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VANISHING COEFFICIENTS IN INFINITE PRODUCT EXPANSIONS

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Abstract

Richmond and Szekeres (1977) have conjectured that certain of the coefficients in the power series expansions of certain infinite products vanish. In this paper, we prove a general family of results of this nature which includes the above conjectures.

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1. Introduction

Richmond and Szekeres (1977) have determined Hardy–Ramanujan–Rademacher expansions for quotients of certain infinite products that have arisen in continued fraction expansions of the Rogers–Ramanujan type. From these results they deduce that if

$$\sum_{m=0}^{\infty} c_m q^m = F(q) \equiv \prod_{n=0}^{\infty} \frac{(1-q^{8n+3})(1-q^{8n+5})}{(1-q^{8n+1})(1-q^{8n+7})},$$

then c_{4m+3} is always zero; furthermore, if

$$\sum_{m=0}^{\infty} d_m q^m = \frac{1}{F(q)},$$

then d_{4m+2} is always zero.

Their results also lead them to conjecture that if

$$\sum_{m=0}^{\infty} a_m q^m = G(q) = \prod_{n=0} \frac{(1-q^{12n+5})(1-q^{12n+7})}{(1-q^{12n+1})(1-q^{12n+11})},$$

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then a_{6m+5} is always zero, and if

$$\sum_{m=0}^{\infty} b_m q^m = \frac{1}{G(q)},$$

then b_{6m+3} is always zero.

We shall prove two theorems concerning the quotient of infinite products

$$F_{k,r}(q) = \prod_{m=0}^{\infty} \frac{(1-q^{2km+r})(1-q^{2km+2k-r})}{(1-q^{2km+k-r})(1-q^{2km+k+r})},$$

where $1 \leqslant r \leqslant k$ are integers. The first theorem gives a nice identity for treating such congruence problems.

THEOREM 1.

$$\left(\sum_{n=0}^{\infty} q^{kn(n+1)/2}\right)^2 F_{k,r}(q) = \sum_{n=1}^{\infty} \frac{q^{(k-r)(n-1)} - q^{(k+r)n-k}}{1-q^{2kn-k}}.$$

THEOREM 2. *If* $1 \leqslant r < k$ *are relatively prime integers of opposite parity and* $F_{k,r}(q) = \sum_{n=0}^{\infty} \varphi_n q^n$ *then* $\varphi_{kn+r(k-r+1)/2}$ *is always zero.*

We remark that cases $k=4$, $r=3$ and $k=4$, $r=1$ are the results proved by Richmond and Szekeres for the C_m, while the cases $k=6$, $r=5$ and $k=6$, $r=1$ establish the two conjectures for the d_m.

2. Proof of Theorem 1

Theorem 1 relies on one of Ramanujan's elegant summations:

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n z^n}{(b)_n} = \prod_{n=0}^{\infty} \frac{\left[\left(1-\frac{b}{a}q^n\right)(1-azq^n)\left(1-\frac{q^{n+1}}{az}\right)(1-q^{n+1})\right]}{\left[\left(1-\frac{q^{n+1}}{a}\right)\left(1-\frac{bq^n}{az}\right)(1-bq^n)(1-zq^n)\right]}, \tag{2.1}$$

where $(a)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$ when n is positive and more generally $(a)_n = \prod_{j>0}(1-aq^j)(1-aq^{j+n})^{-1}$. We must also require for the convergence of the series in (2.1) that $|b/a| < |z| < 1$, $|q| < 1$. For proofs of (2.1) see Andrews (1969), Andrews and Askey (1977) and Ismail (1977).

