PARTITIONS AND DURFEE DISSECTION.

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1. Introduction. The Rogers-Ramanujan identities may be stated analytically as follows:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{n=0}^{\infty} (1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1}, \tag{1.1}$$

$$\sum_{n=0} \frac{q^{n^2+n}}{(q)_n} = \prod_{n=0}^{\infty} (1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1}, \tag{1.2}$$

where $(a)_0 = 1$ and $(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. These identities together with their history are discussed at length in [9, Chapter 7].

Recently the identities (1.1) and (1.2) were generalized [6; 8; 9, p. 111] to

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_a+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \prod_{\substack{n=1 \\ n\not\equiv 0,\pm a \pmod{2k+1}}}^{\infty} (1-q^n)^{-1}, \tag{1.3}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$ with $1 \leqslant a \leqslant k$. The identity (1.3) reduces to (1.1) when $k = a = 2$ and to (1.2) when $k = a + 1 = 2$.

Now the right side of (1.3) is clearly the generating function for $A_{k,a}(n)$, the number of partitions of n into parts $\not\equiv 0, \pm a \pmod{2k+1}$. B. Gordon [12] (see also [1], [3], [9, Chapter 7, Section 3]) has proved that $A_{k,a}(n) = B_{k,a}(n)$, the number of partitions of n of the form $b_1 + b_2 + \cdots + b_s$ where $b_j \geqslant b_{j+1}$, $b_j - b_{j+k-1} \geqslant 2$, and at most $a-1$ of the b_j equal 1.

In [11], R. Askey and I asked for a combinatorial interpretation of the left side of (1.3) which would show directly that it was the generating function for

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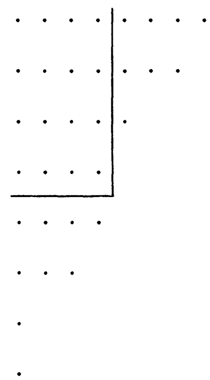
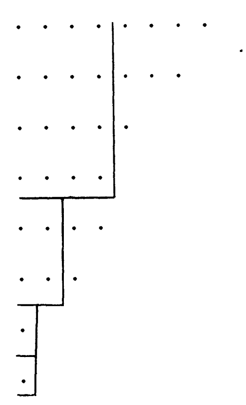
$B_{k,a}(n)$. It turns out that there is a very natural interpretation for the left side of (1.3), especially when $a=k$; however, the interpretation is completely different from any given previously. To give the new interpretation we must recall the definition of the Durfee square. For example, let us consider the partition $9+7+5+4+4+3+1+1$; its graphical representation is given by



and the "Durfee square" or largest square of dots is the 4×4 square indicated. The Durfee square concept has been extended to other rectangles [2]; however, here we wish to discuss "successive Durfee squares." This concept is almost self-explanatory. Once the Durfee square is determined, it splits the given partition into 3 parts: (1) the square itself, (2) a smaller partition to the right of the square, and (3) a smaller partition below the square. We say that the smaller partition to the right of the square is "attached" or "associated" with the square. One can now determine a "second Durfee square" in the smaller partition below the original Durfee square. Clearly third, fourth, etc. Durfee squares can be determined as long as the lower portion of the partition is not exhausted. Thus the partition $9+7+5+4+4+3+1+1$ has four successive Durfee squares:



The nicest result we obtain on successive Durfee squares is the following:

THEOREM 1. *The number of partitions of n with at most $k-1$ successive Durfee squares equals the number of partitions of n into parts* $\not\equiv 0, \pm k \pmod{2k+1}$.

We remark that Theorem 1 directly implies the first Rogers-Ramanujan identity in the case $k=2$. To see this most easily, we note that the generating function for partitions with at most one Durfee square is

$$\sum_{n=0}^{\infty} \frac{q^{n+n+\cdots+n}}{(q)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n},$$

while the generating function for partitions with parts $\not\equiv 0, \pm 2 \pmod 5$ is

$$\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

and (1.1) follows from the assertion that these two functions are identical.

