^n q^{3(n+\frac{1}{2})^2}\frac{1+q^{2n+1}}{1-q^{2n+1}}, \text{ [12; p. 66]}, \tag{2.3.2}$$

and if $q = \exp\{\pi i(h+iz)/k\}$

$$
\begin{aligned}
&q^{\frac{3}{4}}\Omega(q) \\
&= -\tfrac{1}{2}\sum_{\nu=0}^{k-1}\sum_{m=-\infty}^{\infty}\frac{(-1)^{\nu+m}\exp\{3(\nu+\frac{1}{2})^2\pi ih/k - 3\pi z(nk+\nu+\frac{1}{2})^2/k\}}{\tanh\{(\nu+\frac{1}{2})\pi ih/k - (mk+\nu+\frac{1}{2})\pi z/k\}};
\end{aligned} \tag{2.3.3}
$$

by the Poisson summation formula [1; pp. 7-9] we obtain

$$
\begin{aligned}
q^{\frac{3}{4}}\Omega(q) = &-k^{-1}\sum_{\nu \bmod k}(-1)^{\nu}\exp(3\pi ih\mu^2/k)\sum_{n=0}^{\infty} \\
&\cdot\int_{-\infty}^{\infty}\frac{\exp\{(2n+1)\pi i(x-\mu)/k - 3\pi zx^2/k\}}{\tanh(\pi ih\mu/k - \pi zx/k)}dx
\end{aligned} \tag{2.3.4}
$$

as in the previous theorems with $\mu = \nu + \frac{1}{2}$. Hence if $\mathcal{S}_k(x) = \coth(x/k) \cdot \sinh 3x$, then we have $\mathcal{S}_k(x)$ is an entire function of x since k is an integer, and

$$
\begin{aligned}
\mathcal{F}_n(x) &= \frac{\exp\{(2n+1)\pi i(x-\mu)/k - 3\pi zx^2/k\}}{\tanh(\pi ih\mu/k - \pi zx/k)} \\
&= \frac{-\exp\{(2n+1)\pi i(x-\mu)/k - 3\pi zx^2/k\}\mathcal{S}_k(\pi ih\mu - \pi zx)}{(-1)^{\frac{1}{2}h}\sinh 3\pi zx}.
\end{aligned}
$$

Thus $\mathcal{F}_n(x)$ has (at most) simple poles at

$$x_m = mi/3z = 2mi/6z.$$

The residue at x_m is

$$\lambda_{n,m} = -(-1)^{m+\frac{1}{2}h}(3\pi z)^{-1}\exp\{(2n+1)\pi i(x_m-\mu)/k$$
$$-3\pi z x_m{}^2/k\mathcal{S}_k(\pi i h\mu - \pi z x_m),$$

and

$$\lambda_{m+l,m} = \exp\{2\pi i l(x_m-\mu)/k\}\lambda_{m,m}.$$

Now by Cauchy's theorem,

$$\left\{\int_{-\infty}^{\infty} - \int_{-\infty+(2n+1)i/6z}^{\infty+(2n+1)i/6z}\right\}\mathcal{F}_n(x)dx = 2\pi i\{\tfrac{1}{2}\lambda_{n,0}+\lambda_{n,1}+\cdots+\lambda_{n,n}\}.$$

Hence

$$\begin{aligned} q^{\frac{3}{4}}\Omega(q) &= -2\pi i k^{-1}\sum_{\nu \bmod k}(-1)^\nu \\ &\quad\cdot\exp(3\pi i h\mu^2/k)\sum_{n=0}^{\infty}\{\tfrac{1}{2}\lambda_{n,0}+\lambda_{n,1}+\cdots+\lambda_{n,n}\} \\ &\quad - k^{-1}\sum_{\nu \bmod k}(-1)^\nu\exp(3\pi i h\mu^2/k)\sum_{n=0}^{\infty}\int_{-\infty+(2n+1)i/6z}^{\infty+(2n+1)i/6z}\mathcal{F}_n(x)\,dx. \\ &= \Sigma_1+\Sigma_2. \end{aligned} \tag{2.3.5}$$

Again Σ_1 yields the main term of the transformation this time involving $f(q)$, and Σ_2 as always yields the "error" term.

In Σ_1 we have

$$\begin{aligned} &\sum_{n=0}^{\infty}\{\lambda_{n,0}+\cdots+\lambda_{n,n}\} \\ &= \tfrac{1}{2}\frac{\lambda_{0,0}}{1-\exp\{2\pi i(x_m-\mu)/k\}}+\sum_{m=1}^{\infty}\frac{\lambda_{m,m}}{1-\exp\{2\pi i(x_m-\mu)/k\}}\,; \end{aligned} \tag{2.3.6}$$

so

$$\begin{aligned} \Sigma_1 &= -\pi i k^{-1}\sum_{\nu \bmod k}(-1)^\nu \\ &\quad\cdot\exp(3\pi i h\mu^2/k)\sum_{m=-\infty}^{\infty}\frac{\lambda_{m,m}}{1-\exp\{2\pi i(x_m-\mu)/k\}}\,. \end{aligned} \tag{2.3.6a}$$

We now take a closer look at $\lambda_{m,m}$. Since $\lambda_{m,m}$ was a residue of $\mathcal{F}_m(x)$ we note that $\lambda_{m,m}\neq 0$ if and only if $\tanh(\pi i h\mu/k - \pi z x/k) = 0$, i.e. $\sin\{h\mu/k - m/3k\} = 0$. Thus we must have

$$3h(2\nu+1) - 2m \equiv 0 \pmod{6k}$$

since μ is the abbreviation for $\nu+\frac{1}{2}$. If $m \not\equiv 0 \pmod 3$, then the above congruence is impossible. Hence $m = 3j$.

$$\tfrac{1}{2}h(2\nu+1)-j\equiv 0 \pmod{k}$$

or

$$2\nu+1\equiv -2h'j \pmod{k}$$

or

$$\nu\equiv\tfrac{1}{2}(k+1)(-2h'j-1) \pmod{k}$$
$$\equiv -h'j-\tfrac{1}{2}(k+1) \pmod{k};$$

provided $hh'\equiv -1 \pmod{k}$.

Thus when considering $\sum_{\nu \bmod k}$ we see that the only non-zero term is the term $\nu\equiv -h'j-\frac{1}{2}(k+1) \pmod{k}$.

We now evaluate $\lambda_{m,m}$ when $m=3j$ and $\nu\equiv -h'j-\frac{1}{2}(k+1) \pmod{k}$.

$$\lambda_{m,m}=\lim_{x\to x_m}(x-x_m)\mathfrak{F}_m(x)$$

(2.3.7)

$$=-ik(\pi z)^{-1}\exp\{\pi ij(h'+iz^{-1})/\mathrm{k}-3\pi j^2/kz+6\pi ih'j^2/k\}.$$

Hence

(2.3.8) $$\Sigma_1=(2z)^{-1}(-1)^{\frac{1}{2}(k-1)}\exp(3\pi ihk/4)F(\exp\{2\pi i(h'+iz^{-1})/k\}),$$

where $F(q)$ is defined in the first line of the proof of Theorem 2.1.

We now evaluate Σ_2. With $n=3p+\delta$ and $\mu=\nu+\frac{1}{2}$, we have

(2.3.9)

$$\Sigma_2=-(2k)^{-1}\sum(-1)^{\nu}\exp(3\pi ih\mu^2/k)\sum_{\delta=-1}^{1}\sum_{p=-\infty}^{\infty}$$
$$\cdot\int_{-\infty}^{\infty}\frac{\exp\{-(2n+1)\pi i\mu/k-(2n+1)^2\pi/12kz-3\pi zx^2/k\}}{\tanh\{\pi ih\mu/k-\pi zx/k-\pi i(6p+2\delta+1)/6k\}}dx.$$

If we replace $h\mu-p$ by ν, i.e. $\nu\to -h'(\nu+p)-\frac{1}{2}(k+1)$, then we have $\sum_{p=-\infty}^{\infty}$ free of x and ν. We define

(2.3.10)

$$U_\delta(z)=\sum_{p=-\infty}^{\infty}(-1)^p\exp\{\pi ih'(3p^2+(2\delta+1)p)/k$$
$$-(6p+2\delta+1)^2\pi/12kz\}.$$

As before

$$U_1(z)=0$$

(2.3.11) $$U_{-1}(z)=U_0(z)=\frac{\exp(-\pi/12kz)}{P(\exp\{2\pi i(h'+iz^{-1})/k\})}.$$

Thus

$$\Sigma_2=\frac{ik^{-1}(-1)^{\frac{1}{2}(k-1)}\exp(3\pi ihk/4-\pi/12kz)}{P(\exp\{2\pi i(h'+iz^{-1})/k\})}\sum_{\nu \bmod k}exp(-3\pi ih'\nu^2/k)$$

$$(2.3.12) \qquad \cdot \exp(\pi i h'\nu/k) \int_{-\infty}^{\infty} \frac{\exp(-3\pi z x^2/k)}{\tanh(\pi i\nu/k - \pi z x/k - \pi i/6k)} dx.$$

