Inventiones math. 53, 193–225 (1979)

Inventiones mathematicae

Plane Partitions (III): The Weak Macdonald Conjecture

George E. Andrews*

The Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.

Dedicated to the memory of Alfred Young and F.J.W. Whipple

1. Introduction

A plane partition π of n is an array of positive integers a_{ij} such that $\sum a_{ij}=n$ (more briefly $|\pi|=n$) where $a_{ij} \geqq \max(a_{i+1,j}, a_{i,j+1})$. Generally we represent a plane partition by the array

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1r} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2s} \\ \vdots \\ a_{j1} & \cdots & \cdots & a_{jw}. \end{matrix}$$

Thus the plane partitions of 3 are

$$\begin{matrix} 3, & 21, & 2, & 111, & 11, & 1 \\ & & 1 & & 1 & 1 \\ & & & & & 1. \end{matrix}$$

There are some well-known theorems and open questions related to plane partitions in which the number of rows and columns and the size of the parts is restricted. I.G. Macdonald [10] has devised a notation that allows a uniform consideration of these questions. First one considers the "Ferrers graph" $D(\pi)$ of a plane partition π; this is the set of integer points (i,j,k) in the first octant that satisfy $1 \leqq k \leqq a_{ij}$. Next define the height of $p=(i,j,k)$ to be $ht(p)=i+j+k-2$. Defining

$$\mathscr{B}_{l,m,n}=[1,l]\times[1,m]\times[1,n]\subset \mathbb{Z}^3, \tag{1.1}$$

* Partially supported by National Science Foundation Grant MCS 75-19162

0020-9910/79/0053/0193/$06.60

Macdonald notes [10] that MacMahon's [12; p. 293] main result on plane partitions asserts that

$$\sum_{D(\pi)\subseteq \mathscr{B}_{l,m,n}} q^{|\pi|} = \prod_{p\in\mathscr{B}_{l,m,n}} \frac{1-q^{1+ht(p)}}{1-q^{ht(p)}}. \tag{1.2}$$

Furthermore, MacMahon [11] made a conjecture about symmetric plane partitions. Let G_2 be the two element subgroup of the symmetric group on three letters that contains the identity and the transposition $(i,j,k)\to(j,i,k)$.

Macdonald [10] points out that MacMahon's conjecture reduces to

$$\sum_{\substack{D(\pi)\subseteq\mathscr{B}_{l,l,n}\\ D(\pi)\text{ is }G_2\text{ invariant}}} q^{|\pi|} = \prod_{\xi\in\mathscr{B}_{l,l,n}/G_2} \frac{(1-q^{|\xi|+ht(\xi)})}{(1-q^{ht(\xi)})}, \tag{1.3}$$

where $|\xi|=\mathrm{Card}(\xi)$, $ht(\xi)=\sum_{p\in\xi} ht(p)$. Macdonald (unpublished) and I [1], [4] have independently proved this conjecture.

Next Macdonald [10] considers G_3, the three element group of cyclic permutations of (i,j,k), and he conjectures

$$\sum_{\substack{D(\pi)\subseteq\mathscr{B}_{m,m,m}\\ D(\pi)\text{ is }G_3\text{ invariant}}} q^{|\pi|} = \prod_{\xi\in\mathscr{B}_{m,m,m}/G_3} \frac{(1-q^{|\xi|+ht(\xi)})}{(1-q^{ht(\xi)})}. \tag{1.4}$$

While the formulae (1.2), (1.3) and (1.4) beautifully illustrate the parallel nature of these assertions, one becomes aware very quickly that an immense number of cancellations occur in each of these products. Hence we choose to reformulate (1.4) less elegantly but more tractably as follows:

Macdonald's Conjecture. Let $M(m,n)$ denote the number of plane partitions π such that $|\pi|=n$, $D(\pi)$ is invariant under G_3 and $D(\pi)\subseteq\mathscr{B}_{m,m,m}$. Then

$$\sum_{n\geq 0} M(m,n)\,q^n = \prod_{i=1}^{n} \left\{\frac{(1-q^{3i-1})}{(1-q^{3i-2})} \prod_{j=i}^{m} \frac{(1-q^{3(m+i+j-1)})}{(1-q^{3(2i+j-1)})}\right\}. \tag{1.5}$$

If we let $m\to\infty$, then we obtain

Limiting Macdonald Conjecture.

$$\sum_{n\geq 0} M(\infty,n)\,q^n = \prod_{n=1}^{\infty} \frac{(1-q^{3n-1})}{(1-q^{3n-2})(1-q^{3n})^{\left[\frac{n+1}{3}\right]}}. \tag{1.6}$$

The above conjectures are still open. Our main object in this paper is to prove the following:

Weak Macdonald Conjecture (Theorem 9 below). The total number of plane partitions π such that $D(\pi)\subseteq\mathscr{B}_{m,m,m}$ and $D(\pi)$ is G_3 invariant is

$$\prod_{i=1}^{m} \left\{\frac{3i-1}{3i-2} \prod_{j=i}^{m} \frac{m+i+j-1}{2i+j-1}\right\}$$

(i.e. (1.5) holds at $q=1$).

In our treatment of the Weak Macdonald Conjecture we are led to consider what we term descending plane partitions. To define these objects we must first consider shifted plane partitions, a topic treated extensively in [9] (see also [14] and [15]). A shifted plane partition ψ of n is an array of positive integers a_{ij} *defined only for* $j \geqq i$ such that $\sum a_{ij} = n$ (more briefly $|\psi| = n$), $a_{ij} \geqq a_{i,j+1}$, and for $j > i$, $a_{ij} \geqq a_{i+1,j}$. Thus the seven shifted plane partitions of 4 are

$$4, \quad 31, \quad 22, \quad 211, \quad \begin{matrix} 21, \\ \ \ 1 \end{matrix} \quad 1111, \quad \begin{matrix} 111 \\ \ \ 1 \end{matrix}.$$

We shall call shifted plane partitions "strict" if there is strict decrease down each column. Finally we define a *descending plane partition* to be a strict shifted plane partition such that the first entry in each row does not exceed the number of parts in the preceding row but is larger than the number of parts in its own row. Thus

$$\begin{matrix} 5 & 4 & 4 & 3 \\ & 3 & 3 & \\ & & 2 & \end{matrix}$$

is a descending plane partition of 24.

Descending Plane Partitions Conjecture. Let $De(m,n)$ denote the number of descending plane partitions of n in which each part is $\leqq m$. Then

$$\sum_{n \geqq 0} De(m,n)\, q^n = \prod_{i=1}^{m} \prod_{j=i}^{m} \frac{(1-q^{m+i+j-1})}{(1-q^{2i+j-1})}. \tag{1.7}$$

Limiting Descending Plane Partitions Conjecture.

$$\sum_{n \geqq 0} De(\infty,n)\, q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{\left[\frac{n+1}{3}\right]}}. \tag{1.8}$$

These two conjectures are also open. However as a parallel consequence of our solution to the weak Macdonald conjecture we are able to prove also the following assertion:

Weak Descending Plane Partitions Conjecture (Theorem 10 below). The total number of descending plane partitions whose parts do not exceed m is $\prod_{i=1}^{m} \prod_{j=i}^{m} \frac{m+i+j-1}{2i+j-1}$.

In Sect. 2 we present a combinatorial study of shifted plane partitions with the object of showing their relationship to the conjectures we have described. The three main results to be proved there are:

Theorem 1. *Let* $\delta(a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r; n)$ denote the number of strict shifted r-rowed plane partitions of n with a_i the largest part in the i-th row and λ_i the number of parts in the i-th row. Then for $a_1 > a_2 > \ldots > a_r > 0$, $\lambda_1 > \lambda_2 > \ldots > \lambda_r > 0$,

$$\sum_{n\geqq 0} \delta(a_1, \ldots, a_r; \lambda_1, \ldots, \lambda_r; n)\, q^n = q^{a_1+\ldots+a_r+\lambda_1+\ldots+\lambda_r-r} \det\left(\begin{bmatrix} a_j+\lambda_i-2 \\ \lambda_i-1 \end{bmatrix}\right), \quad (1.9)$$

where $\begin{bmatrix} A \\ B \end{bmatrix}$ is the Gaussian polynomial defined by

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} \dfrac{(1-q^A)(1-q^{A-1})\ldots(1-q^{A-B+1})}{(1-q^B)(1-q^{B-1})\ldots(1-q)}, & B\geqq 0 \\ 0, & B<0. \end{cases} \quad (1.10)$$

Theorem 3. *For* $m\geqq 2$

$$\sum_{n\geqq 0} De(m, n)\, q^n = \det\left(\delta_{ij} + q^{j+2}\begin{bmatrix} i+j+2 \\ i \end{bmatrix}\right)_{0\leqq i,\, j\leqq m-2}, \quad (1.11)$$

where $\delta_{ij}=0$ *if* $i\neq j$ *and* $\delta_{ii}=1$.

Theorem 4

$$\sum_{n\geqq 0} M(m, n)\, q^n = \det\left(\delta_{ij} + q^{3j+1}\begin{bmatrix} i+j \\ i \end{bmatrix}_{q^3}\right)_{0\leqq i,\, j\leqq m-1} \quad (1.12)$$

where $\begin{bmatrix} A \\ B \end{bmatrix}_{q^3}$ *is the Gaussian polynomial with* q *replaced by* q^3.

Obviously then the Macdonald conjecture is reduced to evaluating the determinant in Theorem 4, while the descending plane partitions conjecture is reduced to evaluating the determinant in Theorem 3. Unfortunately we have not been able to evaluate these determinants except when $q=1$. In Sect. 3 we prove a hypergeometric series identity utilizing the work of Whipple [16, 17]. Then in Sect. 4, we prove the following result on determinants of binomial coefficients:

Theorem 8

$$\det\left(\delta_{ij} + \binom{\mu+i+j}{i}\right)_{0\leqq i,\, j\leqq m-1} = \prod_{j=0}^{m-1} \Delta_j(\mu), \quad (1.13)$$

where

$$\Delta_{2j}(\mu) = \begin{cases} \dfrac{(\mu+2j+2)_j \left(\dfrac{\mu}{2}+2j+\dfrac{3}{2}\right)_{j-1}}{(j)_j \left(\dfrac{\mu}{2}+j+\dfrac{3}{2}\right)_{j-1}}, & j>0 \\ 2, & j=0 \end{cases} \quad (1.14)$$

and

$$\Delta_{2j-1}(\mu) = \frac{(\mu+2j)_{j-1} \left(\dfrac{\mu}{2}+2j+\dfrac{1}{2}\right)_j}{(j)_j \left(\dfrac{\mu}{2}+j+\dfrac{1}{2}\right)_{j-1}}, \quad j>0, \quad (1.15)$$

with $(A)_j = A(A+1)\ldots(A+j-1)$.

It is then a straight forward exercise to show that the weak Macdonald conjecture (Theorem 9) follows from the case $\mu=0$ of Theorem 8, while the weak descending plane partitions conjecture follows from the case $\mu=2$.

One might expect that the proof of Theorem 8 would be easy especially in light of the fact that if the δ_{ij} is removed then it is easy to prove that

$$\det\left(\binom{\mu+i+j}{i}\right)_{0\leqq i,\, j\leqq m-1}=1, \qquad [13;\text{p. }682].$$

However, it will become abundantly obvious in Sects. 3 and 4 that we have not found any simple proof of Theorem 8.