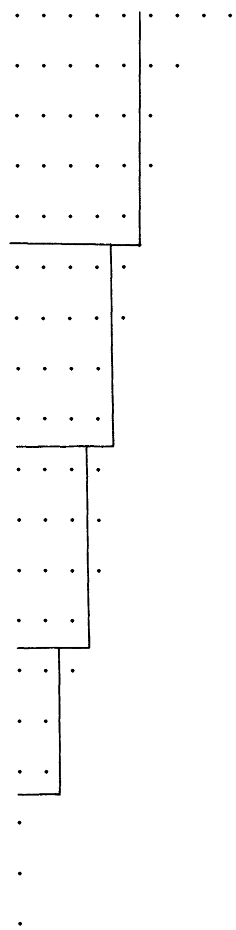
Now, of course, the identity (1.3) contains the two parameters a and k. In Theorem 2 we interpret all the cases of (1.3). To do this we must introduce the concept of a "Durfee dissection," and we do this in the next section.

2. Durfee Dissections. The idea of successive Durfee squares introduced in Section 1 is adequate to state Theorem 1. To treat the identity (1.3) in full generality we must define the "(k,a)-Durfee dissection" of a partition.

We begin by determining the first $a-1$ successive Durfee squares of the partition. Next we determine $k-a$ maximal rectangles (one below the other) for which the horizontal side has one less dot than the vertical side. For example the (5,3)-Durfee dissection of $9+7+6+6+5+5+5+4+4+4+4+4+3+3+2+2+1+1+1$ is (see following page).

We shall say a partition is "(k,a)-admissible" if it has no parts below the last rectangle in its (k,a)-Durfee dissection and furthermore the lower edge of each of the final $k-a$ rectangles of the (k,a)-Durfee dissection is actually a part of the partition (i.e., there are no dots of the Ferrers graph of the partition to the right and on the same row).

Finally we note that "(k,k)-admissible" means only that the partition has at most $k-1$ successive Durfee squares. We are now prepared to interpret (1.3).



3. The Main Theorems.

THEOREM 2. *For* $1 \leqslant a \leqslant k$ *the number of* (k,a)*-admissible partitions of* n *equals the number of partitions of* n *into parts* $\not\equiv 0, \pm a \pmod{2k+1}$.

Proof. As is well known, the right hand side of (1.3) is the generating function for $A_{k,a}(n)$ the number of partitions of n into parts $\not\equiv 0, \pm a$ (mod $2k+1$) (see [9, Chapter 7, Section 3]). Thus to prove Theorem 2 we deduce it from (1.3) by showing that the left hand side is the generating function for $\Delta_{k,a}(n)$, the number of (k,a)-admissible partitions of n.

Let us recall that the Gaussian polynomial

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \frac{(q)_{n+m}}{(q)_n (q)_m}$$

is the generating function for partitions with at most m parts each $\leqslant n$ [9, p. 33]. Now

$$\sum_{n_1\geqslant 0}\cdots\sum_{n_{k-1}\geqslant 0}\frac{q^{N_1^2+N_2^2+\cdots+N_{a-1}^2+N_a+N_{a+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}$$

$$=\sum_{n_1\geqslant 0}\cdots\sum_{n_{k-1}\geqslant 0}\frac{q^{(n_1+n_2+\cdots+n_{k-1})^2}}{(q)_{n_1+n_2+\cdots+n_{k-1}}}$$

$$\times\frac{q^{(n_2+n_3+\cdots+n_{k-1})^2}(q)_{n_1+n_2+\cdots+n_{k-1}}}{(q)_{n_1}(q)_{n_2+n_3+\cdots+n_{k-1}}}$$

$$\times\frac{q^{(n_3+n_4+\cdots+n_{k-1})^2}(q)_{n_2+n_3+\cdots+n_{k-1}}}{(q)_{n_2}(q)_{n_3+n_4+\cdots+n_{k-1}}}$$

$$\times\cdots\times\frac{q^{(n_{a-1}+\cdots+n_{k-1})^2}(q)_{n_{a-2}+\cdots+n_{k-1}}}{(q)_{n_{a-2}}(q)_{n_{a-1}+\cdots+n_{k-1}}}$$