We have established

$$\exp(3\pi i(h+iz)/4k) \frac{\omega(\exp\{\pi i(b+iz)/k\})}{P(\exp\{2\pi i(h+iz)/k\})} = \Sigma_1 + \Sigma_2,$$

hence by (2.0.1)

$$\begin{aligned} &\omega(\exp\{\pi i(h+iz)/k\}) \\ (2.3.13) \quad &= \tfrac{1}{2} z^{-\frac{1}{2}} (-1)^{\frac{1}{2}(k-1)} \exp(3\pi i h k/4 - 3\pi i h/4k) \omega_{h,k} \exp(\pi/12kz + 2\pi z/3k) \\ &\qquad \cdot f(\exp\{2\pi i(h' + iz^{-1})/k\}) \\ &\quad + i(-1)^{\frac{1}{2}(k-1)} \exp(3\pi i h k/4 - 3\pi i h/4k + 2\pi z/3k) k^{-1} z^{\frac{1}{2}} \omega_{h,k} \\ &\quad \cdot \sum_{\nu \bmod k} \exp(-3\pi i h'\nu^2/k + \pi i h'\nu/k) \int_{-\infty}^{\infty} \frac{\exp(-3\pi z x^2/k)}{\tanh(\pi i(\nu - \frac{1}{6})/k - \pi z x/k)} dx. \end{aligned}$$

This is the required transformation formula.

We proced to the final part of the theorem in which we estimate the integral appearing in $E_1^{(\omega)}(h,k;z)$.

$$\begin{aligned} z^{\frac{1}{2}} J_1^{(\omega)}(\nu;k;z) &= z^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{\exp(-3\pi z x^2/k)}{\tanh(\pi i(\nu - \frac{1}{6})/k - \pi z x/k)} dx \\ &= k(\pi z^{\frac{1}{2}})^{-1} \int_{-\infty}^{\infty} \frac{\exp(-3kt^2/\pi z)}{\tanh(\pi i(\nu - \frac{1}{6})/k - t)} dt \end{aligned}$$

by Cauchy's theorem. Now with $z = kN^{-2} - ik\phi$ and $-\frac{1}{2}(k+1) \leqq \nu \leqq \frac{1}{2}(k-1)$, we have

$$| z^{\frac{1}{2}} J_1^{(\omega)}(\nu;k;z)| \leqq k\pi^{-1} |z|^{-\frac{1}{2}}$$

$$\int_{-\infty}^{\infty} \frac{\exp(-3k(\operatorname{Re} z^{-1})t^2/\pi) |\cosh\{\pi i(\nu - \frac{1}{6})/k - t\}|}{|\sinh\{\pi i(\nu - \frac{1}{6})/k - t\}|} dt.$$

Now

$$|\cosh(x+iy)| = (\cos^2 y \cosh^2 x + \sin^2 y \sinh^2 x)^{\frac{1}{2}} \leqq \sqrt{2} \cosh x,$$

and

$$|\sinh(x+iy)| = (\sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y)^{\frac{1}{2}} \geqq |\sin y|.$$

Thus

$$| z^{\frac{1}{2}} J_1^{(\omega)}(\nu;k;z)| \leqq k\sqrt{2}\,\pi^{-1} |z|^{-\frac{1}{2}} |\sin\{\pi(\nu - \tfrac{1}{6})/k\}|^{-1} \int_{-\infty}^{\infty}$$

$$\cdot \exp(-3k(\operatorname{Re} z^{-1})t^2/\pi) \cosh t \, dt$$

$$\leqq k^2\sqrt{2}\,\pi^{-1}\,|\,z\,|^{-\frac{1}{2}}\,|\,2(\nu-\tfrac{1}{6})\,|^{-1}\int_{-\infty}^{\infty}\exp(-3k(\operatorname{Re}z^{-1})t^2/\pi+t)\,dt$$

since $-\frac{1}{2}(k+1)\leqq\nu\leqq\frac{1}{2}(k-1)$;

$$=k^2\pi^{-1}(2z)^{-\frac{1}{2}}\,|\,\nu-\tfrac{1}{6}\,|^{-1}\exp(\pi/12k(\operatorname{Re}z^{-1}))\int_{-\infty}^{\infty}\exp(-3k(\operatorname{Re}z^{-1})t^2/\pi)\,dt.$$

Now

$$\begin{aligned}k(\operatorname{Re}z^{-1})&=k^2N^{-2}\,|\,z\,|^{-2}\\&\geqq k^2N^{-2}(k^2N^{-4}+k^2/k^2N^2)^{-1}\\&\geqq\tfrac{1}{2}k^2\geqq\tfrac{1}{2}.\end{aligned}$$

Hence

$$\begin{aligned}|\,z^{\frac{1}{2}}J_1{}^{(\omega)}(\nu;k;z)\,|&\leqq k^2(2\,|\,z\,|)^{-\frac{1}{2}}\pi^{-1}\,|\,\nu-\tfrac{1}{6}\,|^{-1}\exp(\pi/6)\\&\qquad\cdot\int_{-\infty}^{\infty}\exp(-6k(\operatorname{Re}z^{-1})t^2/\pi)\sqrt{2}\,dt\\&\leqq k^2\,|\,z\,|^{-\frac{1}{2}}\sqrt{2}\,|\,\nu-\tfrac{1}{6}\,|^{-1}\exp(\pi/6)\,(12k(\operatorname{Re}z^{-1}))^{-\frac{1}{2}}\end{aligned}$$

as in Theorem 2.1,

$$\leqq 2^{-\frac{1}{4}}3^{-\frac{1}{2}}\,|\,\nu-\tfrac{1}{6}\,|^{-1}\exp(\pi/6)N^{\frac{1}{2}}k.$$

This concludes the proof of Theorem 2.3.

Proof of Theorem 2.4. As before

$$(2.4.1)\qquad \Omega(q)=\frac{\omega(q)}{P(q)^2}$$

$$(2.4.2)\qquad =q^{-\frac{3}{4}}\sum_{n=0}^{\infty}(-1)^nq^{3(n+\frac{1}{2})^2}\frac{1+q^{2n+1}}{1-q^{2n+1}},\ [12;\ \text{p. }66],$$

and if $q=\exp\{\pi i(h+iz)/k\}$,

(2.4.3)
$$q^{\frac{3}{4}}\Omega(q)=-\tfrac{1}{2}\sum_{\nu=0}^{2k-1}\sum_{m=-\infty}^{\infty}\frac{(-1)^\nu\exp\{3(\nu+\frac{1}{2})^2\pi ih/k-3(2mk+\nu+\frac{1}{2})^2\pi z/k\}}{\tanh\{(\nu+\frac{1}{2})\pi ih/k-(2mk+\nu+\frac{1}{2})\pi z/k\}};$$

by the Poisson summation formula [1; pp. 7-9] we obtain

(2.4.4)
$$\begin{aligned}q^{\frac{3}{4}}\Omega(q)=-(2k)^{-1}&\sum_{\nu\bmod 2k}(-1)^\nu\exp(3\pi ih\mu^2/k)\\&\cdot\sum_{n=0}^{\infty}{}^*\int_{-\infty}^{\infty}\frac{\exp\{2n\pi i(x-\mu)/2k-3\pi zx^2/k\}}{\tanh(\pi ih\mu/k-\pi zx/k)}\,dx\end{aligned}$$

where $\sum_{n=0}^{\infty}{}^*a_n=\frac{1}{2}a_0+\sum_{n=0}^{\infty}a_n$ and $\mu=\nu+\frac{1}{2}$.

Note that if ν is replaced by $\nu+k$, the change introduced is

$$(-1)^k\exp\{3\pi ih(k(\nu+k+1)+k\nu)/k-\pi in\}=(-1)^{2\nu+2k+1+n}$$
$$=(-1)^{n+1}.$$

Hence if n is even the terms ν and $\nu+k$ cancel each other, and if n is odd then these terms are identical. Thus

$$q^{\frac{3}{4}}\Omega(q)=-k^{-1}\sum_{\nu \bmod k}(-1)^{\nu}$$

(2.4.4a)

$$\cdot\exp(3\pi ih\mu^2/k)\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\{(2n+1)\pi i(x-\mu)/k-3\pi zx^2/k\}}{\tanh(\pi ih\mu/k-\pi zx/k)}\,dx.$$

As in Theorem 2.3, we define $\mathcal{S}_k(x)=\coth(x/k)\sinh(3x)$, then if

$$\mathcal{F}_n(x)=\frac{\exp\{(2n+1)\pi i(x-\mu)/k-3\pi zx^2/k\}}{\tanh(\pi ih\mu/k-\pi zx/k)}$$
$$=\frac{(-1)^{\nu+\frac{1}{2}(3h-1)}\exp\{(2n+1)\pi i(x-\mu)/k-3\pi zx^2/k\}\mathcal{S}_k(\pi ih\mu-\pi zx)}{i\cosh(3\pi zx)}$$