We conclude with an examination of the problems involved in proving the two main conjectures.

Since the results of this paper were first announced at Alfred Young Day in Waterloo on June 2, 1978, it is certainly fitting to include him in the dedication. Since the crucial results for the determinant evaluations (Sect. 3) rely heavily on the little known but highly significant work of Whipple [16, 17], we have also included him in the dedication.

2. Shifted and Descending Plane Partitions

The cornerstone of all our work is Theorem 1 (stated in Sect. 1) which allows us to represent various generating functions as determinants. Our proof resembles that of Carlitz [8] for his derivation of (1.2).

Before we treat Theorem 1 we must make some conventions which vary from the ordinary theory of partitions. Namely, we shall admit into consideration two different partitions of zero: (i) the "empty" partition of zero (i.e. that partition that has no parts) will be said to have no parts and largest part equal to zero; (ii) the "non-empty" partition of zero is that partition which has one part and that part is zero. These rather strange conventions greatly simplify the recurrences that we must treat.

Proof of Theorem 1. When $r=1$, $\delta(a_1;\lambda_1;n)$ is the number of partitions of n into exactly λ_1 parts with largest part equal to a_1. Hence by Theorem 3.1 of [3; p. 33] we find

$$\begin{aligned}
&\sum_{n\geqq 0}\delta(a_1;\lambda_1;n)\,q^n\\
&\quad=\begin{cases} q^{a_1+\lambda_1-1}\begin{bmatrix}(a_1-1)+(\lambda_1-1)\\ \lambda_1-1\end{bmatrix}, & \text{if } a_1\geqq 0,\ \lambda_1\geqq 0,\ \textit{not both}=0\\ 1, & a_1=\lambda_1=0.\end{cases}\\
&\quad=\begin{cases} q^{a_1+\lambda_1-1}\begin{bmatrix}a_1+\lambda_1-2\\ \lambda_1-1\end{bmatrix}, & \text{if } a_1\geqq 0,\ \lambda_1\geqq 0,\ \text{not both}=0\\ 1, & \text{if } a_1=\lambda_1=0.\end{cases}
\end{aligned} \tag{2.1}$$

We next define

$$G(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r;q)=\sum_{n\geqq 0}\delta(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r;n)\,q^n. \tag{2.2}$$

Instead of proving (1.9) for $a_1>a_2>\ldots>a_r\geqq 1$, $\lambda_1>\ldots>\lambda_r\geqq 1$, we prove more generally that for $a_1>\ldots>a_r\geqq 0$, $\lambda_1>\ldots>\lambda_r\geqq 0$,

$$G(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r;q)=\det(G(a_i;\lambda_j;q))_{1\leqq i,j\leqq r}. \tag{2.3}$$

We proceed by a double induction first on r then on λ_r. If $r=1$, (2.3) is trivial and (2.1) provides the Gaussian polynomial representation of $G(a_i;\lambda_1;q)$.

Assume that (2.3) is valid up to but not including a fixed r. We now start a second induction on λ_r. If $\lambda_r=0$, we see that

$$G(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_{r-1},0;q) \\ =\begin{cases}G(a_1,\ldots,a_{r-1};\lambda_1,\ldots,\lambda_{r-1};q), & \text{if } a_r=0\\ 0, & \text{if } a_r>0.\end{cases} \tag{2.4}$$

However expanding the determinant in (2.3) along the right hand column we see that with $\lambda_r=0$:

$$\det(G(a_i;\lambda_j;q))_{1\leqq i,j\leqq r}=G(a_r;0;q)\det(G(a_i;\lambda_j;q))_{1\leqq i,j\leqq r}, \tag{2.5}$$

and from (2.1) we see that (2.5) is the same assertion for the right side of (2.3) as (2.4) is for the left side. Consequently by the induction hypothesis on r, (2.3) holds for $\lambda_r=0$. We now assume $\lambda_r>0$; consequently a_r must be >0. Hence for $\lambda_j\geqq 0$, we have (assuming $a_{r+1}=-1$)

$$\sum_{b_i=a_{i+1}+1}^{a_i} G(b_i;\lambda_j-1;q)=q^{-a_i}G(a_i;\lambda_j;q)-q^{-a_{i+1}}G(a_{i+1};\lambda_j;q); \tag{2.6}$$

this identity is the assertion for $\lambda_j>1$ that (by [3; p. 37, Eq. (3.3.9))

$$\sum_{b_i=a_{i+1}+1}^{a_i} q^{b_i+\lambda_j-2}\begin{bmatrix}b_i+\lambda_j-3\\ \lambda_j-2\end{bmatrix}=q^{\lambda_j-1}\left(\begin{bmatrix}a_i+\lambda_j-2\\ \lambda_{j+1}-1\end{bmatrix}-\begin{bmatrix}a_{i+1}+\lambda_j-2\\ \lambda_j-1\end{bmatrix}\right), \tag{2.7}$$

and for $\lambda_j=1$ (and therefore $j=r$) that $0=q^{-a_i}\cdot q^{a_i}-q^{-a_{i+1}}\cdot q^{a_{i+1}}$ if $i<r$, and $1=q^{-a_r}q^{a_r}$.

By directing attention to what remains if we remove each of the first parts in each row of a strict shifted plane partition, we see that (where $a_{r+1}=-1$)

$$G(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r;q) \\ =q^{a_1+\cdots+a_r}\sum_{\substack{a_1\geqq b_1>a_2\\ a_2\geqq b_2>a_3\\ \vdots\\ a_r\geqq b_r>a_r+1}} G(b_1,\ldots,b_r;\lambda_1-1,\ldots,\lambda_r-1;q). \tag{2.8}$$

However it is true that the right side of (2.3) also satisfies this recurrence, a fact which may be seen as follows:

$$q^{a_1+\dots+a_r} \sum_{\substack{a_1 \geqq b_1 > a_2 \\ a_2 \geqq b_2 > a_3 \\ \vdots \\ a_r \geqq b_r > a_{r+1}}} \det(G(b_i; \lambda_j - 1; q))$$

$$= q^{a_1+\dots+a_r} \det(q^{-a_i} G(a_i; \lambda_j; q) - q^{-a_{i+1}} G(a_{i+1}; \lambda_j; q))$$

(applying (2.6) to each row)

$$= q^{a_1+\dots+a_r} \det(q^{-a_i} G(a_i; \lambda_j; q))$$

(noting $G(a_{r+1}; \lambda_j; q) = 0$ since $\lambda_j \geqq \lambda_r > 0$, and starting from the bottom adding each row to the one above it)

$$= \det(G(a_i; \lambda_j; q)). \tag{2.9}$$

Therefore by the fact that each side of (2.3) satisfies the recurrence (2.8) we see that therefore (2.3) holds for every λ_r by the induction on λ_r and therefore for every r by the induction on r. □

The following theorem provides a further application of Carlitz's technique of [8]. It allows us to treat several families of plane partition problems.

Theorem 2. *For $d \leqq 2$, and fixed $a_1 > a_2 > \dots > a_r \geqq 0$*

$$\sum_{a_1+d > \lambda_1 \geqq a_2+d > \dots \geqq a_r+d > \lambda_r \geqq 1} \det(G(a_i; \lambda_j; q)) = q^{a_1+\dots+a_r} \det\left(\begin{bmatrix} a_i + a_j - 2 + d \\ a_j \end{bmatrix}\right) \tag{2.10}$$

where the λ_i are the summation indices.

Proof. In parallel with (2.6), we now require (assuming $a_{r+1} = -d+1$)

$$\sum_{\lambda_j = a_{j+1}+d}^{a_j+d-1} G(a_i; \lambda_j; q) = q^{a_i}\left(\begin{bmatrix} a_i + a_j + d - 2 \\ a_i \end{bmatrix} - \begin{bmatrix} a_i + a_{j+1} + d - 2 \\ a_i \end{bmatrix}\right); \tag{2.11}$$

this result is equivalent to

$$\sum_{\lambda_j = a_{j+1}+d}^{a_j+d-1} q^{a_i+\lambda_j-1} \begin{bmatrix} a_i + \lambda_j - 2 \\ \lambda_j - 1 \end{bmatrix} = q^{a_i}\left(\begin{bmatrix} a_i + a_j + d - 2 \\ a_i \end{bmatrix} - \begin{bmatrix} a_i + a_{j+1} + d - 2 \\ i \end{bmatrix}\right). \tag{2.12}$$

Thus applying (2.11) to each column independently in (2.10), we find that the left side of (2.10) equals

$$q^{a_1+\dots+a_r} \det\left(\begin{bmatrix} a_i + a_j - 2 + d \\ a_i \end{bmatrix} - \begin{bmatrix} a_i + a_{j+1} - 2 + d \\ a_i \end{bmatrix}\right)$$

$$= q^{a_1+\dots+a_r} \det\left(\begin{bmatrix} a_i + a_j - 2 + d \\ a_j \end{bmatrix}\right),$$

since $\begin{bmatrix} a_i + a_{r+1} - 2 + d \\ a_i \end{bmatrix} = \begin{bmatrix} a_i - 1 \\ a_i \end{bmatrix} = 0$ the second determinant is obtained from the first by adding each column to the one on its left starting from the right. □

We are now prepared to prove a theorem more general than Theorem 3 (stated in the Introduction). We let $De(d; m, n)$ denote the number of strict shifted plane partitions of n with largest part $\leqq m+1-d$ such that the largest part on each row is $\leqq$ the number of parts on the previous row less d at the same time being larger than the number of parts on the same row less d. Note that $De(0; m, n) = De(m+1, n)$.

Theorem 3′. *For $d \leqq 2$ and integral,*

$$\sum_{n \geqq 0} De(d; m, n)\, q^n = \det\left(\delta_{ij} + q^{j-d+2} \begin{bmatrix} i+j+2-d \\ i \end{bmatrix}\right)_{0 \leqq i, j \leqq m-1}. \tag{2.13}$$

Proof. From (2.3) and Theorem 2, we see that

$$\sum_{n \geqq 0} De(d; m, n)\, q^n = \sum_{\substack{1+m-d \geqq a_1 > a_2 > \ldots > a_r \geqq 2-d \\ r \text{ arbitrary}}} \det\left(q^{a_j} \begin{bmatrix} a_i + a_j - 2 + d \\ a_j \end{bmatrix}\right).$$

But this is merely the formula for the expansion along the main diagonal of

$$\det\left(\delta_{ij} + q^j \begin{bmatrix} i+j-2+d \\ j \end{bmatrix}\right)_{2-d \leqq i, j \leqq 1+m-d}$$
$$= \det\left(\delta_{ij} + q^{j-d+2} \begin{bmatrix} i+j-d+2 \\ i \end{bmatrix}\right)_{0 \leqq i, j \leqq m-1}.$$

We next utilize Theorem 2 to prove Theorem 4 (stated in the Introduction).

Proof of Theorem 4. We begin with an arbitrary plane partition π that is invariant under G_3. We form a new semi-graphical representation for π as follows:

(1) Associate a 1 with each point (i, i, i) in the positive 1st octant that lies in $D(\pi)$.

(2) Associate a 3 with each point (i, j, k) in the positive first octant that lies in $D(\pi)$ for which $k < \min(j, i)$ or $k = j < i$.