$$\times\frac{q^{(n_a+\cdots+n_{k-1})(n_a+\cdots+n_{k-1}+1)}(q)_{n_{a-1}+\cdots+n_{k-1}}}{(q)_{n_{a-1}}(q)_{n_a+\cdots+n_{k-1}}}$$

$$\times\cdots\times\frac{q^{n_{k-1}(n_{k-1}+1)}(q)_{n_{k-2}+n_{k-1}}}{(q)_{n_{k-2}}(q)_{n_{k-1}}}$$

$$=\sum_{n_1\geqslant 0}\cdots\sum_{n_{k-1}\geqslant 0}\frac{q^{N_1^2}}{(q)_{N_1}}q^{N_2^2}\begin{bmatrix}N_1\\ n_1\end{bmatrix}$$

$$\times q^{N_3^2}\begin{bmatrix}N_2\\ n_2\end{bmatrix}\cdots q^{N_{a-1}^2}\begin{bmatrix}N_{a-2}\\ n_{a-2}\end{bmatrix}$$

$$\times q^{N_a(N_a+1)}\begin{bmatrix}N_{a-1}\\ n_{a-1}\end{bmatrix}\cdots q^{N_{k-1}(N_{k-1}+1)}\begin{bmatrix}N_{k-2}\\ n_{k-2}\end{bmatrix}.$$

Now $q^{N_1^2}/(q)_{N_1}$ is clearly the generating function for that portion of the partition attached to a first Durfee square of side N_1. To see this, work with the conjugate of the partition consisting of the dots of the Ferrers graph that are attached to the first Durfee square. The graphical representation of the conjugate of a partition is obtained by reflecting in the main diagonal, the graph of the original partion. For $2\leqslant j\leqslant a-1$

$$q^{N_j^2}\begin{bmatrix}N_{j-1}\\ n_{j-1}\end{bmatrix}$$

generates those parts of the partition associated with a jth Durfee square of side

N_j. Note that the conjugate of the partition attached to the jth Durfee square has $\leqslant n_{j-1}$ parts each $\leqslant N_{j-1}-n_{j-1}$. For if not, we could enlarge either the $(j-1)$st or the jth maximal Durfee square respectively. Finally, for $a \leqslant j \leqslant k-1$,

$$q^{N_j^2}\begin{bmatrix} N_{j-1} \\ n_{j-1} \end{bmatrix} q^{N_j}$$

generates those parts of the partition associated with a $N_j \times (N_j+1)$ Durfee rectangle where the lower edge of the rectangle is a part of the partition. This last fact follows because $\begin{bmatrix} N_{j-1} \\ n_{j-1} \end{bmatrix}$ enumerates only those partitions attached to the jth Durfee rectangle whose conjugate partition has parts all $\leqslant N_{j-1}-n_{j-1}=N_j$. Thus from this analysis we see that all the (k,a)-admissible partitions are generated by the left side of (1.3), since as $n_1,\ldots,n_{k-1}$ independently range over all nonnegative $(k-1)$-tuples of integers, we see that $N_1,\ldots,N_{k-1}$ range over all nonnegative $(k-1)$-tuples subject to $N_1 \geqslant N_2 \geqslant \cdots \geqslant N_{k-1}$, which is precisely what is required to produce the squares and rectangles in the (k,a)-Durfee dissection. Thus Theorem 2 is established.

Theorem 1 now follows immediately from Theorem 2 and the final remark in Section 2.