$\mathcal{F}_n(x)$ *has* (at most) simple poles at $x_m=(2m+1)i/6z$. The residue at x_m is

$$\lambda_{n,m}=$$
$$\frac{(-1)^{m+\nu+\frac{1}{2}(3h-1)}\exp\{(2n+1)\pi i(x-\mu)/k-3\pi zx^2/k\}\mathcal{S}_k(\pi ih\mu-\pi zx_m)}{3\pi zi},$$

and

$$\lambda_{m+l,m}=\exp\{2\pi il(x_m-\mu)/k\}\lambda_{m,m}.$$

Now by Cauchy's theorem,

$$\left\{\int_{-\infty}^{\infty}-P\int_{-\infty+x_n}^{\infty+x_n}\right\}\mathcal{F}_n(x)\,dx=2\pi i\{\lambda_{n,0}+\lambda_{n,1}+\cdots+\tfrac{1}{2}\lambda_{n,n}\}.$$

Hence

$$q^{\frac{3}{4}}\Omega(q)=-2\pi ik^{-1}\sum_{\nu \bmod k}(-1)^{\nu}$$

(2.4.5)

$$\cdot\exp(3\pi ih\mu^2/k)\sum_{n=0}^{\infty}\{\lambda_{n,0}+\lambda_{n,1}+\cdots+\tfrac{1}{2}\lambda_{n,n}\}$$
$$-k^{-1}\sum_{\nu \bmod k}(-1)^{\nu}\exp(3\pi ih\mu^2/k)\sum_{n=0}^{\infty}P\int_{-\infty+x_n}^{\infty+x_n}\mathcal{F}_n(x)\;dx$$
$$=\Sigma_1+\Sigma_2.$$

Σ_1 yields the main term of the transformation which involves $\omega(q)$ this time, and Σ_2 yields the "error" term.

In Σ_1 we have

$$(2.4.6)\quad \sum_{m=0}^{\infty}\{\lambda_{n,0}+\lambda_{n,1}+\cdots+\tfrac{1}{2}\lambda_{n,n}\}=\tfrac{1}{2}\sum_{m=0}^{\infty}\lambda_{m,m}\frac{1+\exp\{2\pi i(x_m-\mu)/k\}}{1-\exp\{2\pi i(x_m-\mu)/k\}}$$

We now take a closer look at $\lambda_{m,m}$. Since $\lambda_{m,m}$ was a residue of $\mathcal{F}_m(x)$, we note that $\lambda_{m,m}\neq 0$ if and only if

$$\tanh(\pi ih\mu/\mathrm{k}-\pi zx_m/k)=0,\ \text{i.e.}\ \sin\{\pi(h\mu/k-(2m+1)/6k)=0.$$

Thus we must have

$$3h(2\nu+1)-(2m+1)\equiv 0\ (\mathrm{mod}\ 6k)$$

If $2m+1\not\equiv 0\ (\mathrm{mod}\ 3)$, then the above congruence is impossible. Thus $2m+1=6j+3$, and

$$h(2\nu+1)-(2j+1)\equiv 0\ (\mathrm{mod}\ 2k),$$

or $$2\nu+1\equiv -h''(2j+1)\ (\mathrm{mod}\ 2k),$$

or $$\nu\equiv -\tfrac{1}{2}\{h''(2j+1)+1\}\ (\mathrm{mod}\ k).$$

Thus when considering $\sum_{\nu \bmod k}$ we see that the only non-zero term is the term $\nu\equiv -\frac{1}{2}\{h''(2j-1)-1\}\ (\mathrm{mod}\ k)$.

We now evaluate $\lambda_{m,m}$ when $m=3j+1$ and $\nu=-\frac{1}{2}\{h''(2j+1)+1\}$.

$$\lambda_{m,m}=\lim_{x\to x_m}\mathcal{F}_m(x)$$

$$(2.4.7)\qquad =-k(\pi z)^{-1}\exp\{-3\pi(2j+1)^2/4kz+\tfrac{1}{2}3(2j+1)^2\pi ih''/k\}.$$

Hence

$$(2.4.8)\quad \Sigma_1=iz^{-1}(-1)^{\frac{1}{2}(h''+1)}\exp\{-3\pi ih''k''/2+3\pi i(h''+iz^{-1}(/4k\}\cdot\Omega(\exp\{\pi i(h''+iz^{-1})/k\}).$$

We now evaluate Σ_2. In Σ_2 if $n=3p+\delta$, we have

$$\Sigma_2=-(2k)^{-1}\sum_{\delta=-1}^{1}\sum_{p=-\infty}^{\infty}(-1)^{\nu}\exp\{3\pi ih\mu^2/k-\pi(2n+1)^2/12kz-(2n+1)\pi i\mu/k\}$$

$$(2.4.9)\qquad \cdot\int_{-\infty}^{\infty}\frac{\exp(-3\pi zx^2/k)}{\tanh\{\pi ih\mu/k-\pi zx/k-\pi i(6p+2\delta+1)/6k\}}dx.$$

If we replace $(2\nu+1)h-2p$ by $(2\nu+1)$ $(\bmod 2k)$, i.e.

$$\nu\to-\tfrac{1}{2}\{h''(2\nu+1+2p)+1\}\,(\bmod k),$$

then we have $\sum_{p=-\infty}^{\infty}$ free of x and ν. We define

$$(2.4.10)\quad U_\delta(z)=\sum_{p=-\infty}^{\infty}(-1)^p\exp\{\pi ih''(3p^2(2\delta+1)p)/k \\ -\pi(6p+2\delta+1)^2/12kz\}$$

$$(2.4.11)\qquad =\begin{cases}0 \text{ if } \delta=1 \\ \dfrac{\exp(-\pi/12kz)}{P(\exp\{2\pi i(h''+iz^{-1})/k\})} & \text{if } \delta=-1 \text{ or } 0, \text{ as before.}\end{cases}$$

Thus

(2.4.12)

$$\Sigma_2=\frac{-k^{-1}(-1)^{\frac{1}{2}(h''+1)}\exp(-\pi/12kz-3\pi ih''k''/2)}{P(\exp\{2\pi i(h''+iz^{-1})/k\})}\sum_{\nu \bmod k}(-1)^\nu$$

$$\cdot\exp(-3\pi ih''\mu^2/k+\pi ih''\mu/k)$$

$$\int_{-\infty}^{\infty}\frac{\exp(-3\pi zx^2/k)}{\tanh(\pi ih\mu/k-\pi zx/k-\pi i/6k)}\,dx.$$

We have established

$$\exp\{3\pi i(h+iz)/4k)\frac{\omega(\exp\{\pi i(h+iz)/k\})}{P(\exp\{2\pi i(h+iz)/k\})}=\Sigma_1+\Sigma_2;$$

hence by (2.0.1)

$$(2.4.13)\quad \begin{aligned}&\omega(\exp\{\pi i(h+iz)/k\})\\ &=i(-1)^{\frac{1}{2}(h''+1)}\exp(-3\pi ih''k''/2+3\pi ih''/4k-3\pi ih/4k)\omega_{h,k}\\ &\quad\cdot z^{-\frac{1}{2}}\exp(-2\pi z^{-1}/3k+2\pi z/3k)\omega(\exp\{\pi i(h''+iz^{-1})/k\})\\ &\quad -k^{-1}(-1)^{\frac{1}{2}(h''+1)}\exp(-3\pi ih''k''/2-3\pi ih/4k)\omega_{h,k}z^{\frac{1}{2}}\exp(2\pi z/3k)\\ &\quad\cdot\sum_{\nu \bmod k}(-1)^\nu\exp(-3\pi ih''\mu^2/k+\pi ih''\mu/k)\\ &\quad\cdot\int_{-\infty}^{\infty}\frac{\exp(-3\pi zx^2/k)}{\tanh(\pi ih\mu/k-\pi zx/k-\pi i/6k)}\,dx.\end{aligned}$$

Thus when h is odd, the general linear transformation sends ω into ω plus a correction $E_2^{(\omega)}(h,k;z)$; this concludes the proof of Theorem 2.4.

We now proceed to the task of simplifying the expressions for $\epsilon_{h,k}^{(f)}$, $\eta_{h,k}^{(f)}$, $\epsilon_{h,k}^{(\omega)}$, $\eta_{h,k}^{(\omega)}$.