This process clearly associates with each π invariant under G_3 a set of two dimensional arrays (for each $k \geqq 1$). For example the arrays

$k=1$						$k=2$	
1	3	3	3	3		1	3
	3	3	3				
	3	3					

correspond to

5	4	3	1	1
3	3	2	1	
3	2	1		
2				
1				.

Notice that the 1's and 3's are associated with corresponding representatives of orbits of size 1 and 3 respectively. Consequently the sum total of the 1's and 3's in all the layers equals $|\pi|$. Now we add up the entries in each row of each layer and write the results as a shifted plane partition. Thus the above example yields

$$\begin{matrix} 13 & 9 & 6 \\ & 4 & \end{matrix} \;.$$

note that the first entries in each row are of the form $3a_{ii}+1=3a_i+1$, while the remaining entries are of the form $3a_{ij}$. Furthermore in order that the shifted plane partition π' come from a plane partition π invariant under G_3 it is clearly necessary and sufficient that

$$a_1>\lambda_1-2\geqq a_2>\lambda_2-2\geqq\ldots\geqq a_r>\lambda_r-2\geqq -1,$$

where λ_i is the number of parts on the i-th row of π'. Note that $a_r=0$ is permissable since it corresponds to the r-th layer having the single entry 1. Applying (2.3) and Theorem 2, we thus see that

$$\begin{aligned}
&\sum_{n\geqq 0} M(m,n)\,q^n \\
&\quad=\sum_{m+1\geqq a_1+2>\lambda_1\geqq a_2+2>\ldots>a_r+2>\lambda_r\geqq 1} q^r\, G(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r;q^3)\\
&\quad=\sum_{m+1\geqq a_1+2>\lambda_1\geqq a_2+2>\ldots\geqq a_r+2>\lambda_r\geqq 1} q^r \det(G(a_i;\lambda_j;q^3))\\
&\quad=\sum_{m-1\geqq a_1>a_2>\ldots>a_r\geqq 0} \det\left(q^{3a_i+1}\begin{bmatrix}a_i+a_j\\ a_j\end{bmatrix}_{q^3}\right)\\
&\quad=\det\left(\delta_{ij}+q^{3j+1}\begin{bmatrix}i+j\\ i\end{bmatrix}_{q^3}\right)_{0\leqq i,\,j\leqq m-1}.
\end{aligned}$$

3. Hypergeometric Series

In order to evaluate the determinant in Theorem 8, we require the following hypergeometric series identity which follows from some fundamental transformations due to F.J.W. Whipple [16, 17].

Theorem 5. *Let*

$$M(i,j,m;a,\omega)=\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}} \times\sum_{s\geqq 0}\binom{m+i+j+a-s-1}{i-j+a+s-1}\frac{(j-s)_{2s}(-m-3j-a-\omega+1)_s\,4^{-s}}{s!\left(-\frac{m}{2}-2j-a-\omega+2\right)_s\left(-j-\frac{m}{2}+\frac{1}{2}\right)_s}, \tag{3.1}$$

and

$$A(i,j,m;a,\omega)=\frac{(2j-a-i)!}{(2j+\omega-i-1)!}(-1)^{i+a}\sum_{s\geqq 0}\binom{j-s-1}{i-j+a+s-1}$$
$$\times\frac{(j-s)_{2s+a+\omega-1}}{s!\,(m+4j-2)(m+4j-4)\dots(m+4j-2s)}$$
$$\times\frac{(-m-3j-a-\omega+1)_s}{(m+2j-1)(m+2j-3)\dots(m+2j-2s+1)}. \tag{3.2}$$

Then for a, ω, i and j integers with $a+\omega\geqq 1$, $2j-a\geqq i$,

$$M(i,j,m;\,a,\omega)=A(i,j,m;\,a,\omega). \tag{3.3}$$

Remark. For a restatement of Theorem 5 as a purely hypergeometric identitiy see Sect. 5, Eq. (5.1).

Proof. We shall require formulas involving the generalized hypergeometric function:

$${}_{s+1}F_s\left[\begin{matrix}a_0, a_1, \dots, a_s;\ t\\ b_1, \dots, b_s\end{matrix}\right]=\sum_{n\geqq 0}\frac{(a_0)_n(a_1)_n\dots(a_s)_n\,t^n}{n!\,(b_1)_n\dots(b_s)_n}, \tag{3.4}$$

where

$$(A)_n=A(A+1)\dots(A+n-1)=\frac{\Gamma(A+n)}{\Gamma(A)},$$

[7; p. 8].

The two formulas we need are as follows (M and N nonnegative integers):

$${}_4F_3\left[\begin{matrix}A, B, C, -M;\ 1\\ R, S, A+B+C-R-S+1-M\end{matrix}\right]$$
$$=\frac{(R+S-A-B)_M(S-C)_M}{(R+S-A-B-C)_M(S)_M}\times{}_4F_3\left[\begin{matrix}C, R-B, R-A, -M;\ 1\\ R, R+S-A-B,\ C-S+1-M\end{matrix}\right],$$
[16; Eq. (10.11)], (3.5)

$${}_4F_3\left[\begin{matrix}-N, B, C, D;\ 1\\ 1-N-B, 1-N-C, W\end{matrix}\right]$$
$$=\frac{(W-D)_N}{(W)_N}\,{}_5F_4\left[\begin{matrix}D, 1-N-B-C, -\frac{N}{2}, \frac{1}{2}-\frac{N}{2}, 1-N-W;\ 1\\ 1-N-B, 1-N-C, \frac{1}{2}(1+D-W-N), 1+\frac{1}{2}(D-W-N)\end{matrix}\right]$$
[17; Eq. (6.6)], [7; p. 33, Eq. (1)]. (3.6)

Identity (3.5) is Whipple's relation between two Saalschutzian or balanced ${}_4F_3$'s; the hypergeometric series given by (3.4) is called Saalschutzian or balanced if $1+a_0+a_1+\dots+a_s=b_1+b_2+\dots+b_s$. It should be noted in passing that (3.5) is a little known generalization of the celebrated Pfaff-Saalschutz summation [7; p. 9] which follows from (3.5) if we set $R=A$. Identity (3.6) is a transformation of a nearly poised ${}_4F_3$ series of the second kind into a balanced or Saalschutzian ${}_5F_4$; the hypergeometric series given by (3.4) is called "nearly poised of the second kind" if $1+a_0=a_1+b_1=a_2+b_2=\ldots=a_{s-1}+b_{s-1}$.

$$M(i,j,m;\,a,\omega)=\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}}$$

$$\times\sum_{s\geqq 0}\binom{m+i+j+a-s-1}{i-j+a+s-1}\frac{(j-s)_{2s}(-m-3j-a-\omega+1)_s\cdot 4^{-s}}{s!\left(-\frac{m}{2}-2j-a-\omega+2\right)_s\left(-j-\frac{m}{2}+\frac{1}{2}\right)_s}$$

$$=\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}}\sum_{s\geqq 0}\frac{(m+i+j+a-1)!\,(j-s)_{2s}(-m-3j-a-\omega+1)_s\,2^{-s}}{(i-j+a-1)!\,(m+2j-2s)!\left(-\frac{m}{2}-2j-a-\omega+2\right)_s}$$

$$\times\frac{1}{(-m-i-j-a+1)_s(i-j+a)_s(m+2j-1)(m+2j-3)\ldots(m+2j-2s+1)}$$

$$=\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}}\binom{m+i+j+a-1}{i-j+a-1}$$

$$\times{}_4F_3\left[\begin{matrix}-j+1,j,\,-m-3j-a-\omega+1,\,-\frac{m}{2}-j;\,1\\ i-j+a,\,-\frac{m}{2}-2j-a-\omega+2,\,-m-i-j-a+1\end{matrix}\right]$$

$$=\binom{m+i+j+a-1}{i-j+a-1}\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}}$$

$$\times\frac{\left(i-j+\frac{m}{2}+a+1\right)_{j-1}(-j-a-\omega+2)_{j-1}}{(i+m+a+1)_{j-1}\left(-\frac{m}{2}-2j-a-\omega+2\right)_{j-1}}$$

$$\times{}_4F_3\left[\begin{matrix}i-2j+a,\,-j+1,\,-j-\frac{m}{2},\,m+i+2j+2a+\omega-1;\,1\\ i-j+a,\,i-j+\frac{m}{a}+a+1,\,a+\omega\end{matrix}\right]$$

(by (3.5) with $A=j$, $B=-m-3j-a-\omega+1$,

$$C=-j-\frac{m}{2}, R=i-j+a, S=-\frac{m}{2}-2j-a-\omega+2\Big)$$

$$=\binom{m+i+j+a-1}{i-j+a-1}\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}}$$

$$\times\frac{\left(i-j+\frac{m}{2}+a+1\right)_{j-1}(-j-a-\omega+2)_{j-1}}{(i+m+a+1)_{j-1}\left(-\frac{m}{2}-2j-a-\omega+2\right)_{j-1}}\frac{(-m-i-2j-a+1)_{2j-i-a}}{(a+\omega)_{2j-i-a}}$$

$$\times{}_4F_3\left[\begin{matrix}m+i+2j+2a+\omega-1,\ -j+\frac{i+a}{2},\ -j+\frac{i+a+1}{2},\ 1+i-2j-\omega;\ 1\\ i-j+a,\ i-j+\frac{m}{2}+a+1,\ 1+\frac{1}{2}(m+2i+2a-1)\end{matrix}\right]$$

$\Big($by (3.6) with $N=2j-i-a$, $B=-j+1$, $C=-j-\frac{m}{2}$,

$D=m+i+2j+2a+\omega-1$, $W=a+\omega\Big)$

$$=\binom{m+i+j+a-1}{i-j+a-1}\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}\left(i-j+\frac{m}{2}+a+1\right)_{j-1}(-j-a-\omega+2)_{j-1}}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}(i+m+a+1)_{j-1}\left(-\frac{m}{2}-2j-a-\omega+2\right)_{j-1}}$$

$$\times\frac{(-m-i-2j-a+1)_{2j-i-a}\,\Gamma\left(-\frac{m}{2}-j-\frac{i+a}{2}+\frac{1}{2}\right)\Gamma\left(i-j+\frac{m}{2}+a+1\right)}{(a+\omega)_{2j-i-a}\,\Gamma\left(\frac{i+a+1+m}{2}\right)\Gamma\left(\frac{i+a+2+m}{2}\right)}$$

$$\times\frac{\Gamma\left(-\frac{m}{2}-j-\frac{i+a}{2}+1\right)\Gamma\left(\frac{m}{2}+j+\frac{1}{2}\right)}{\Gamma\left(1-\frac{m}{2}-2j\right)\Gamma\left(-\frac{m}{2}-i-a+\frac{1}{2}\right)}$$

$$\times{}_4F_3\left[\begin{matrix}-j+\frac{i+a}{2},\ -j+\frac{i+a+1}{2}, j+a+\omega-1,\ -m-3j-a-\omega+1;1\\ i-j+a,\ -\frac{m}{2}-2j+1,\ -\frac{m}{2}-j+\frac{1}{2}.\end{matrix}\right]$$

$\Big($by (3.5) once we have noted that it is symmetric in C and $-M$ so that we need only require one of them to be a nonpositive integer; we therefore may take A