Let us now replace q by q^{2k}, then set $z = q^{k+r}$, $a = q^{-k}$, $b = q^k$ and multiply both sides by $(1-q^k)^{-1}$. The resulting formula is

$$\left(\prod_{n=1}^{\infty} \frac{(1-q^{2kn})}{(1-q^{2kn-k})}\right)^2 F_{k,r}(q) = \frac{1}{1-q^k} \sum_{n=-\infty}^{\infty} \frac{(1-q^{-k})q^{(k+r)n}}{(1-q^{2kn-k})}$$

$$= -q^{-k} \sum_{n=-\infty}^{\infty} \frac{q^{(k+r)n}}{1-q^{2kn-k}}$$

$$= -q^{-k}\sum_{n=1}^{\infty}\frac{q^{(k+r)n}}{1-q^{2kn-k}} - \sum_{n=1}^{\infty}\frac{q^{-r(n-1)-kn}}{1-q^{-2kn+k}}$$

$$= \sum_{n=1}^{\infty}\frac{q^{(k-r)(n-1)}-q^{(k+r)n-k}}{1-q^{2kn-k}}.$$

Theorem 1 follows immediately when we recall Gauss's formula (Andrews (1976), p. 23):

$$\sum_{n=0}^{\infty} x^{n(n+1)/2} = \prod_{n=1}^{\infty}\frac{(1-x^{2n})}{(1-x^{2n-1})}.$$

3. Proof of Theorem 2

From Theorem 1 it is immediate that to prove Theorem 2, we need only prove that the coefficient of $q^{kn+r(k-r+1)/2}$ in

$$\sum_{n=1}^{\infty}\frac{q^{(k-r)(n-1)}-q^{(k+r)n-k}}{1-q^{2kn-k}} \tag{3.1}$$

is identically zero since $\sum_{n=0}^{\infty} q^{kn(n+1)/2}$ is a function of q^k. Now the terms of the series that contribute something to these coefficients either have

$$(k-r)(n-1) \equiv r(k-r+1)/2 \pmod k \tag{3.2}$$

or

$$(k+r)n-k \equiv r(k-r+1)/2 \pmod k. \tag{3.3}$$

Since we are assuming k and r are relatively prime, we see that $n=(k-r+1)/2$ is the unique solution of (3.3) lying in $[1, k-1]$ while $n=(k+r+1)/2$ is the unique solution of (3.2) lying in $[2, k]$. Hence the portion of (3.1) that contains powers $q^{kn+r(k-r+1)/2}$ is given by

$$\sum_{n=0}^{\infty}\frac{q^{(k-r)(kn+(k+r-1)/2)}}{1-q^{2k(kn+(k+r+1)/2)-k}} - \sum_{n=0}^{\infty}\frac{q^{(k+r)(kn+(k-r+1)/2)-k}}{1-q^{2k(kn+(k-r+1)/2)-k}} = T_1 - T_2. \tag{3.4}$$

Now

$$\begin{aligned} T_2 &= \sum_{n=0}^{\infty} q^{(k+r)(kn+(k-r+1)/2)-k}\sum_{m=0}^{\infty} q^{2km(kn+(k-r+1)/2)-km} \\ &= \sum_{m=0}^{\infty}\frac{q^{km(k-r+1)-km+(k+r)(k-r+1)/2-k}}{1-q^{2k^2m+(k+r)k}} \\ &= \sum_{m=0}^{\infty}\frac{q^{(k-r)(km+(k+r-1)/2)}}{1-q^{2k(km+(k+r+1)/2)-k}} \\ &= T_1. \end{aligned} \tag{3.5}$$

Hence the expression (3.4) is identically zero. Thus all the coefficients

$$\varphi_{kn+r(k-r+1)/2}$$

in the expansion of $F_{k,r}(q)$ are identically zero. This concludes the proof of Theorem 2.

References

G. E. Andrews (1969), 'On Ramanujan's summation of ${}_1\psi_1(a; b; z)$, *Proc. Amer. Math. Soc.* **22**, 552–553.

G. E. Andrews (1976), *The theory of partitions* (Encyclopedia of Math. and its Applications, Vol. 2, Addison-Wesley, Reading, Mass.).

G. E. Andrews and R. A. Askey (1977), 'A simple proof of Ramanujan's summation of the ${}_1\psi_1$ summation', *Aequationes Math.* (to appear).

M. Ismail (1977), 'A simple proof of Ramanujan's ${}_1\psi_1$ summation', *Proc. Amer. Math. Soc.* 63, 185–186.

B. Richmond and G. Szekeres (1977), 'The Taylor coefficients of certain infinite products', *Acta Szeged* (to appear).

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