COROLLARY 2.5. *We have*

$$\epsilon_{h,k}^{(f)} = \begin{cases} (-1)^{\frac{1}{2}(k+1)}\omega_{h,2k} & k \text{ odd} \quad (2.5.1\text{a}) \\ i(-1)^{\frac{1}{2}(h-1+k)}\omega_{h,2k} & k \text{ even} \quad (2.5.1\text{b}) \end{cases}$$

$$\eta_{h,k}^{(f)} = (-1)^{\frac{1}{2}(k-1)}\exp\{3(k+1)^2\pi ih'/2k\}\omega_{\frac{1}{2}h,k} \tag{2.5.2}$$

$$\epsilon_{h,k}^{(\omega)} = (-1)^{\frac{1}{2}(k-1)}\exp\{2\pi ih3(k^2-1)/8k\}\omega_{h,k} \tag{2.5.3}$$

$$\eta_{h,k}^{(\omega)} = \begin{cases} i(-1)^{\frac{1}{2}(h'''+1)}\exp\{3\pi i(h'''-h)/4k\}\omega_{h,k}, \ k \text{ even}, \\ \qquad hh''' \equiv -1 \pmod{8k}; \quad (2.5.4\text{a}) \\ (-1)^{\frac{1}{2}(k+1)}\exp\{2\pi ih'(5k^2+3)/8k \\ \qquad + 2\pi ih3(k^2-1)/8k\}\omega_{h,k} \ k \text{ odd}. \quad (2.5.4\text{b}) \end{cases}$$

Remark. Our results agree with those of Dragonette insofar as is possible to check; however, it appears she has made a slight error in line (7.8) [2; p. 499]. She gives the incorrect evaluation of $\epsilon_{h,k}^{(\omega)}$; if one uses line (7.2) of her paper, one easily derives a result on agreement with ours.

Proof. The proofs of these results involve routine simplifications of the expressions already obtained. For (2.5.1), we use the fact that h is odd, $hh'' = -1 - 2kk''$, and $\exp(\pi ih''/2) = -\exp(\pi ih/2)$. For (2.5.2), we use the fact that h is even, k is odd, and $k' \equiv -k - khh' \pmod 8$. (2.5.3) is deduced directly. For (2.5.4a), we use the fact that h is odd, k is even, and $h'' \equiv h''' \pmod 4$. For (2.5.4b), we use the fact that h is odd, k is odd, and $k'' \equiv -\frac{1}{2}(-k - khh'') \pmod 8$.

III. The modular transformation formulae of φ, ψ, and ν. Having obtained the transformation formulae for f and ω, we obtain the corresponding formulae for ϕ, ψ, and v by means of certain identities involving the mock theta functions [2; p. 475, (1.2(a)), (1.2(d))].

LEMMA 3.1. *If* $\theta_2(0,q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}$, *and* $\theta_4(0,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}$ *with* $hh' + kk' = -1$ *and* g.c.d.$(h,k) = 1$, *then we have for* $(\operatorname{Re} z) > 0$

$$\theta_2^2(0, \exp\{\pi i(h+iz)/k\} = iz^{-1}(-1)^{\frac{1}{2}(h-1)}\exp(\pi i h k'/2)$$

$$\cdot\,\theta_2^2(0, \exp\{\pi i(h'+iz^{-1})/k\}) \quad h \text{ odd},\ k \text{ even},\ h' \text{ odd}; \tag{3.1.1}$$

$$\theta_4^2(0, \exp\{\pi i(h+iz)/k\} = z^{-1}(-1)^{\frac{1}{2}(k-1)}\exp(-\pi i h' k/2)$$

$$\cdot\,\theta_2^2(0, \exp\{\pi i(h'+iz^{-1})/k\}) \quad h \text{ even},\ k \text{ odd},\ k' \text{ odd}; \tag{3.1.2}$$

$$\theta_4^2(0, \exp\{\pi i(h+iz)/k\}) = iz^{-1}(-1)^{\frac{1}{2}(h-1)}\exp(-\pi i h k/2)$$

$$\cdot\,\theta_4^2(0, \exp\{\pi i(h'+iz^{-1})/k\}) \quad h \text{ odd},\ k' \text{ even},\ h' \text{ odd}; \tag{3.1.3}$$

$$\theta_2^2(0, \exp\{\pi i(h+iz)/k\}) = z^{-1}(-1)^{\frac{1}{2}(k-1)}\exp(\pi i h k/2)$$

$$\cdot\,\theta_4^2(0, \exp\{\pi i(h'+iz^{-1})/k\}) \quad k \text{ odd},\ k' \text{ odd},\ h' \text{ even}; \tag{3.1.4}$$

Proof. These formulae are obtained directly from [11] using Table XX on page 241 and Table XLII on page 262. In Table XX, we let $-h = a$, $k = b$, $k' = c$, $h' = d$, $\tau = (h+iz)/k$, $T = (h'+iz^{-1})/k$. We obtain

(3.1.1) with k' even from line 1 of Table XX, and lines (2), (6), (7), (8) of XLII;

(3.1.1) with k' odd from line 2 of Table XX, and lines (2), (6), (7), (8) of XLII;

(3.1.2) with h' odd from line 6 of Table XX, and lines (2), (6), (7), (8) of XLII;

(3.1.2) with h' even from line 5 of Table XX, and lines (2), (6), (7), (8) of XLII;

(3.1.3) with k even from line 1 of Table XX, and lines (4), (6), (7), (8) of XLII;

(3.1.3) with k odd from line 3 of Table XX, and lines (4), (6), (7), (8) of XLII;

(3.1.4) with h odd from line 4 of Table XX, and lines (4), (6), (7), (8) of XLII;

(3.1.4) with h even from line 5 of Table XX, and lines (4), (6), (7), (8) of XLII.

In the next two lemmas, we study two theta functions closely related to the mock theta functions.

LEMMA 3.2. *If* $G_1(q) = \theta_4(0,q) \prod_{r=1}^{\infty} (1+q^r)^{-1}$ *and*

$$G_2(q) = \tfrac{1}{2} q^{-\frac{1}{4}} \theta_2(0,q) \prod_{r=1}^{\infty} (1+q^{2r}),$$

then

$$G_1(\exp\{\pi i(h+iz)/k\}) = \begin{cases} 2^{\frac{3}{2}} z^{-\frac{1}{2}} (-1)^{h'} \exp(-\pi i h'/k)\, \eta_{h,k}^{(f)} \exp(-\pi/3kz \\ \qquad -\pi z/24k) G_2(\exp\{\pi i(h'+iz^{-1})/k\}) & h \text{ even} \\ z^{-\frac{1}{2}} (-1)^k \epsilon_{h,k}^{(f)} \exp(\pi/24kz \\ \qquad -\pi z/24k) G_1(\exp\{\pi i(h''+iz^{-1})/k\}) & h \text{ odd} \end{cases}$$

where in the top line $hh' \equiv -1 \pmod{k}$ and in the bottom line $hh'' \equiv -1 \pmod{2k}$. $\epsilon_{h,k}^{(f)}$ appears in Theorem 2.2, and $\eta_{h,k}^{(f)}$ appears in Theorem 2.1.

Proof. We note that

$$\begin{aligned} G_1(q) &= \theta_4(0,q) \prod_{r=1}^{\infty} (1+q^r)^{-1} \\ &= \theta_4(0,q) \prod_{r=1}^{\infty} (1+q^r)^{-1} \prod_{r=1}^{\infty} (1-q^r) \prod_{r=1}^{\infty} (1-q^r)^{-1} \\ &= \theta_4^2(0,q) P(q) \qquad [5;\ \text{p. } 185] \end{aligned}$$

$$\begin{aligned} G_2(q) &= \tfrac{1}{2} q^{-\frac{1}{4}} \theta_2(0,q) \prod_{r=1}^{\infty} (1+q^{2r}) \\ &= \tfrac{1}{2} q^{-\frac{1}{4}} \theta_2(0,q) \prod_{r=1}^{\infty} (1+q^{2r}) \prod_{r=1}^{\infty} (1-q^{4r}) \prod_{r=1}^{\infty} (1-q^{4r})^{-1} \\ &= \tfrac{1}{4} q^{-\frac{1}{2}} \theta_2^2(0,q) P(q^4) \qquad [5;\ \text{p. } 185]. \end{aligned}$$

The remainder of the proof is routine. Since $G_1(q) = \theta_4^2(0,q)P(q)$, we apply Lemma 3.1 to $\theta_4^2(0,q)$, and (2.0.1) to $P(q)$; the desired result follows directly using the expressions for $\epsilon_{h,k}^{(f)}$ and $\eta_{h,k}^{(f)}$ found in Corollary 2.5.

LEMMA 3.3. *If G_1 and G_2 are as in the previous lemma, then*

$$G_2(\exp\{\pi i(h+iz)/k\}) = \begin{cases} (2z)^{-\frac{1}{2}}(-1)^h \exp(\pi i h/k)\epsilon_{2h,k}^{(\omega)}\exp(\pi z/3k \\ \qquad + \pi z^{-1}/24k)G_1[(-1)^{h'}\exp\{\pi i(h'+iz^{-1})/k\}] \\ \qquad\qquad k \text{ odd} \\ z^{-\frac{1}{2}}(-1)^{k'}\exp\{\pi i(h-h')/k\}\eta_{h,\frac{1}{2}k}^{(\omega)}\exp(\pi z/3k \\ \qquad - \pi z^{-1}/3k)G_2(\exp\{\pi i(h'+iz^{-1})/k\}) \\ \qquad\qquad k \text{ even} \end{cases}$$

where $hh'+kk'=-1$.