$= 1+i-2j-\omega,\ B = m+i+2j+2a+\omega-1,\ C = -j+\frac{i+a+1}{2},\ -M = -j+\frac{i+a}{2},$
$R = i-j+a,\ S = i-j+\frac{m}{2}+a+1\Big).$

The $_4F_3$ series that we now have is basically what we want for $\Lambda(i,j,m;a,\omega)$. First we must simplify the massive expression in front of the $_4F_3$. In our treatment we shall assume that m is not an integer; the case when m is an integer (i.e. the interesting case for us) will then be obtained by continuity. We shall utilize (3.5) as well as Legendre's duplication formula [18; p. 240]:

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2}), \tag{3.7}$$

and the functional equation [18; p. 239]:

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{3.8}$$

$$\binom{m+i+j+a-1}{i-j+a-1}\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}\left(i-j+\frac{m}{2}+a+1\right)_{j-1}(-j-a-\omega+2)_{j-1}}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}(i+m+a+1)_{j-1}\left(-\frac{m}{2}-2j-a-\omega+2\right)_{j-1}}$$
$$\times\frac{(-m-i-2j-a+1)_{2j-i-a}\,\Gamma\left(-\frac{m}{2}-j-\frac{i+a}{2}+\frac{1}{2}\right)\Gamma\left(i-j+\frac{m}{2}+a+1\right)}{(a+\omega)_{2j-i-a}\,\Gamma\left(\frac{i+a+1+m}{2}\right)\Gamma\left(\frac{i+a+2+m}{2}\right)}$$
$$\times\frac{\Gamma\left(-\frac{m}{2}-j-\frac{i+a}{2}+1\right)\Gamma\left(\frac{m}{2}+j+\frac{1}{2}\right)}{\Gamma\left(1-\frac{m}{2}-2j\right)\Gamma\left(-\frac{m}{2}-i-a+\frac{1}{2}\right)}$$
$$=\binom{m+i+j+a-1}{i-j+a-1}\frac{\left(i-j+\frac{m}{2}+a+1\right)_{j-1}(a+\omega)_{j-1}}{(i+m+a+1)_{j-1}\left(\frac{m}{2}+j+a+\omega\right)_{j-1}}$$
$$\times\frac{(-1)^{i+a}(m+2i+2a)_{2j-i-a}(a+\omega-1)!}{(2j-i+\omega-1)!}$$
$$\times\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}\Gamma\left(-\frac{m}{2}-j-\frac{i+a}{2}+\frac{1}{2}\right)}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}\Gamma\left(\frac{i+a+1+m}{2}\right)\Gamma\left(\frac{i+a+2+m}{2}\right)}$$

$$\times \frac{\Gamma\left(i-j+\frac{m}{2}+a+1\right)\Gamma\left(-\frac{m}{2}-j-\frac{i+a}{2}+1\right)\Gamma\left(\frac{m}{2}+j+\frac{1}{2}\right)}{\Gamma\left(1-\frac{m}{2}-2j\right)\Gamma\left(-\frac{m}{2}-i-a+\frac{1}{2}\right)}$$

$$=\frac{(-1)^{i+a}\Gamma(m+i+a+1)\left(i-j+\frac{m}{2}+a+1\right)_{j-1}}{(i-j+a-1)!\,\Gamma(m+2j+1)\left(\frac{m}{2}+j+a+\omega\right)_{j-1}}$$

$$\times\frac{(a+\omega+j-2)!\,(m+2i+2a)_{2j-i-a}}{(2j-i+\omega-1)!}$$

$$\times\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}\Gamma\left(-\frac{m}{2}-j-\frac{i+a}{2}+\frac{1}{2}\right)\Gamma\left(i-j+\frac{m}{2}+a+1\right)}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}\Gamma\left(\frac{i+a+1+m}{2}\right)\Gamma\left(\frac{i+a+2+m}{2}\right)}$$

$$\times\frac{\Gamma\left(-\frac{m}{2}-j-\frac{i+a}{2}+1\right)\Gamma\left(\frac{m}{2}+j+\frac{1}{2}\right)}{\Gamma\left(1-\frac{m}{2}-2j\right)\Gamma\left(-\frac{m}{2}-i-a+\frac{1}{2}\right)}$$

$$=(-1)^{i+a}\binom{j+a+\omega-2}{i-j+a-1}\frac{\Gamma(i+a+m+1)}{\Gamma(m+2j+1)}$$

$$\times\frac{\left(i-j+\frac{m}{2}+a+1\right)_{j-1}(m+2i+2a)_{2j-i-a}}{\left(\frac{m}{2}+j+a+\omega\right)_{j-1}}$$

$$\times\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}\Gamma\left(-\frac{m}{2}-j-\frac{i+a}{2}+\frac{1}{2}\right)\Gamma\left(i-j+\frac{m}{2}+a+1\right)}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}\Gamma\left(\frac{i+a+1+m}{2}\right)\Gamma\left(\frac{i+a+2+m}{2}\right)}$$

$$\times\frac{\Gamma\left(-\frac{m}{2}-j-\frac{i+a}{2}+1\right)\Gamma\left(\frac{m}{2}+j+\frac{1}{2}\right)}{\Gamma\left(1-\frac{m}{2}-2j\right)\Gamma\left(-\frac{m}{2}-i-a+\frac{1}{2}\right)}$$

$$=(-1)^{i+a}\binom{j+a+\omega-2}{i-j+a-1}\frac{\Gamma(m+i+a+1)\,\Gamma\left(i+\frac{m}{2}+a\right)\Gamma\left(\frac{m}{2}+j+a+\omega\right)}{\Gamma(m+2j+1)\,\Gamma\left(\frac{m}{2}+2j+a+\omega-1\right)\Gamma(m+2i+2a)}$$

$$\times\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}\Gamma(m+2j+i+a)\,\Gamma\left(-\frac{m}{2}-j-\frac{i+a}{2}+\frac{1}{2}\right)}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}\Gamma\left(\frac{i+a+1+m}{2}\right)\Gamma\left(\frac{i+a+2+m}{2}\right)}$$

$$\times \frac{\Gamma\left(-\frac{m}{2}-j-\frac{i+a}{2}+1\right)\Gamma\left(\frac{m}{2}+j+\frac{1}{2}\right)}{\Gamma\left(1-\frac{m}{2}-2j\right)\Gamma\left(-\frac{m}{2}-i-a+\frac{1}{2}\right)}$$

$$=(-1)^{i+a}\binom{j+a+\omega-2}{i-j+a-1}2^{2(i+a+m+j)-1}\Gamma(-m-2j-i-a+1)$$

$$\times \frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}\Gamma\left(i+\frac{m}{2}+a\right)\Gamma\left(\frac{m}{2}+j+a+\omega\right)}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}\Gamma(m+2j+1)\Gamma\left(\frac{m}{2}+2j+a+\omega-1\right)}$$

$$\times \frac{\Gamma(m+2j+i+a)\Gamma\left(\frac{m}{2}+j+\frac{1}{2}\right)}{\Gamma(m+2i+a)\Gamma\left(1-\frac{m}{2}-2j\right)\Gamma\left(-\frac{m}{2}-i-a+\frac{1}{2}\right)}$$

$$=(-1)^{i+a}\binom{j+a+\omega-2}{i-j+a-1}\frac{2\pi\sin\pi\left(i+\frac{m}{2}+a+\frac{1}{2}\right)}{\sin\pi(m+2j+i+a)}$$

$$\times \frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}\Gamma\left(\frac{m}{2}+j+a+\omega\right)}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}\Gamma\left(\frac{m}{2}+j+1\right)\Gamma\left(\frac{m}{2}+2j+a+\omega-1\right)\Gamma\left(1-\frac{m}{2}-2j\right)}$$

$$=(-1)^{i+a}\binom{j+a+\omega-2}{i-j+a-1}\frac{2\sin\pi\left(i+\frac{m}{2}+a+\frac{1}{2}\right)\sin\frac{\pi m}{2}}{\sin\pi(m+i+a)}$$

$$=(-1)^{i+a}\binom{j+a+\omega-2}{i-j+a-1}\frac{2(-1)^{i+a}\cos\frac{\pi m}{2}\sin\frac{\pi m}{2}}{(-1)^{i+a}\sin\pi m}$$

$$=(-1)^{i+a}\binom{j+a+\omega-2}{i-j+a-1}\qquad(\text{since }\sin 2\theta=2\sin\theta\cos\theta).$$

Hence we have

$$M(i,j,m;\,a,\omega)=(-1)^{i+a}\binom{j+a+\omega-2}{i-j+a-1}$$

$$\times {}_4F_3\left[\begin{matrix}-j+\frac{i+a}{2},\ -j+\frac{i+a+1}{2},\ j+a+\omega-1,\ -m-3j-a-\omega+1;\ 1\\ i-j+a,\ -\frac{m}{2}-2j+1,\ -\frac{m}{2}-j+\frac{1}{2}\end{matrix}\right]$$

$$=(-1)^{i+a}\sum_{s\geq 0}\frac{(j+a+\omega+s-2)!\,(-2j+i+a)_{2s}}{s!\,(i-j+a+s-1)!\,(2j-i+\omega-1)!}$$

$$\times \frac{4^{-s}(-m-3j-a-\omega+1)_s}{\left(-\frac{m}{2}-2j+1\right)_s\left(-\frac{m}{2}-j+\frac{1}{2}\right)_s}=\frac{(-1)^{i+a}(2j-i-a)!}{(2j-i+\omega-1)!}$$

$$\times \sum_{s\geqq 0}\binom{j-s-1}{i-j+a+s-1}\frac{4^{-s}(-m-3j-a-\omega+1)_s}{s!\left(-\frac{m}{2}-2j+1\right)_s\left(-\frac{m}{2}-j+\frac{1}{2}\right)_s}$$

$$\times\frac{(j+a+\omega+s-2)!}{(j-s-1)!}$$

$$=\frac{(-1)^{i+a}(2j-i-a)!}{(2j-i+\omega-1)!}\sum_{s\geqq 0}\binom{j-s-1}{i-j+a+s-1}\frac{(-m-3j-a-\omega+1)_s}{s!}$$

$$\times\frac{(j-s)_{2s+a+\omega-1}}{(m+4j-2)(m+4j-4)\dots(m+4j-2s)(m+2j-1)(m+2j-3)\dots(m+2j-2s+1)}$$

$$=\Lambda(i,j,m;\,a,\omega),\text{ as desired.}$$

In actual fact our above treatment would appear to be valid only provided $i-2j+a$ is a nonpositive integer and $i-j+a$ is any complex (or real) number other than a nonpositive integer. The result then follows for $i-j+a$ nonpositive integral by continuity. Also we had to assume m nonintegral to allow $\sin \pi m$ in a denominator; however again the excluded values on m are admissible by continuity. □

Corollary 5a

$$\Lambda(i,j,m;\,a,\omega)=\begin{cases}\binom{j+a+\omega-2}{j-1} & \text{for } i=2j-a\\ 0 & \text{for } i>2j-a.\end{cases}$$

Proof. In (3.2) we see that if $i=2j-a$ then only the first term does not vanish while if $i>2j-a$ then all terms vanish. □

Corollary 5b. *For j a positive integer,*

$$M(2j+\omega,j,m;\,a,\omega)$$
$$=\binom{m+2j+a+\omega-1}{a+\omega-1}\frac{(m+2j+a+\omega)_j\left(\frac{m}{2}+2j\right)_{a+\omega-1}\left(2j+\frac{m}{2}+a+\omega\right)_{j-1}}{(j+a+\omega-1)_j\left(\frac{m}{2}+j+1\right)_{a+\omega-1}\left(\frac{m}{2}+j+a+\omega\right)_{j-1}}.$$