4. Generalizations of the Theorems of Euler and Cauchy. In a recent paper [10], we discussed and proved analytically the following identity:

$$\sum_{n_1 \geqslant 0} \cdots \sum_{n_{k-1} \geqslant 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2} z^{N_1+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}(zq)_{n_{k-1}}} = \frac{1}{(zq)_\infty}, \tag{4.1}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$, and $(zq)_\infty = \lim_{n\to\infty}(zq)_n$. When $k=2$ this reduces to Cauchy's generalization of an identity of Euler (see [9, p. 20]). The coefficient of $z^n q^n$ on the right hand side of (4.1) is well known to be the number of partitions of n into m parts [9, p. 16]. From our analysis in Section 3 we see that

$$\frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2} z^{N_1+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} \tag{4.2}$$

is the generating function for partitions with at most $k-1$ successive Durfee squares (whose sides are $N_1, N_2, \ldots, N_{k-1}$). The factor $(zq)_{n_{k-1}}^{-1}$ generates partitions whose parts are all $\leqslant N_{k-1} = n_{k-1}$ [9, p. 16]. Hence the product of the expression (4.2) with $(zq)_{n_{k-1}}^{-1}$ generates all partitions with the first $k-1$ successive Durfee squares having edges $N_1, N_2, \ldots, N_{k-1}$. Note that in this case we split the Ferrers graph of a partition up into two parts: (1) the first $N_1 + \cdots + N_{k-1}$ rows, and (2) the remaining rows (each having $\leqslant N_{k-1} = N_{k-1}$ dots). Finally we sum over all nonnegative $(k-1)$-tuples $n_1, \ldots, n_{k-1}$, and we obtain the generating function for all partitions. Hence the left side of (4.1) equals the right side.

5. Conclusion. It is my belief that the type of analysis of partitions described here may lead to other interesting and new partition identities. Obviously the Durfee square generalizations treated in [2] could now be utilized in Durfee dissections. Thus, for example, (4.1) could be extended in the way that identities (2.11) and (2.12) of [2] extend the case $k=2$ of (4.1). Also the various special cases of the multiple series extension of the q-analog of Whipple's theorem [8, Theorem 4] suggest that similar analyses could be made of the series-product identities related to the general Rogers-Ramanujan theorem of [5].

Finally we again draw attention to the fact that there are now known to be four different generalizations of the Rogers-Ramanujan identities. Besides Gordon's result stated in Section 1 and our Theorem 2, there is an interpretation involving the successive ranks [4] (see also [9, Chapter 9]), and one involving the Alder polynomials [7]. In the case $k=2$ it is a simple matter to pass from any one of these interpretations to another. When $k>2$, D. Bressoud has recently found how to interpret each term in the series of (1.3) in terms of the partitions enumerated by $B_{k,a}(n)$; he has thus answered the question Askey and I posed in [11].

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REFERENCES.

[1] G. E. Andrews, An analytic proof of the Rogers-Ramanujan-Gordon identities, *Amer. J. Math.* 88 (1966), pp. 844–846.

[2] ———, Generalizations of the Durfee square, *J. London Math. Soc. (2)* 3 (1971), pp. 563–570.

[3] ———, Partition identities, *Advances in Math.* 9 (1972), pp. 10–51.

[4] ———, Sieves in the theory of partitions, *Amer. J. Math.* 94 (1972), pp. 1214–1230.

[5] ———, On the general Rogers-Ramanujan theorem, *Mem. Amer. Math. Soc.*, No. 152, 1974.

[6] ———, An analytic generalization of the Rogers-Ramanujan identities for odd moduli, *Proc. Nat. Acad. Sci. U.S.A.* 71 (1974), pp. 4082–4085.

[7] ———, On the Alder polynomials and a new generalization of the Rogers-Ramanujan identities, *Trans. Amer. Math. Soc.* **204** (1975), pp. 40–64.

[8] ———, Problems and prospects for basic hypergeometric functions, in *The Theory and Applications of Special Functions* (R. Askey, Ed.), Academic, New York, 1975, pp. 191–224.

[9] ———, *The Theory of Partitions*, Encyclopedia of Mathematics and Its Applications, Vol. 2, Addison-Wesley, Reading, 1977.

[10] ———, Multiple q-series identities, to appear.

[11] G. E. Andrews and R. Askey, Enumeration of partitions: the role of Eulerian series and q-orthogonal polynomials, in *Higher Combinatorics* (M. Aigner, Ed.), Reidel, Dordrecht, 1977.

[12] B. Gordon, A combinatorial generalization of the Rogers-Ramanujan identities, *Amer. J. Math.* **83** (1961), pp. 393–399.