Proof. The proof of this lemma is exactly like that of Lemma 3.2. Since $G_2(q)=\frac{1}{4}q^{-\frac{1}{2}}\theta_2{}^2(0,q)P(q^4)$, we apply Lemma 3.1 to $\theta_2{}^2(0,q)$ and (2.0.1) to $P(q^4)$; the desired result follows directly using the expressions $\epsilon_{h,k}^{(\omega)}$ and $\eta_{h,k}^{(\omega)}$ found in Corollary 2.5.

To obtain the linear transformation formulae for ϕ, ψ, and v we utilize the following identities partially stated by Ramanujan [9; p. 354] and proved by Watson [12; p. 63].

$$\text{(A)} \qquad 2\phi(q) = f(-q) + G_1(-q)$$

$$\text{(B)} \qquad 4\psi(q) = -f(-q) + G_1(-q)$$

$$\text{(C)} \qquad \mp q^{-1}v(\pm q) = \omega(q^2) \mp q^{-1}G_2(q)$$

where G_1 and G_2 are given in Lemma 3.2,

Theorem 3.4. *Let* g.c.d.$(h,k)=1$, $hh'+kk'=-1$, *also if h is odd* $hh''+2kk''=-1$, *and* $\operatorname{Re} z>0$ *with the principal branch of $z^{\frac{1}{2}}$ taken. Then*

$$\phi[\exp\{\pi i(h+iz)/k\}] = \begin{cases} \text{(3.4.1)}\quad z^{-\frac{1}{2}}\epsilon_{h,k}^{(\phi)}\exp(-\pi z/24k+\pi z^{-1}/24k)\,. \\ \qquad \cdot\phi(\exp\{\pi i(h''+iz^{-1})/k\}) \\ \qquad + \frac{1}{2}E_2^{(f)}(h+k,k;z) \quad (k \text{ even}) \\ \text{(3.4.2)}\quad 2^{\frac{1}{2}}z^{-\frac{1}{2}}\eta_{h,k}^{(\phi)}\exp(-\pi z/24k-\pi z^{-1}/3k) \\ \qquad \cdot v[(-1)^{h'-1}\exp\{\pi i(h'+iz^{-1})/k\}] \\ \qquad + \frac{1}{2}E_1^{(f)}(h+k,k;z) \quad (k \text{ odd, } h \text{ odd}) \\ \text{(3.4.3)}\quad 2z^{-\frac{1}{2}}\sigma_{h,k}^{(\phi)}\exp(-\pi z/24k+\pi z^{-1}/24k) \\ \qquad \cdot\psi[(-1)^{h'-1}\exp\{\pi i(h'+iz^{-1})/k\}] \\ \qquad + \frac{1}{2}E_2^{(f)}(h+k,k;z) \quad (h \text{ even}) \end{cases}$$

$$\psi[\exp\{\pi i(h+iz)/k\}] = \begin{cases} \text{(3.4.4)}\quad \frac{1}{2}z^{-\frac{1}{2}}\epsilon_{h,k}^{(\psi)}\exp(-\pi z/24k+\pi z^{-1}/24k) \\ \qquad \cdot\phi[(-1)^{h'}\exp\{\pi i(h'+iz^{-1})/k\}] \\ \qquad - \frac{1}{4}E_2^{(f)}(h+k,k;z) \quad (h \text{ even}) \\ \text{(3.4.5)}\quad (2z)^{-\frac{1}{2}}\eta_{h,k}^{(\psi)}\exp(-\pi z/24k-\pi z^{-1}/3k) \\ \qquad \cdot v[(-1)^{h'}\exp\{\pi i(h'+iz^{-1})/k\}] \\ \qquad - \frac{1}{4}E_1^{(f)}(h+k,k;z) \quad (h \text{ odd, } k \text{ odd}) \\ \text{(3.4.6)}\quad z^{-\frac{1}{2}}\sigma_{h,k}^{(\psi)}\exp(-\pi z/24k+\pi z^{-1}/24k) \\ \qquad \cdot\psi(\exp\{\pi i(h''+iz^{-1})/k\}) \\ \qquad - \frac{1}{4}E_2^{(f)}(h+k,k;z) \quad (k \text{ even}) \end{cases}$$

$$
\begin{array}{ll}
(3.4.7) & \\
(3.4.8)\ v(\exp\{\pi i(h+iz)/k\}) = & \\
(3.4.9) &
\end{array}
\begin{cases}
(2z)^{-\frac{1}{2}}\epsilon_{h,k}{}^{(v)}\exp(\pi z/3k+\pi z^{-1}/3k) \\
\quad \cdot\phi[(-1)^{h'-1}\exp\{\pi i(h'+iz^{-1})/k\}] \\
\quad -\exp\{\pi i(h+iz)/k\}E_1{}^{(\omega)}(2h,k;2z) \\
\quad (h \text{ odd, k odd}) \\
z^{-\frac{1}{2}}\eta_{h,k}{}^{(v)}\exp(\pi z/3k-\pi z^{-1}/3k) \\
\quad \cdot v[(-1)^{k'}\exp\{\pi i(h'+iz^{-1})/k\}] \\
\quad -\exp\{\pi i(h+iz)/k\}E_2{}^{(\omega)}(h,\tfrac{1}{2}k;z) \\
\quad (k \text{ even}) \\
2^{\frac{1}{2}}z^{-\frac{1}{2}}\sigma_{h,k}{}^{(v)}\exp(\pi z/3k+\pi z^{-1}/24k) \\
\quad \cdot\psi[(-1)^{h'-1}\exp\{\pi i(h'+iz^{-1})k] \\
\quad -\exp\{\pi i(h+iz)/k\}E_1{}^{(\omega)}(2h+2k,k;2z) \\
\quad (h \text{ even})
\end{cases}
$$

with

$\epsilon_{h,k}{}^{(\phi)} = \epsilon_{h+k,k}{}^{(f)}$ (k even)

$\eta_{h,k}{}^{(\phi)} = (-1)^{h'}\exp(-\pi ih'/k)\eta_{h+k,k}{}^{(f)}$ ($h+k$ even)

$\sigma_{h,k}{}^{(\phi)} = -\epsilon_{h+k,k}{}^{(f)}$ (h even)

$\epsilon_{h,k}{}^{(\psi)} = -\epsilon_{h+k,k}{}^{(f)}$ (h even)

$\eta_{h,k}{}^{(\psi)} = (-1)^{h'}\exp(-\pi ih'/k)\eta_{h+k,k}{}^{(f)}$ ($h+k$ even)

$\sigma_{h,k}{}^{(\psi)} = \epsilon_{h+k,k}{}^{(f)}$ (k even)

$\epsilon_{h,k}{}^{(v)} = -\exp(\pi ih/k)\epsilon_{2h,k}{}^{(\omega)}$ ($h+k$ even)

$\eta_{h,k}{}^{(v)} = (-1)^{k'}\exp\{\pi i(h-h')/k\}\eta_{h,\frac{1}{2}k}{}^{(\omega)}$ (k even)

$\sigma_{h,k}{}^{(v)} = \exp(\pi ih/k)\epsilon_{2h,k}{}^{(\omega)}$ (h even)

where $\epsilon_{h,k}{}^{(f)}$, $\eta_{h,k}{}^{(f)}$, $\epsilon_{h,k}{}^{(\omega)}$, $\eta_{h,k}{}^{(\omega)}$ *and* $E_1{}^{(f)}(h,k;z)$, $E_2{}^{(f)}(h,k;z)$, $E_1{}^{(\omega)}(h,k;z)$, $E_2{}^{(\omega)}(h,k;z)$ *are given in Theorems 2.1, 2.2, 2.3 and 2.4 respectively.*

Proof. We shall derive (3.4.1) only. The remaining formulae are proved in a similar manner.

$$
(3.4.10)\quad 2\phi(\exp\{\pi i(h+iz)/k\}) \\
= f(\exp\{\pi i[(h+k)+iz]/k\} + G_1(\exp\{\pi i[(h+k)+iz]/k\}).
$$

Now for k even, we use the fact that $(h+k)'' \equiv h''+k \pmod{2k}$ since $(h+k)(h''+k) = hh''+k(h+h''+k) \equiv -1 \pmod{2k}$. Hence by (A)

$$
(3.4.11)\quad 2\phi(-\exp\{\pi i[(h+k)''+iz^{-1}]/k\}) \\
= f(\exp\{\pi i[(h+k)''+iz^{-1}]/k\}) + G_1(\exp\{\pi i[(h+k)''+iz^{-1}]/k\}).
$$

Multiplying (3.4.11) by $z^{-\frac{1}{2}}\epsilon_{h+k,k}{}^{(f)}\exp(-\pi z/24k+\pi z^{-1}/24k)$, then subtracting from (3.4.10) and using Theorem 2.2 and Lemma 3.2, we get

$$
\begin{aligned}
&\phi(\exp\{\pi i(h+iz)/k\}) \\
(3.4.12) \qquad &= z^{-\frac{1}{2}}\epsilon_{h+k,k}{}^{(f)}\exp(-\pi z/24k+\pi z^{-1}/24k) \\
&\quad\cdot\phi(\exp\{\pi i(h''+iz^{-1})/k\})+\tfrac{1}{2}E_2{}^{(f)}(h+k,k;z)
\end{aligned}
$$

which is (3.4.1).