Proof. After the first transformation applied to $M(i,j,m;\,a,\omega)$ in the proof of Theorem 5 we find that

$$M(2j+\omega,j,m;\,a,\omega)$$
$$=\binom{m+3j+a+\omega-1}{j+a+\omega-1}\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}\left(j+\frac{m}{2}+a+\omega+1\right)_{j-1}}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}(2j+m+a+\omega+1)_{j-1}}$$

$$\times\frac{(-j-a-\omega+2)_{j-1}}{\left(-\frac{m}{2}-2j-a-\omega+2\right)_{j-1}}\,{}_3F_2\left[\begin{matrix}-j+1,\ -j-\frac{m}{2},\ m+4j+2a+2\omega-1;\ 1\\ j+a+\omega,\ j+\frac{m}{2}+a+\omega+1\end{matrix}\right]$$

$$= \binom{m+3j+a+\omega-1}{j+a+\omega-1} \frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1} \left(j+\frac{m}{2}+a+\omega+1\right)_{j-1}}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1} (2j+m+a+\omega+1)_{j-1}}$$

$$\times \frac{(a+\omega)_{j-1} \left(2j+\frac{m}{2}+a+\omega\right)_{j-1} (m+2j+a+\omega+1)_{j-1}}{\left(\frac{m}{2}+j+a+\omega\right)_{j-1} (j+a+\omega)_{j-1} \left(\frac{m}{2}+j+a+\omega+1\right)_{j-1}} \quad \text{(by [7; p. 9])}$$

$$= \frac{(m+2j+a+\omega-1)!\,(m+2j+a+\omega)_j}{(a+\omega-1)!\,(j+a+\omega-1)(m+2j)!}$$

$$\times \frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1} \left(2j+\frac{m}{2}+a+\omega\right)_{j-1}}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1} \left(\frac{m}{2}+j+a+\omega\right)_{j-1} (j+a+\omega)_{j-1}}$$

$$= \binom{m+2j+a+\omega-1}{a+\omega-1} \frac{(m+2j+a+\omega)_j \left(\frac{m}{2}+2j\right)_{a+\omega-1} \left(2j+\frac{m}{2}+a+\omega\right)_{j-1}}{(j+a+\omega-1)_j \left(\frac{m}{2}+j+1\right)_{a+\omega-1} \left(\frac{m}{2}+j+a+\omega\right)_{j-1}},$$

as desired. □

While Theorem 5 and its corollaries play the central role in the next section, it turns out that the following relation among three balanced ${}_4F_3$'s is crucial in order to conclude the proof of Theorem 7.

Theorem 6. *For* $0 \leqq i \leqq 2j-2$ *with* $i-2j+2$ *integral,*

$${}_4F_3\left[\begin{matrix} i-2j+1,\ 1-j,\ -j-\frac{\mu}{2}+\frac{1}{2},\ i+\mu+2j;\ 1 \\ i-j+1,\ i-j+\frac{\mu}{2}+\frac{3}{2},\ 1 \end{matrix}\right] + \frac{(2i+\mu+1)\left(\frac{\mu}{2}+j-\frac{1}{2}\right) j(j-1)}{\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)(i-j+1)(i-j+2)}$$

$$\times {}_4F_3\left[\begin{matrix} i-2j+2,\ 2-j,\ -j-\frac{\mu}{2}+\frac{3}{2},\ i+\mu+2j+1;\ 1 \\ i-j+3,\ i-j+\frac{\mu}{2}+\frac{5}{2},\ 2 \end{matrix}\right]$$

$$+ \frac{\left(\frac{\mu}{2}+i+\frac{1}{2}\right)(i+1)(2j-i-2)j}{\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)(i-j+1)(i-j+2)}$$

$$\times {}_4F_3\left[\begin{matrix} i-2j+3,\ 1-j,\ -j-\frac{\mu}{2}+\frac{3}{2},\ i+\mu+2j+1;\ 1 \\ i-j+3,\ i-j+\frac{\mu}{2}+\frac{5}{2},\ 2 \end{matrix}\right] = 0. \tag{3.9}$$

Proof. If we denote the index of summation in each of the above ${}_4F_3$'s by s and if we shift s to $s-1$ in the second ${}_4F_3$, then we find that the expression L on the left of (3.9) is

$$L=\sum_{s\geqq 0}\frac{(i-2j+1)_s(1-j)_s\left(-j-\frac{\mu}{2}+\frac{1}{2}\right)_s(i+\mu+2j)_s}{s!\,(i-j+1)_s\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_s s!}$$

$$-(2i+\mu+1)\sum_{s\geqq 0}\frac{s\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_s(i+2j+\mu+1)_{s-1}(-j)_{s+1}(i-2j+2)_{s-1}}{s!\,(i-j+1)_{s+1}\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_s s!}$$

$$+\left(\frac{\mu}{2}+i+\frac{1}{2}\right)(i+1)\sum_{s\geqq 0}\frac{(i-2j+2)_{s+1}(-j)_{s+1}\left(-j-\frac{\mu}{2}+\frac{3}{2}\right)_s(i+\mu+2j+1)_s}{s!\,(i-j+1)_{s+2}\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{s+1}(s+1)!}.$$

Since $(2i+\mu+1)=(i+\mu+2j)+(i-2j+1)$, we combine the first sum with the $(i+\mu+2j)$ portion of the second sum:

$$\sum_{s\geqq 0}\frac{(i-2j+2)_{s-1}(1-j)_s\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_s(i+\mu+2j)_s}{s!\,(i-j+1)_{s+1}\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_s s!}$$
$$\times\{(i-2j+1)(i-j+s+1)-(-j)s\},$$

and since the expression inside { } is just $(s+i-2j+1)(i-j+1)$ we see that this last expression equals

$$\sum_{s\geqq 0}\frac{(i-2j+2)_s(1-j)_s\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_s(i+\mu+2j)_s}{s!\,(i-j+2)_s\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_s s!}.$$

Therefore the expression on the left of (3.9) may be written as follows:

$$L=\sum_{s\geqq 0}\frac{(i-2j+2)_s(1-j)_s\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_s(i+\mu+2j)_s}{s!\,(i-j+2)_s\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_s s!}$$

$$-\sum_{s\geqq 0}\frac{s\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_s(i+2j+\mu+1)_{s-1}(-j)_{s+1}(i-2j+1)_s}{s!\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_s(i-j+1)_{s+1}\,s!}$$

$$+\tfrac{1}{2}(i+1)(\mu+2i+1)$$

$$\times \sum_{s\geqq 0} \frac{(i-2j+2)_{s+1}(-j)_{s+1}\left(-j-\frac{\mu}{2}+\frac{3}{2}\right)_s (i+\mu+2j+1)_s}{s!\,(i-j+1)_{s+2}\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{s+1}(s+1)!}.$$

Let us now shift the index of summation in the second sum from s to $s+1$ and then combine it with the third sum. Hence

$$L=\sum_{s\geqq 0} \frac{(i-2j+2)_s(1-j)_s\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_s (i+\mu+2j)_s}{s!\,(i-j+2)_s\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_s s!}$$

$$+\sum_{s\geqq 0} \frac{\left(-j-\frac{\mu}{2}+\frac{3}{2}\right)_s (i+\mu+2j+1)_s(-j)_{s+1}(i-2j+2)_s}{s!\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{s+1}(i-j+1)_{s+2}(s+1)!}$$

$$\times\left[-\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)(-j+s+1)(i-2j+1)+(i+1)\right.$$

$$\left.\times\left(\frac{\mu}{2}+i+\frac{1}{2}\right)(i-2j+s+2)\right].$$

The expression inside "[ ]" factors into

$$(i+1-j)\left\{(1+s-j)(\mu+2j+i)+(i+1)\left(\frac{\mu}{2}+i+\frac{1}{2}\right)\right\}.$$

Hence we conclude that L (the expression on the left side of (3.9)) is given by

$$L=\sum_{s\geqq 0} \frac{(i-2j+2)_s(1-j)_s\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_s (i+\mu+2j)_s}{s!\,(i-j+2)_s\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_s s!}$$

$$+\sum_{s\geqq 0} \frac{(i-2j+2)_s(-j)_{s+1}\left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_s (i+\mu+2j+1)_s}{s!\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{s+1}(i-j+2)_{s+1}(s+1)!}$$

$$\times\left\{(1+s-j)(\mu+2j+i)+(i+1)\left(\frac{\mu}{2}+i+\frac{1}{2}\right)\right\}. \tag{3.10}$$

Equation (3.10) now provides the starting point for the final assault. We propose to prove that if for $0\leqq n\leqq 2j-2-i$

$$L_n = \sum_{s=2j-2-i-n}^{2j-i-2} \frac{(i-2j+2)_s(1-j)_s\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_s(i+\mu+2j)_s}{s!\,(i-j+2)_s\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_s s!}$$
$$+\sum_{s=2j-2-i-n}^{2j-i-2} \frac{(i-2j+2)_s(-j)_{s+1}\left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_s(i+\mu+2j+1)_s}{s!\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{s+1}(i-j+2)_{s+1}(s+1)!}$$
$$\times\left\{(1+s-j)(\mu+2j+i)+(i+1)\left(\frac{\mu}{2}+i+\frac{1}{2}\right)\right\},$$

then

$$L_n = \frac{-(-1)^i(i+\mu+2j+1)_{2j-i-3-n}(i-2j+2)_{n+1}}{n!\,(2j-i-2-n)!\,(i-j+2)_n}$$
$$\times\frac{(1-j)_n\left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_n\left(\frac{\mu}{2}+3j-n-\frac{3}{2}\right)}{\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{n+1}}. \tag{3.11}$$

If we can prove (3.11) for $0 \leqq n \leqq 2j-i-2$ this will imply the main result since $L = L_{2j-i-2}$, and by (3.11) we see that $L_{2j-i-2} = 0$.

We proceed to treat (3.11) by induction on n. For $n=0$,

$$L_0 = \frac{(i-2j+2)_{2j-i-2}(1-j)_{2j-i-2}\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_{2j-i-2}(i+\mu+2j)_{2j-i-2}}{(2j-i-2)!\,(i-j+2)_{2j-i-2}\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{2j-i-2}(2j-i-2)!}$$
$$+\frac{(i-2j+2)_{2j-i-2}(-j)_{2j-i-1}\left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_{2j-i-2}(i+\mu+2j+1)_{2j-i-2}}{(2j-i-2)!\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{2j-i-1}(i-j+2)_{2j-i-1}(2j-i-1)!}$$
$$\times\left\{(j-i-1)(\mu+2j+i)+(i+1)\left(\frac{\mu}{2}+i+\frac{1}{2}\right)\right\}$$
$$=\frac{(-1)^i\left(\frac{\mu}{2}+j-\frac{1}{2}\right)(i+\mu+2j)_{2j-i-2}}{(2j-i-2)!\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)}$$
$$-\frac{(-1)^i(i+\mu+2j+1)_{2j-i-2}}{(2j-i-1)!\left(\frac{\mu}{2}+j-\frac{1}{2}\right)}\left\{(\mu+2j-1)\left(j-\frac{i}{2}-\frac{1}{2}\right)\right\}$$

$$=\frac{-(-1)^i(i+\mu+2j+1)_{2j-i-3}}{(2j-i-2)!\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)}$$

$$\times\left[-(i+\mu+2j)\left(\frac{\mu}{2}+j-\frac{1}{2}\right)+(\mu+4j-2)\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)\right]$$

$$=\frac{-(-1)^i(i+\mu+2j+1)_{2j-i-3}(i-2j+2)\left(\frac{\mu}{2}+3j-\frac{3}{2}\right)}{(2j-i-2)!\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)},$$

which is (3.11) in the case $n=0$.

We now assume that (3.11) is proved for all nonnegative integers up to but not including a specific n in the interval $(0, 2j-i-2]$. Hence by the induction hypothesis

$$L_n=\frac{(i-2j+2)_{2j-2-i-n}(1-j)_{2j-2-i-n}}{(2j-2-i-n)!\,(i-j+2)_{2j-2-i-n}}$$

$$\times\frac{\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_{2j-2-i-n}(1+\mu+2j)_{2j-2-i-n}}{\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{2j-2-i-n}(2j-2-i-n)!}$$

$$+\frac{(i-2j+2)_{2j-2-i-n}(-j)_{2j-1-i-n}}{(2j-2-i-n)!\,(i-j+2)_{2j-1-i-n}}$$

$$\times\frac{\left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_{2j-2-i-n}(i+\mu+2j+1)_{2j-2-i-n}}{\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{2j-1-i-n}(2j-i-n-1)!}$$

$$\times\left\{(j-i-n-1)(\mu+2j+i)+(i+1)\left(\frac{\mu}{2}+i+\frac{1}{2}\right)\right\}$$

$$-\frac{(-1)^i(i+\mu+2j+1)_{2j-i-2-n}(i-2j+2)_n}{(n-1)!\,(2j-i-1-n)!\,(i-j+2)_{n-1}}$$

$$\times\frac{(1-j)_{n-1}\left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_{n-1}\left(\frac{\mu}{2}+3j-n-\frac{1}{2}\right)}{\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_n}$$

$$=\frac{-(-1)^i(i+\mu+2j)_{2j-2-i-n}(1-j)_n\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_{n+1}(i-2j+2)_n}{n!\,(2j-2-i-n)!\,(i-j+2)_n\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{n+1}}$$

$$+\frac{(-1)^i(i+\mu+2j+1)_{2j-2-i-n}(-j)_n\left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_{n-1}(i-2j+2)_n}{n!\,(2j-i-n-1)!\,(i-j+2)_n\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_n}$$

$$\times\left\{(j-i-n-1)(\mu+2j+i)+(i+1)\left(\frac{\mu}{2}+i+\frac{1}{2}\right)\right\}$$

$$-\frac{(-1)^i(i+\mu+2j+1)_{2j-i-2-n}(i-2j+2)_n}{(n-1)!\,(2j-1-i-n)(i-j+2)_{n-1}}$$

$$\times\frac{(1-j)_{n-1}\left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_{n-1}\left(\frac{\mu}{2}+3j-n-\frac{1}{2}\right)}{\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_n}$$

$$=\frac{-(-1)^i(i+\mu+2j)_{2j-2-i-n}(1-j)_n\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_{n+1}(i-2j+2)_n}{n!\,(2j-2-i-n)!\,(i-j+2)_n\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{n+1}}$$

$$-\frac{(-1)^i(i+\mu+2j+1)_{2j-2-i-n}(-j)_n\left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_{n-1}(i-2j+2)_{n-1}}{n!\,(2j-i-n-2)!\,(i-j+2)_n\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_n}$$

$$\times\left\{(j-i-n-1)(\mu+2j+i)+(i+1)\left(\frac{\mu}{2}+i+\frac{1}{2}\right)\right\}$$

$$+\frac{(-1)^i(i+\mu+2j+1)_{2j-i-2-n}(i-2j+2)_{n-1}}{(n-1)!\,(2j-i-n-2)!\,(i-j+2)_{n-1}}$$

$$\times\frac{(1-j)_{n-1}\left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_{n-1}\left(\frac{\mu}{2}+3j-n-\frac{1}{2}\right)}{\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_n}$$

$$=\frac{-(-1)^i(i+\mu+2j+1)_{2j-i-3-n}}{n!\,(2j-i-n-2)!\,(i-j+2)_n}$$

$$\times\frac{(i-2j+2)_{n-1}(1-j)_{n-1}\left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_{n-1}}{\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{n+1}}$$

$$\times\left[(i+\mu+2j)(n-j)\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)(i-2j+n+1)\left(-\frac{\mu}{2}-j+n+\frac{1}{2}\right)\right.$$

$$+(\mu+4j-n-2)(-j)\left(i-j+\frac{\mu}{2}+\frac{3}{2}+n\right)$$

$$\times\left\{(j-i-n-1)(\mu+2j+i)+(i+1)\left(\frac{\mu}{2}+i+\frac{1}{2}\right)\right\}$$

$$\left.-(\mu+4j-n-2)\left(\frac{\mu}{2}+3j-n-\frac{1}{2}\right)n(i-j+n+1)\left(i-j+\frac{\mu}{2}+\frac{3}{2}+n\right)\right].$$

Now we must simplify the expression inside the square brackets. When we multiply it out, we find that there are 752 separate terms initially. After cancellation we find that

$$
\begin{aligned}
L_n = {} & \frac{(-1)^i (i+\mu+2j+1)_{2j-i-3-n} (i-2j+2)_{n-1} (1-j)_{n-1} \left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_{n-1}}{n!\,(2j-i-n-2)!\,(i-j+2)_n \left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{n+1}} \\
& \times [\tfrac{3}{2}j-\tfrac{3}{2}n+\tfrac{9}{4}ji+\tfrac{39}{2}jin-15jin\mu+\tfrac{3}{2}jin\mu^2-3ji\mu \\
& +33jin^2-10jin^2\mu+14jin^3+\tfrac{3}{4}ji\mu^2+\tfrac{67}{4}jn-15jn\mu \\
& +\tfrac{9}{4}jn\mu^2-2j\mu+\tfrac{3}{4}ji^2+5ji^2n-3ji^2n\mu-ji^2\mu+5ji^2n^2 \\
& +\tfrac{1}{4}ji^2\mu^2+\tfrac{163}{4}jn^2-20jn^2\mu+\tfrac{5}{4}jn^2\mu^2+34jn^3-7jn^3\mu \\
& +9jn^4+\tfrac{1}{2}j\mu^2-\tfrac{9}{4}in+3in\mu-\tfrac{3}{4}in\mu^2-\tfrac{15}{2}in^2+5in^2\mu \\
& -\tfrac{1}{2}in^2\mu^2-7in^3+2in^3\mu-2in^4+2n\mu-\tfrac{1}{2}n\mu^2-12j^2i-47j^2in \\
& +16j^2in\mu+10j^2i\mu-34j^2in^2-j^2i\mu^2-59j^2n+32j^2n\mu-2j^2n\mu^2 \\
& +10j^2\mu-3j^2i^2-7j^2i^2n+2j^2i^2\mu-82j^2n^2+18j^2n^2\mu-31j^2n^3-\tfrac{3}{2}j^2\mu^2 \\
& +21j^3i+34j^3in-8j^3i\mu+83j^3n-20j^3n\mu-16j^3\mu+3j^3i^2+51j^3n^2 \\
& +j^3\mu^2-12j^4i-40j^4n+8j^4\mu-\tfrac{3}{4}i^2n+i^2n\mu-\tfrac{1}{4}i^2n\mu^2-2i^2n^2 \\
& +i^2n^2\mu-i^2n^3+5n^2\mu-\tfrac{3}{4}n^2\mu^2+4n^3\mu-\tfrac{1}{4}n^3\mu^2+n^4\mu-\tfrac{21}{2}j^2 \\
& +27j^3-30j^4+12j^5-\tfrac{25}{4}n^2-\tfrac{35}{4}n^3-5n^4-n^5] \\
= {} & \frac{(-1)^i (i+\mu+2j+1)_{2j-i-3-n} (i-2j+2)_{n-1} (1-j)_{n-1} \left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_{n-1}}{n!\,(2j-i-n-2)!\,(i-j+2)_n \left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{n+1}} \\
& \times \left[(i-2j+n+1)(i-2j+n+2)(n-j)\left(-\frac{\mu}{2}-j+n+\frac{1}{2}\right)\left(\frac{\mu}{2}+3j-n-\frac{3}{2}\right)\right] \\
= {} & \frac{-(-1)^i (i+\mu+2j+1)_{2j-i-3-n} (i-2j+2)_{n+1}}{n!\,(2j-i-n-2)!\,(i-j+2)_n} \\
& \times \frac{(1-j)_n \left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_n \left(\frac{\mu}{2}+3j-n-\frac{3}{2}\right)}{\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{n+1}},
\end{aligned} \tag{3.12}
$$

and so (3.11) is proved by mathematical induction. As we remarked earlier (3.11) implies (3.9) the assertion of our theorem. □

4. Proof of the Weak Conjectures

We begin with a matrix identity that will immediately imply Theorem 8.

Theorem 7. *The following matrix identity holds* (i *and* j *lie in* $[0, m-1]$)

$$
\left(\delta_{ij} + \binom{\mu+i+j}{i}\right) \times (e_{ij}) = (f_{ij}), \tag{4.1}
$$

where

$$e_{ij}=0 \quad \textit{for} \quad i>j \tag{4.2}$$

$$e_{ii}=1, \tag{4.3}$$

and more generally for $2j \geqq i$

$$e_{i,2j}=\sum_{s\geqq 0}(-1)^i\binom{j-s}{2j-2s-i}\frac{(j-s)_{2s}(-\mu-3j-1)_s 4^{-s}}{s!\left(-\frac{\mu}{2}-j\right)_s\left(-\frac{\mu}{2}-2j+\frac{1}{2}\right)_s}, \tag{4.4}$$

and for $2j-1 \geqq i$

$$e_{i,2j-1}=\frac{(-1)^{i-1}}{2}\sum_{s\geqq 0}\left(\binom{j-s}{2j-1-i-2s}+\binom{j-s-1}{2j-1-i-2s}\right)\frac{(j-s)_{2s}(-\mu-3j+1)_s 4^{-s}}{s!\left(-\frac{\mu}{2}-j+1\right)_s\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)_s}\times\frac{\mu+3j-1-3s}{\mu+3j-1}; \tag{4.5}$$

furthermore

$$f_{ij}=0 \quad \textit{for} \quad j>i, \tag{4.6}$$

and

$$f_{ii}=\Delta_i(\mu), \tag{4.7}$$

where $\Delta_i(\mu)$ *is defined in the statement of Theorem* 8 *in Section* 1.

Proof. Since there is no assertion about the f_{ij} for $j<i$, we need only show that the assumption of (4.2), (4.3), (4.4) and (4.5) implies (4.6) and (4.7). We note that (4.2) and (4.3) are implicit in (4.4) and (4.5).