IV. The exponential sums. Before evaluating the exponential sums that arise for the mock theta functions, we give the following lemma which will be useful in the next section.

LEMMA 4.1. *If* g.c.d.$(D,k)=1$, $d\mid 24$, *and* $0\leqq\sigma_1<\sigma_2\leqq k$,

$$
\begin{aligned}
&A_k(D;d;F,G;\sigma_1;\sigma_2) \\
&\qquad=\sum_{\substack{h \bmod dk\\ \sigma_1\leqq Dh^*\leqq\sigma_2\\ \text{g.c.d.}(h,dk)=1}}\omega_{h,k}\exp(-2\pi iFh/dk\ 2\pi iGh^*/dk),
\end{aligned}
$$

where $\omega_{h,k}$ *appears in* (2.0.1), *and* $hh^*\equiv-1\pmod{dk}$, *then*

$A_k(D;d;F,G;\sigma_1;\sigma_2)=O(((24F+d)/d,k)^{\frac{1}{2}}k^{\frac{1}{2}+\epsilon})$, *as* $k\to+\infty$, *where the constant implied by the O-symbol is absolute.*

Proof. Many papers contain theorems like this for sums related to the standard Kloosterman sum (e.g. [3], [7]). However, by [9; p. 284] we have

$$\omega_{h,k}=-i\left(\frac{-k}{h}\right)\exp\{2\pi i[(4k^2+3k-1)h-(k^2-1)\tilde{h}]/24k\}$$

if k is even, g.c.d.$(h,24)=\Theta$ and $h\tilde{h}\equiv-1\ (\bmod 24k/\Theta)$;

$$\omega_{h,k}=\exp\{-\pi i(k-1)/4\}\left(\frac{-h}{k}\right)\exp\{2\pi i[(k^2-1)h-(k^2-1)\tilde{h}]/24k\}$$

if k is odd, g.c.d.$(h,24)=\Theta$ and $h\tilde{h}\equiv-1\ (\bmod 24k/\Theta)$. Thus we see that the sum we are considering is in fact related to a generalized Kloosterman sum; such sums have recently been studied in detail in [13]. A generalized Klosterman sum is of the form

$$\sum_{\substack{h \bmod k \\ \text{g.c.d.}(h,k)=1}} \chi(h)\exp\{2\pi i(ah-h')/k\}$$

where $\chi(h)$ is a quadratic character. This sum satisfies the same estimates as the original Kloosterman sum; one may easily prove that such sums are multiplicative, and one has

$$\left| \sum_{\substack{h \bmod p^\alpha \\ \text{g.c.d.}(h,p)=1}} \chi(h)\exp\{2\pi i(ah-h')/p^\alpha\} \right| \leqq 2p^{\frac{1}{2}\alpha}, \qquad \alpha > 1$$

by [6; p. 283] and [10; p. 102, line (54)].

This information gives by the method of Lehner [7; §10]

$$| A_k(D;d;F,G;\sigma_1,\sigma_2) = \begin{cases} O\{(-24F/d+4k^2+3k-1, 24k)^{\frac{1}{2}}k^{\frac{1}{2}+\epsilon}\} & (k \text{ even}) \\ O\{(-24F/d+k^2-1, 24k)^{\frac{1}{2}}k^{\frac{1}{2}+\epsilon}\} & (k \text{ odd}) \end{cases}$$
$$= O\{(24F/d+1, k)^{\frac{1}{2}}k^{\frac{1}{2}+\epsilon}\}.$$

We now go to the exponential sums in question. The following equations are direct consequences of Corolary 2.5. We shall only exhibit the exponential sums necessary to deal with Theorem 5.1. Examination of Theorem 3.4 shows that analogous results hold for the exponential sums relevant in the proof of Theorem 5.2. If we let

$$(4.2) \qquad \lambda_k^{(f)}(\epsilon;\sigma_1;\sigma_2;n) = \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,2k)=1 \\ \sigma_1 \leqq h' \leqq \sigma_2}} \epsilon_{h,k}^{(f)}\exp(-\pi i h n/k),$$

then

$$\lambda_k^{(f)}(\epsilon;\sigma_1;\sigma_2;n) = \begin{cases} (-1)^{\frac{1}{2}(k+1)}\sum_{j=0}^{1} A_{2k}(1;1;n,0;\sigma_1-jk,\sigma_2-jk) & (k \text{ odd}) \\ (-1)^{\frac{1}{2}k}\sum_{j=0}^{1} \mathrm{A}_{2k}(1,1;n-\frac{1}{2}k,0;\sigma_1-jk,\sigma_2-jk) & (k \text{ even}) \end{cases}$$

If we let

$$(4.3) \qquad \lambda_k^{(f)}(\eta;\sigma_1;\sigma_2;n) = \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,k)=1 \\ 2|h \\ \sigma_1 \leqq h' \leqq \sigma_2}} \eta_{h,k}^{(f)}\exp(-\pi i h n/k),$$

then

$$\lambda_k^{(f)}(\eta;\sigma_1;\sigma_2;n) = (-1)^{\frac{1}{2}(k-1)}A_k(\tfrac{1}{2}(k+1);1;n,3(k+1)^3/8;\sigma_1;\sigma_2).$$

The following two exponential sums will also arise when we estimate the error terms in the asymptotic expansions. If we let

$$\text{(4.4)}\qquad \Lambda_{k,1}{}^{(f)}(\delta;\sigma_1,\sigma_2;n)=\sum_{\substack{h \bmod 2k\\ \text{g.c.d.}(h,2k)=1\\ 2\mid h\\ \sigma_1\leqq h'\leqq\sigma_2\\ \sigma_1\leqq Dh^*\leqq\sigma_2}} \delta_1{}^{(f)}(\nu;\tfrac{1}{2}h;2h';k)\exp(-\quad ihn/k),$$

then

$$\Lambda_{k,1}{}^{(f)}(\delta;\sigma_1,\sigma_2;n)=(-1)^{\nu}\sum_{\substack{h \bmod k\\ \text{g.c.d.}(2h,k)=1\\ \sigma_1\leqq\frac{1}{2}(k+1)h'\leqq\sigma_2}} \delta_1{}^{(f)}(\nu;h;h';k)\exp(-2\pi ihn/k)$$

$$=(-1)^{\nu}A_k(\tfrac{1}{2}(k+1);1;n,\tfrac{1}{2}(\nu-3\nu^2);\sigma_1,\sigma_2).$$

If we let

$$\text{(4.5)}\qquad \Lambda_{k,2}{}^{(f)}(\delta;\sigma_1,\sigma_2;n)=\sum_{\substack{h \bmod 2k\\ \text{g.c.d.}(h,2k)=1\\ \sigma_1\leqq h'\leqq\sigma_2}} \delta_1{}^{(f)}(\nu;h;h'';2k)\exp(-\pi ihn/k),$$

then

$$\Lambda_{k,2}{}^{(f)}(\delta;\sigma_1,\sigma_2;n)$$
$$=(-1)^{\nu}\sum_{j=0}^{1}A_{2k}(1;1;n,\tfrac{1}{2}(\nu-3\nu^2);\sigma_1-jk;\sigma_2-jk).$$

We now treat the exponential sums which actually appear in the asymptotic expansions.

THEOREM 4.6. *If*

$$\text{(4.6.1)}\qquad \lambda(k)=\tfrac{1}{2}\sum_{\substack{h \bmod 2k\\ \text{g.c.d.}(h,2k)=1}} \epsilon_{h,k}{}^{(f)}\exp(-\pi ihn/k);$$

$$\text{(4.6.2)}\qquad \lambda^{(\phi)}(k)=\tfrac{1}{2}\sum_{\substack{h \bmod 2k\\ \text{g.c.d.}(h,2k)=1\\ 2\mid k}} \epsilon_{h,k}{}^{(\phi)}\exp(-\pi ihn/k);$$

$$\text{(4.6.3)}\qquad \lambda^{(\psi)}(k)=\tfrac{1}{2}\sum_{\substack{h \bmod 2k\\ \text{g.c.d.}(h,k)=1\\ 2\mid h}} \epsilon_{h,k}{}^{(\psi)}\exp(-\pi ihn/k);$$

$$\text{(4.6.4)}\qquad \lambda^{(\nu)}(k)=\tfrac{1}{2}\sum_{\substack{h \bmod 2k\\ \text{g.c.d.}(h,k)=1\\ 2\nmid h,\,2\nmid k}} \epsilon_{h,k}{}^{(\nu)}\exp(-\pi ihn/k);$$