We begin by treating $f_{i,2j}$:

$$\begin{aligned}
f_{i,2j}&=\sum_{l\geqq 0}\left\{\delta_i+\binom{\mu+i+l}{i}\right\}e_{l,2j}\\
&=\sum_{l\geqq 0}\left\{\delta_i+\binom{\mu+i+l}{i}\right\}\\
&\quad\times\sum_{s\geqq 0}(-1)^l\binom{j-s}{2j-2s-l}\frac{(j-s)_{2s}(-\mu-3j-1)_s 4^{-s}}{s!\left(-\frac{\mu}{2}-j\right)_s\left(-\frac{\mu}{2}-2j+\frac{1}{2}\right)_s}\\
&=\sum_{s\geqq 0}\frac{(j-s)_{2s}(-\mu-3j-1)_s 4^{-s}}{s!\left(-\frac{\mu}{2}-j\right)_s\left(-\frac{\mu}{2}-2j+\frac{1}{2}\right)_s}\sum_{l\geqq 0}(-1)^l\left\{\binom{\mu+i+l}{i}+\delta_{il}\right\}\binom{j-s}{2j-2s-l}\\
&=\sum_{s\geqq 0}\frac{(j-s)_{2s}(-\mu-3j-1)_s 4^{-s}}{s!\left(-\frac{\mu}{2}-j\right)_s\left(-\frac{\mu}{2}-2j+\frac{1}{2}\right)_s}\left\{\binom{\mu+i+j-s}{i-j+s}+(-1)^i\binom{j-s}{2j-2s-i}\right\}
\end{aligned}$$

(by the Chu-Vandermonde summation [7; p. 3])

$$
\begin{aligned}
&=(M(i+1,j,\mu+1;0,1)-M(i,j,\mu+1;0,1))\\
&\quad+(\Lambda(i,j,\mu+1;0,1)-\Lambda(i+1,j,\mu+1;0,1))\\
&=(M(i+1,j,\mu+1;0,1)-\Lambda(i+1,j,\mu+1;0,1))\\
&\quad-(M(i,j,\mu+1;0,1)-\Lambda(i,j,\mu+1;0,1))\\
&=\begin{cases}0 & \text{for } 0\leqq i\leqq 2j-1 \text{ by Theorem 5}\\ M(2j+1,j,\mu+1;0,1) & \text{for } i=2j \text{ by Theorem 5 and Corollary 5a}\end{cases}\\
&=\begin{cases}0 & \text{for } 0\leqq i\leqq 2j-1\\ 2 & \text{for } i=j=0\\ \dfrac{(\mu+2j+2)_j\left(\frac{\mu}{2}+2j+\frac{3}{2}\right)_{j-1}}{(j)_j\left(\frac{\mu}{2}+j+\frac{3}{2}\right)_{j-1}} & \text{if } i=2j>0\end{cases}\\
&=\begin{cases}0 & \text{for } 0\leqq i\leqq 2j-1\\ \Delta_{2j}(\mu) & \text{for } i=2j,\end{cases}
\end{aligned}
\tag{4.8}
$$

where $\Delta_{2j}(\mu)$ is defined by (1.14).

Next we treat $f_{i,\,2j-1}$:

$$
\begin{aligned}
f_{i,\,2j-1}&=\sum_{l\geqq 0}\left\{\delta_{il}+\binom{\mu+i+l}{i}\right\}e_{l,\,2j-1}\\
&=\tfrac{1}{2}\sum_{l\geqq 0}\left\{\binom{m+i+l}{i}+\delta_{il}\right\}\sum_{s\geqq 0}(-1)^{l-1}\left(\binom{j-s}{2j-1-l-2s}+\binom{j-s-1}{2j-1-l-2s}\right)\\
&\quad\times\frac{(j-s)_{2s}(-\mu-3j+1)_s4^{-s}}{s!\left(-\frac{\mu}{2}-j+1\right)_s\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)_s}\;\frac{\mu+3j-1-3s}{\mu+3j-1}\\
&=-\tfrac{1}{2}\sum_{s\geqq 0}\frac{(j-s)_{2s}(-\mu-3j+1)_s4^{-s}}{s!\left(-\frac{\mu}{2}-j+1\right)_s\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)_s}\;\frac{\mu+3j-1-3s}{\mu+3j-1}\\
&\quad\times\sum_{l\geqq 0}(-1)^l\left\{\binom{m+i+l}{i}+\delta_{il}\right\}\left\{\binom{j-s}{2j-1-l-2s}+\binom{j-s-1}{2j-1-l-2s}\right\}\\
&=-\tfrac{1}{2}\sum_{s\geqq 0}\left\{(-1)^i\binom{j-s}{2j-2s-i-1}+(-1)^i\binom{j-s-1}{2j-2s-i-1}\right.\\
&\quad\left.-\binom{\mu+i+j-s-1}{i-j+s}-\binom{\mu+i+j-s}{i-j+s+1}\right\}\\
&\quad\times\frac{(j-s)_{2s}(-\mu-3j+1)_s4^{-s}(-\mu-3j+1+3s)}{s!\left(-\frac{\mu}{2}-j+1\right)_s\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)_s(-\mu-3j+1)}\\
&=-\tfrac{1}{2}\sum_{s\geqq 0}\left\{(-1)^i\binom{j-s}{2j-2s-i-1}+(-1)^i\binom{j-s-1}{2j-2s-i-1}\right.\\
&\quad\left.-\binom{\mu+i+j-s-1}{i-j+s}-\binom{\mu+i+j-s}{i-j+s+1}\right\}
\end{aligned}
$$

$$\times \frac{(j-s)_{2s}(-\mu-3j+1)_s 4^{-s}}{s!\left(-\frac{\mu}{2}-j+1\right)_s\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)_s}$$

$$-\tfrac{3}{2}\sum_{s\geqq 0}\left\{(-1)^i\binom{j-s-1}{2j-2s-i-3}+(-1)^i\binom{j-s-2}{2j-2s-i-3}\right.$$

$$\left.-\binom{\mu+i+j-s-2}{i-j+s+1}-\binom{\mu+i+j-s-1}{i-j+s+2}\right\}$$

$$\times \frac{(j-s-1)_{2s+2}(-\mu-3j+2)_s 4^{-s-1}}{s!\left(-\frac{\mu}{2}-j+1\right)_{s+1}\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)_{s+1}} \tag{4.9}$$

(here the single sum has been split into two by $(-\mu-3j+1+3s)=(-\mu-3j+1)+(3s)$)

$$=-\tfrac{1}{2}\sum_{s\geqq 0}\left\{(-1)^i\binom{j-s}{2j-2s-i-1}+(-1)^i\binom{j-s-1}{2j-2s-i-1}\right.$$

$$\left.-\binom{\mu+i+j-s-1}{i-j+s}-\binom{\mu+i+j-s}{i-j+s+1}\right\}$$

$$\times \frac{(j-s)_{2s}(-\mu-3j+1)_s 4^{-s}}{s!\left(-\frac{\mu}{2}-j+1\right)_s\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)_s}$$

$$-\tfrac{3}{2}\sum_{s\geqq 0}\left\{(-1)^i\binom{j-s-1}{2j-2s-i-3}+(-1)^i\binom{j-s-2}{2j-2s-i-3}\right\}$$

$$\times \frac{(j-s-1)_{2s+2}(-\mu-3j+2)_s 4^{-s-1}}{s!\left(-\frac{\mu}{2}-j+1\right)_{s+1}\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)_{s+1}}$$

$$+\tfrac{3}{2}\sum_{s\geqq 0}\left\{\binom{\mu+i+j-s-2}{i-j+s+1}+\binom{\mu+i+j-s-1}{i-j+s+2}\right\}$$

$$\times \frac{(j-s-1)_{2s+2}(-\mu-3j+2)_s 4^{-s-1}}{s!\left(-\frac{\mu}{2}-j+1\right)_{s+1}\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)_{s+1}}$$

$$=-S_1-S_2+S_3 \text{ say.}$$

Now

$$\begin{aligned}S_1=\tfrac{1}{2}(&-(\Lambda(i+1,j,\mu-1;0,1)-\Lambda(i+2,j,\mu-1;0,1))\\&-\Lambda(i+1,j,\mu-1;0,1)-M(i+1,j,\mu-1;0,1)\\&-M(i+2,j,\mu-1;0,1))\end{aligned}$$

$$=\begin{cases}-\frac{3}{2}\Lambda(i+1,j,\mu-1;0,1) & \text{for } 0\leqq i\leqq 2j-2\\ -\frac{3}{2}\Lambda(i+1,j,\mu-1;0,1)-\frac{1}{2}M(2j+1,j,\mu-1;0,1) & \text{for } i=2j-1\end{cases}$$

$$=\begin{cases}-\frac{3}{2}\Lambda(i+1,j,\mu-1;0,1) & \text{for } 0\leqq i\leqq 2j-2\\ -\frac{3}{2}-\frac{1}{2}M(2j+1,j,\mu-1;0,1) & \text{for } i=2j-1 \text{ (by Corollary 5a).}\end{cases}$$

As for S_2, we see that

$$S_2=+\tfrac{3}{2}(i+1)(-1)^i \times \sum_{s\geqq 0}\binom{j-s-1}{2j-2s-i-3}\frac{(j-s)_{2s+1}(-\mu-3j+2)_s\,4^{-s-1}}{s!\left(-\frac{\mu}{2}-j+1\right)_{s+1}\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)_{s+1}}$$

$$=-\tfrac{3}{2}(i+1)\frac{(2j-i-2)\,\Lambda(i,j,\mu-3;\,3,\,-1)}{4\left(-\frac{\mu}{2}-j+1\right)\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)}.$$

Finally we treat S_3:

$$S_3=\tfrac{3}{2}(\mu+2i+1)\sum_{s\geqq 0}\frac{(\mu+i+j-s-2)!\,(j-s-1)_{2s+2}}{(i-j+s+2)!\,(\mu+2j-2s-3)!\,s!} \times\frac{(-\mu-3j+2)_s\,4^{-s-1}}{\left(-\frac{\mu}{2}-j+1\right)_{s+1}\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)_{s+1}}$$

$$=\frac{3(\mu+2i+1)\,2^{-1}(\mu+i+j-2)!\,j(-j+1)}{2\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)(i-j+2)!\,(\mu+2j-2)!} \times\sum_{s\geqq 0}\frac{(-j+2)_s(j+1)_s\left(-\frac{\mu}{2}-j+\frac{3}{2}\right)_s(-\mu-3j+2)_s}{s!\left(-\frac{\mu}{2}-2j+\frac{5}{2}\right)_s(i-j+3)_s(-\mu-i-j+2)_s}$$

$$=\frac{3(\mu+2i+1)(\mu+i+j-2)!\,j(j-1)}{2(\mu+4j-3)(i-j+2)!\,(\mu+2j-2)!} \times {}_4F_3\left[\begin{matrix}-j+2,\,j+1,\,-\frac{\mu}{2}-j+\frac{3}{2},\,-\mu-3j+2;\,1\\ -\frac{\mu}{2}-2j+\frac{5}{2},\,i-j+3,\,-\mu-i-j+2\end{matrix}\right]$$

$$=\frac{3(\mu+2i+1)(\mu+i+j-2)!\,j(j-1)}{2(\mu+4j-3)(i-j+2)!\,(\mu+2j-2)!}\,\frac{\left(i-j+\frac{\mu}{2}+\frac{5}{2}\right)_{j-2}(-j+1)_{j-2}}{(\mu+i+1)_{j-2}\left(-\frac{\mu}{2}-2j+\frac{5}{2}\right)_{j-2}} \times {}_4F_3\left[\begin{matrix}-\frac{\mu}{2}-j+\frac{3}{2},\,i+2j+\mu+1,\,-j+2,\,i-2j+2;\,1\\ i-j+\frac{\mu}{2}+\frac{5}{2},\,i-j+3,\,2\end{matrix}\right].$$