(4.6.5) $\lambda^{(\omega)}(k) = \frac{1}{2} \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,k)=1 \\ 2 \mid h}} \epsilon_{h,k}{}^{(\omega)} \exp(-\pi ihn/k)$; *then*

$$(4.6.6) \qquad \lambda(k) = \begin{cases} \frac{1}{2}(-1)^{\frac{1}{2}(k+1)} A_{2k}(n) & (k \text{ odd}) \\ \frac{1}{2}(-1)^{\frac{1}{2}k} A_{2k}(n-\frac{1}{2}k) & (k \text{ even}), \end{cases}$$

$$(4.6.7) \qquad \lambda^{(\phi)}(k) = \tfrac{1}{2}(-1)^{n+\frac{1}{2}k} A_{2k}(n-\tfrac{1}{2}k)$$

$$(4.6.8) \qquad \lambda^{(\psi)}(k) = \tfrac{1}{2}(-1)^{n+\frac{1}{2}(k-1)} A_{2k}(n)$$

$$(4.6.9) \qquad \lambda^{(\nu)}(k) = \tfrac{1}{2}(-1)^{n+\frac{1}{2}(k-1)} A_k\{(n-1)[\tfrac{1}{2}(k+1)]^2 - 3(k^2-1)/8\}$$

$$(4.6.10) \qquad \lambda^{(\omega)}(k) = \tfrac{1}{2}(-1)^{\frac{1}{2}(k-1)} A_k\{\tfrac{1}{2}n(k+1) - 3(k^2-1)/8\}$$

where $A_k(n) = \sum_{\substack{h \bmod k \\ \text{g.c.d.}(h,k)=1}} \omega_{h,k} \exp(-2\pi ihn/k)$ *and* $\omega_{h,k}$ *appears in* (2.0.1).

Proof. (4.6.6) is a special case of (4.2). (4.6.7) follows immediately from the definition of $\epsilon_{h,k}{}^{(\phi)}$ in Theorem 3.4. (4.6.8) also follows immediately from the definition of $\epsilon_{h,k}{}^{(\psi)}$ in Theorem 3.4. We deal with (4.6.10) next. We have

$$\lambda^{(\omega)}(k) = \frac{1}{2} \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,k)=1 \\ 2 \mid h}} \epsilon_{h,k}{}^{(\omega)} \exp(-\pi ihn/k)$$

$$= \frac{1}{4}(-1)^{\frac{1}{2}(k-1)} \sum_{j=0}^{1} \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,k)=1}} \omega_{h,k} \exp\{\pi ihj + 2\pi ih3(k^2-1)/8k - \pi ihn/k\}$$

[by Corollary 2.5]

$$= \frac{1}{4}(-1)^{\frac{1}{2}(k-1)} \sum_{j=0}^{1} \sum_{\substack{h \bmod k \\ \text{g.c.d.}(h,k)=1}} \omega_{h,k} \exp\{\pi ihj + 2\pi ih3(k^2-1)/8k - \pi ihn/k\} \cdot (1 + (-1)^{jk+n})$$

$$= \frac{1}{2}(-1)^{\frac{1}{2}(k-1)} \sum_{\substack{h \bmod k \\ \text{g.c.d.}(h,k)=1}} \omega_{h,k} \exp(-2\pi ih\{\tfrac{1}{2}n(k+1) - 3(k^2-1/8\}/k)$$

$$= \tfrac{1}{2}(-1)^{\frac{1}{2}(k-1)} A_k\{\tfrac{1}{2}n(k+1) - 3(k^2-1)/8\}.$$

$$\lambda^{(\nu)}(k) = -\frac{1}{2} \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,k)=1 \\ 2 \nmid h,\, 2 \nmid k}} \epsilon_{2h,k}{}^{(\omega)} \exp(\pi ih/k - \pi ihn/k) \qquad \text{[by Theorem 3.4]}$$

$$= \frac{1}{2}(-1)^n \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(2h,k)=1}} \epsilon_{2h,k}{}^{(\omega)} \exp\{-\pi ih(n-1)/k\}$$

$$= \tfrac{1}{4}(-1)^n \sum_{j=0}^{1} \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,k)=1}} \epsilon_{2h,k}{}^{(\omega)} \exp(-\pi i h(n-1)/k + \pi i h j)$$

$$= \tfrac{1}{2}(-1)^n \sum_{\substack{h \bmod k \\ \text{g.c.d.}(h,k)=1}} \epsilon_{2h,k}{}^{(\omega)} \exp\{-\pi i h(n-1)(k+1)/k\}$$

$$= \tfrac{1}{2}(-1)^n \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,k)=1 \\ 2 \mid h}} \epsilon_{h,k}{}^{(\omega)} \exp\{-\pi i h(n-1)\tfrac{1}{2}(k+1)\}$$

$$= \tfrac{1}{2}(-1)^{n+\frac{1}{2}(k-1)} A_k\{(n-1)[\tfrac{1}{2}(k+1)]^2 - 3(k^2-1)/8\} \text{ by } (4.6.10).$$

This concludes Theorem 4.6.

V. The asymptotic formulae. In [2], Dragonette has given asymptotic expansions for the power series coefficients of f, ω, ϕ, ψ, and υ (as discussed in § I). In this section we shall improve her results.

Since Dragoneete obtained the main terms for the expansion of $A(n)$ using the Hardy-Ramanujan-Rademacher method, we shall refer the reader to her paper for many of the details. Our method centers upon the use of estimates of generalized Kloosterman sums together with our precise knowledge of the error terms involved in the modular transformation formulae. We now prove the theorem stated in the introduction.

Proof of Theorem 5.1. For the initial Farey dissection we refer the reader to [2; pp. 489-90]. Dragonette obtains there

$$(5.1.1) \quad A(n) = \tfrac{1}{2} \sum_{k=1}^{N} \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,2k)=1}} \exp(-\pi i h n/k) \int_{-\theta'_{h,k}}^{\theta''_{h,k}} \{z^{-\frac{1}{2}} \epsilon_{h,k}{}^{(f)}$$

$$\cdot \exp(\pi z n/k - \pi z/24k + \pi z^{-1}/24k) f(\exp\{\pi i(h'' + i z^{-1})/k\}) + E_2{}^{(f)}(h,k;z)\} d\phi$$

$$+ \tfrac{1}{2} \sum_{k=1}^{N} \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,k)=1 \\ 2 \mid h}} \exp(-\pi i h n/k) \int_{-\theta'_{h,k}}^{\theta''_{h,k}} \{2^{\frac{3}{2}} z^{-\frac{1}{2}} \eta_{h,k}{}^{(f)}$$

$$\cdot \exp(\pi z n/k - \pi z/24k - 4\pi z^{-1}/3k) \omega(\exp\{2\pi i(h' + i z^{-1})/k\}) + E_1{}^{(f)}(h,k;z)\} d\phi$$

$$(5.1.2) \qquad = \sum_{k=1}^{N} \frac{\lambda(k) \exp\{\pi(n-1/24)^{\frac{1}{2}}/k6^{\frac{1}{2}}\}}{k^{\frac{1}{2}}(n-1/24)^{\frac{1}{2}}} + O\Big(\sum_{k=1}^{N} (n,k)^{\frac{1}{2}} k^{\epsilon-1}\Big) + O(n^{\epsilon})$$

$$+\tfrac{1}{2}\sum_{k=1}^{N}\sum_{\substack{h \bmod 2k\\ \text{g.c.d.}(h,2k)=1}} \exp(-\pi ihn/k)\int_{-\theta'_{h,k}}^{\theta''_{h,k}} E_2^{(f)}(h,k;z)d\phi$$

$$+\tfrac{1}{2}\sum_{k=1}^{N}\sum_{\substack{h \bmod 2k\\ \text{g.c.d.}(h,2k)=1}} \exp(-\pi ihn/k)\int_{-\theta'_{h,k}}^{\theta''_{h,k}} E_1^{(f)}(h,k;z)d\phi$$

$$(5.1.3)\qquad = \Sigma_1 + O(n^{\epsilon}) + \Sigma_2 + \Sigma_3,$$

We remark that (5.1.3) is obtained from (5.1.1) by almost the same method that Dragonette uses in obtaining [2; p. 493, line (5.5)] from [2; p. 490, line (4.5)]. The main alteration in the proof is our use of estimates for truncated generalized Kloosterman sums given by Lemma 4.1 and equations (4.2) and (4.3); this is done in exactly the way that Rademacher modifies the Hardy-Ramanujan method to obtain Fourier coefficients for the modular invariant, $J(\tau)$, [8]. Also we exhibit the contributions of $E_1^{(f)}(h,k;z)$ and $E_2^{(f)}(h,k;z)$ whereas Dragonette uses the estimate

$$|E_j^{(f)}(h,k;z)| = O(k\log k) \qquad j=1,2.$$

We shall show in fact that

$$|\Sigma_j| = O(n^{\epsilon}) \qquad j=2,3.$$

For reference we give the following resumè of the notation used above.

$$\theta'_{h,k} = k^{-1}(k_1+k)^{-1}, \qquad \theta''_{h,k} = k^{-1}(k_2+k)^{-1}$$

where $h_1/k_1 < h/k < h_2/k_2$ are adjacent Farey fractions in the Farey series of order N. As a simple consequence of the theory of Farey series we have

$$k_1 \equiv -h' \pmod{k}, \qquad k_2 \equiv h' \pmod{k}, \text{ and}$$
$$N-k < k_1 \leqq N, \qquad N-k < k_2 \leqq N.$$
$$z = kN^{-2} - ik\phi.$$

Also throughout our work here we take $N = [n^{\frac{1}{2}}]$. Using the notation of Theorems 2.1 and 2.2, we have

$$\Sigma_2 = \tfrac{1}{2}\sum_{k=1}^{N} k^{-1}\sum_{\substack{h \bmod 2k\\ \text{g.c.d.}(h,2k)=1}} \exp(-\pi ihn/k)\sum_{\nu \bmod k}\delta_1^{(f)}(\nu;h;h'';2k)$$
$$\cdot\int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} J_1^{(f)}(\nu;2k;4z)\,d\phi$$

$$= \tfrac{1}{2}\sum_{k-1}^{N} k^{-1} \sum_{\nu=1}^{k} \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,2k)=1}} \delta_1^{(f)}(\nu; h; h''; 2k) \exp(-\pi i h n/k)$$

$$\cdot \left\{ \int_{-k^{-1}(N+k)^{-1}}^{k^{-1}(N+k)^{-1}} + \int_{-k^{-1}(k_1+k)^{-1}}^{-k^{-1}(N+k)^{-1}} + \int_{k^{-1}(N+k)^{-1}}^{k^{-1}(k_2+k)^{-1}} \right\}$$

$$= \Sigma_{21} + \Sigma_{22} + \Sigma_{23}.$$

$$|\Sigma_{21}| = O\left(\sum_{k=1}^{N} k^{-1} (n,k)^{\frac{1}{2}} k^{\frac{1}{2}+\epsilon'} (kN)^{-1} \sum_{\nu=1}^{k} N^{\frac{1}{2}} k \,|\, k - 2(\nu - \tfrac{1}{6}) |^{-1} \right)$$

(by Theorem 2.1 and (4.5))

$$= O\left(N^{-\frac{1}{2}} \sum_{k=1}^{N} (n,k)^{\frac{1}{2}} k^{-\frac{1}{2}-2\epsilon''}\right)$$

$$= O\left(N^{-\frac{1}{2}} \sum_{\substack{d \mid n \\ d \leqq n^{\frac{1}{2}}}} d^{\frac{1}{2}} \sum_{\substack{k=1 \\ \text{g.c.d.}(n,k)=1}}^{n^{\frac{1}{2}}} k^{-\frac{1}{2}+2\epsilon''}\right)$$

$$= O\left(N^{-\frac{1}{2}} \sum_{d \mid n} d^{\frac{1}{2}} \sum_{\substack{k'=1 \\ d \leqq n^{\frac{1}{2}}}}^{n^{\frac{1}{2}}/d} d^{-\frac{1}{2}+2\epsilon''} k'^{-\frac{1}{2}+2\epsilon''}\right)$$

$$= O\left(N^{-\frac{1}{2}} \sum_{\substack{d \mid n \\ d \leqq n^{\frac{1}{2}}}} d^{2\epsilon''} n^{\frac{1}{4}+\epsilon''} d^{-\frac{1}{2}-2\epsilon''}\right)$$

$$= O\left(n^{\epsilon''} \sum_{\substack{d \mid n \\ d \leqq n^{\frac{1}{2}}}} d^{-\frac{1}{2}}\right)$$

$$= O(n^{\epsilon}).$$

Σ_{22} and Σ_{23} are of the same structure so we treat only Σ_{23}.

$$\Sigma_{23} = \tfrac{1}{2}\sum_{k=1}^{N} k^{-1} \sum_{\nu=1}^{k} \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,2k)=1}} \delta_1^{(f)}(\nu; h; h''; 2k) \exp(-\pi i h n/k)$$

$$\cdot \sum_{l=k_2+k}^{N+k-1} \int_{k^{-1}(l+1)^{-1}}^{k^{-1}l^{-1}} z^{\frac{1}{2}} J_1^{(f)}(\nu; 2k; 4z)\, d\phi$$

$$= \tfrac{1}{2}\sum_{k=1}^{N} k^{-1} \sum_{\nu=1}^{k} \sum_{l=N+1}^{N+k-1} \int_{k^{-1}(l+1)^{-1}}^{k^{-1}l^{-1}} z^{\frac{1}{2}} J_1^{(f)}(\nu; 2k; 4z)\, d\phi$$

$$\cdot \sum_{\substack{h \bmod 2k \\ \text{g.c.d.}(h,2k)=1 \\ N < k_2+k \leqq l}} \delta_1^{(f)}(\nu; h; h''; 2k) \exp(-\pi i h n/k).$$

The inner sum is now a truncated generalized Kloosterman sum by (4.5) and subject to the estimate of Lemma 4.1. Hence

$$
\begin{aligned}
|\Sigma_{23}| &= O\Big(\sum_{k=1}^{N} k^{-1} \sum_{\nu=1}^{k} N^{\frac{1}{2}} k \,|\, k-2(\nu-\tfrac{1}{6}) \,|^{-1} (kN)^{-1} (n,k)^{\frac{1}{2}} k^{\frac{1}{2}+\epsilon'}\Big) \\
&= O\Big(N^{-\frac{1}{2}} \sum_{k=1}^{N} (n,k)^{\frac{1}{2}} k^{-\frac{1}{2}+\epsilon''} \log k\Big) \\
&= O(N^{\epsilon'''}) \quad \text{(just as for } \Sigma_{21}) \\
&= O(n^{\epsilon}).
\end{aligned}
$$

Thus we have $|\Sigma_2| = O(n^{\epsilon})$ for any $\epsilon > 0$. Σ_3 is handled in exactly the same manner. Hence

$$A(n) = \sum_{k=1}^{[n^{\frac{1}{2}}]} \frac{\lambda(k) \exp\{\pi(n-1/24)^{\frac{1}{2}}/k6^{\frac{1}{2}}\}}{k^{\frac{1}{2}}(n-1/24)^{\frac{1}{2}}} + O(n^{\epsilon}).$$

Coupling this with (4.6.6), we obtain the required result.

In an exactly analogous manner one may obtain the following

THEOREM 5.1. *Let $M(q)$ be one of the functions ϕ, ψ, υ, ω. Then if*

$$M(q) = \sum_{n=0}^{\infty} A^{(M)}(n) q^n,$$

$$A^{(\phi)}(n) = \sum_{\substack{k=1 \\ 2 \mid k}}^{[n^{\frac{1}{2}}]} \frac{\lambda^{(\phi)}(k) \exp\{\pi(n-1/24)^{\frac{1}{2}}/k6^{\frac{1}{2}}\}}{k^{\frac{1}{2}}(n-1/24)^{\frac{1}{2}}} + O(n^{\epsilon}),$$

$$\lambda^{(\phi)}(k) = \tfrac{1}{2}(-1)^{n+\frac{1}{2}k} A_{2k}(n-\tfrac{1}{2}k) \qquad (\text{by } (4.6.7))$$

$$A^{(\psi)}(n) = \sum_{\substack{k=1 \\ (k,2)=1}}^{[n^{\frac{1}{2}}]} \frac{\lambda^{(\psi)}(k) \exp\{\pi(n-1/24)^{\frac{1}{2}}/k6^{\frac{1}{2}}\}}{2k^{\frac{1}{2}}(n-1/24)^{\frac{1}{2}}} + O(n^{\epsilon}),$$

$$\lambda^{(\psi)}(k) = \tfrac{1}{2}(-1)^{n+\frac{1}{2}(k-1)} A_{2k}(n) \qquad (\text{by } (4.6.8))$$

$$A^{(\upsilon)}(n) = \sum_{\substack{k=1 \\ (k,2)=1}}^{[n^{\frac{1}{2}}]} \frac{\lambda^{(\upsilon)}(k) \exp\{\pi(n-1/3)^{\frac{1}{2}}/k6^{\frac{1}{2}}\}}{(2k)^{\frac{1}{2}}(n+1/3)^{\frac{1}{2}}} + O(n^{\epsilon}),$$

$$\lambda^{(\upsilon)}(k) = \tfrac{1}{2}(-1)^{n+\frac{1}{2}(k-1)} A_k\{(n-1)[\tfrac{1}{2}(k+1)]^2 - 3(k^2-1)/8\} \qquad (\text{by } 4.6.9))$$

$$A^{(\omega)}(n) = \sum_{\substack{k=1 \\ (k,2)=1}}^{[n^{\frac{1}{2}}]} \frac{\lambda^{(\omega)}(k)\exp\{\pi(n+2/3)^{\frac{1}{2}}/k3^{\frac{1}{2}}\}}{k^{\frac{1}{2}}(n+2/3)^{\frac{1}{2}}} + O(n^{\epsilon}),$$

$$\lambda^{(\omega)}(k) = \tfrac{1}{2}(-1)^{\frac{1}{2}(k-1)}A_k\{n\tfrac{1}{2}(k+1) - 3(k^2-1)/8\} \quad (\text{by } 4.6.10)).$$

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