Now when $i=2j-1$ we see that $S_2=0$. Consequently

$$
\begin{aligned}
f_{2j-1,\,2j-1} &= \tfrac{3}{2}+\tfrac{1}{2}M(2j+1,j,\mu-1;\,0,1)\\
&\quad +\tfrac{3}{2}\sum_{s\geqq 0}\frac{\left(-\frac{\mu}{2}-j+\frac{1}{2}\right)_{s+1}(\mu+4j-1)_{s+1}(-j+1)_{s+1}}{(s+1)!\left(j+\frac{\mu}{2}+\frac{1}{2}\right)_{s+1}(j+1)_{s+1}}\\
&= \tfrac{1}{2}M(2j+1,j,\mu-1;\,0,1)\\
&\quad +\tfrac{3}{2}\,{}_3F_2\left[\begin{matrix}-j+1,\ \mu+4j-1,\ -\frac{\mu}{2}-j+\frac{1}{2};\ 1\\ j+\frac{\mu}{2}+\frac{1}{2},\ j+1\end{matrix}\right]\\
&= \frac{1}{2}\frac{(\mu+2j)_j\left(2j+\frac{\mu}{2}+\frac{1}{2}\right)_{j-1}}{(j)_j\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j-1}}+\frac{3}{2}\frac{(\mu+2j)_{j-1}\left(-\frac{\mu}{2}-3j+\frac{3}{2}\right)_{j-1}}{\left(j+\frac{\mu}{2}+\frac{1}{2}\right)_{j-1}(-2j+1)_{j-1}}\\
&= \frac{1}{2}\frac{(\mu+2j)_{j-1}\left(\frac{\mu}{2}+2j+\frac{1}{2}\right)_{j-1}}{(j)_j\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j-1}}(\mu+3j-1+3j)\\
&= \frac{(\mu+2j)_{j-1}\left(\frac{\mu}{2}+2j+\frac{1}{2}\right)_j}{(j)_j\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j-1}}=\Delta_{2j-1}(\mu)
\end{aligned}
$$

by (1.15). Thus we have now established (4.7) for all $i\geqq 0$.

Finally for $i<2j-1$, we see that

$$
\begin{aligned}
f_{i,\,2j-1} &= -S_1-S_2+S_3\\
&= \tfrac{3}{2}\Lambda(i+1,j,\mu-1;\,0,1)\\
&\quad +\frac{3}{2}\frac{(i+1)(2j-i-2)\,\Lambda(i,j,\mu-3;\,3,-1)}{4\left(-\frac{\mu}{2}-j+1\right)\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)}\\
&\quad +\frac{3(\mu+2i+1)(\mu+i+j-2)!\,j(j-1)\left(i-j+\frac{\mu}{2}+\frac{5}{2}\right)_{j-2}(1-j)_{j-2}}{(\mu+4j-3)(i-j+2)!\,(\mu+2j-2)!\,(\mu+i+1)_{j-2}\left(-\frac{\mu}{2}-2j+\frac{5}{2}\right)_{j-2}}\\
&\quad \times{}_4F_3\left[\begin{matrix}-\frac{\mu}{2}-j+\frac{3}{2},\ i+2j+\mu+1,\ -j+2,\ i-2j+2;\ 1\\ i-j+\frac{\mu}{2}+\frac{5}{2},\ i-j+3,\ 2\end{matrix}\right].
\end{aligned}
$$

For the two Λ-functions we use the second ${}_4F_3$ expression that arises in the string of equations equating $M(i,j,m;a,\omega)$ with $\Lambda(i,j,m;a,\omega)$ in the proof of Theorem 5. Hence

$$f_{i,\,2j-1} = \frac{3(\mu+i)!\,(1-j)_{j-1}\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)_{j-1}}{2(i-j)!\,(\mu+2j-1)!\left(-\frac{\mu}{2}-2j+\frac{3}{2}\right)_{j-1}}$$

$$\times\left\{{}_4F_3\left[\begin{matrix} i-2j+1,\, 1-j,\, -j-\frac{\mu}{2}+\frac{1}{2},\, i+\mu+2j;\, 1 \\ i-j+1,\, i-j+\frac{\mu}{2}+\frac{3}{2},\, 1 \end{matrix}\right]\right.$$

$$+\frac{(2i+\mu+1)\left(\frac{\mu}{2}+j-\frac{1}{2}\right)j(j-1)}{\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)(i-j+1)(i-j+2)}$$

$$\times\,{}_4F_3\left[\begin{matrix} i-2j+2,\, 2-j,\, -j-\frac{\mu}{2}+\frac{3}{2},\, i+\mu+2j+1;\, 1 \\ i-j+3,\, i-j+\frac{\mu}{2}+\frac{5}{2},\, 2 \end{matrix}\right]$$

$$+\frac{\left(\frac{\mu}{2}+i+\frac{1}{2}\right)(i+1)(2j-i-2)j}{\left(i-j+\frac{\mu}{2}+\frac{3}{2}\right)(i-j+1)(i-j+2)}$$

$$\left.\times\,{}_4F_3\left[\begin{matrix} i-2j+3,\, 1-j,\, -j-\frac{\mu}{2}+\frac{3}{2},\, i+\mu+2j+1;\, 1 \\ i-j+3,\, i-j+\frac{\mu}{2}+\frac{5}{2},\, 2 \end{matrix}\right]\right\}$$

$$=0, \qquad \text{by Theorem 6.}$$

It appears at first glance that we can invoke Theorem 6 for i and j integers only for $i \geqq j$. However the above expression is indentically zero provided $i-2j+2$ is a nonpositive integer and $i-j$ is any complex (or real) number other than a negative integer. The conclusion for all integral i with $0 \leqq i < 2j-1$ then follows by continuity.

As a direct corollary of Theorem 7, we immediately deduce:

Theorem 8.

$$\det\left(\delta_{ij}+\binom{\mu+i+j}{i}\right)_{0\leqq i,\,j\leqq m-1} = \prod_{j=0}^{m-1} \Delta_j(\mu),$$

where

$$\Delta_{2j}(\mu)=\begin{cases}\dfrac{(\mu+2j+2)_j\left(\frac{\mu}{2}+2j+\frac{3}{2}\right)_{j-1}}{(j)_j\left(\frac{\mu}{2}+j+\frac{3}{2}\right)_{j-1}}, & j>0\\ 2, & j=0,\end{cases}\tag{4.10}$$

and

$$\Delta_{2j-1}(\mu)=\frac{(\mu+2j)_{j-1}\left(\frac{\mu}{2}+2j+\frac{1}{2}\right)_j}{(j)_j\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j-1}}, \quad j>0\tag{4.11}$$

Proof. By Theorem 7, we see that

$$\begin{aligned}\det\left(\delta_{ij}+\binom{\mu+i+j}{i}\right)_{0\leqq i,\, j\leqq m-1}&=\frac{\det(f_{ij})}{\det(e_{ij})}\\ &=f_{00}f_{11}\cdots f_{m-1,m-1}\\ &=\prod_{j=0}^{m-1}\Delta_j(\mu).\quad\square\end{aligned}$$

As a consequence of Theorem 8, we can now prove the weak forms of both conjectures given in the introduction.

Theorem 9 (The Weak Macdonald Conjecture). *The total number of plane partitions π such that $D(\pi)\subseteq\mathscr{B}_{m,m,m}$ and $D(\pi)$ is G_3 invariant is*

$$\prod_{i=1}^{m}\left\{\frac{3i-1}{3i-2}\prod_{j=i}^{m}\frac{m+i+j-1}{2i+j-1}\right\}.$$

Proof. We let γ_m denote the product given in Theorem 9. Then by Theorem 4 with $q=1$ and Theorem 8, we see that we must prove that

$$\gamma_1=\Delta_0(0),\tag{4.12}$$

$$\frac{\gamma_m}{\gamma_{m-1}}=\Delta_{m-1}(0).\tag{4.13}$$

Now (4.12) is clear since $\gamma_1=\frac{2}{1}\cdot\frac{2}{2}=2=\Delta_0(0)$. As for (4.13), we see that

$$\begin{aligned}\frac{\gamma_m}{\gamma_{m-1}}&=\frac{3m-1}{3m-2}\times\prod_{i=1}^{m}\frac{2m+i-1}{2i+m-1}\times\prod_{i=1}^{m-1}\frac{2m+i-2}{m+2i-2}=\frac{3m-1}{3m-2}\,\frac{(2m)_m(2m-1)_{m-1}}{(m)_{2m-2}(3m-1)}\\ &=\frac{3m-1}{3m-2}\times\frac{(2m)_{m-1}(2m-1)_{m-1}}{(m)_{2m-2}}=\frac{(3m-1)}{(3m-2)}\times\frac{(2m)_{m-1}}{(m)_{m-1}},\end{aligned}$$

and by (1.15)

$$\begin{aligned}\Delta_{2j-1}(0)&=\frac{(2j)_{j-1}(2j+\frac{1}{2})_j}{(j)_j(j+\frac{1}{2})_{j-1}}=\frac{(3j-\frac{1}{2})(4j)_{2j-1}}{(3j-1)(2j)_{2j-1}}\\ &=\frac{6j-1}{6j-2}\,\frac{(4j)_{2j-1}}{(2j)_{2j-1}}=\frac{\gamma_{2j}}{\gamma_{2j-1}},\end{aligned}$$

while by (1.14)

$$\begin{aligned}\Delta_{2j}(0)&=\frac{(2j+2)_j(2j+\frac{3}{2})_{j-1}}{(j)_j(j+\frac{3}{2})_{j-1}}=\frac{3j+1}{3j+\frac{1}{2}}\times\frac{(2j+2)_{j-1}(2j+\frac{3}{2})}{(j)_j(j+\frac{3}{2})_{j-1}}\\&=\frac{(3j+1)}{(3j+\frac{1}{2})}\times\frac{(2j+1)_j(2j+\frac{3}{2})_j}{(j+1)_j(j+\frac{1}{2})_j}=\frac{(6j+2)}{(6j+1)}\times\frac{(4j+2)_{2j}}{(2j+1)_{2j}}=\frac{\gamma_{2j+1}}{\gamma_{2j}}.\end{aligned}$$

Thus Theorem 9 is established. □

Theorem 10. *The total number of descending plane partitions whose parts do not exceed m is* $\prod_{i=1}^{m}\prod_{j=1}^{m}\frac{m+i+j-1}{2i+j-1}$.

Proof. We let δ_m denote the product given in Theorem 10. Then by Theorem 3′ with $d=0$, $q=1$ and Theorem 8, we see that we must establish

$$\delta_2=\Delta_0(2), \tag{4.14}$$

$$\frac{\delta_m}{\delta_{m-1}}=\Delta_{m-2}(2). \tag{4.15}$$

Now (4.14) is clear since $\delta_2=\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{5}=2=\Delta_0(2)$. As for (4.15), we see that

$$\frac{\delta_m}{\delta_{m-1}}=\frac{3m-2}{3m-1}\cdot\frac{\gamma_m}{\gamma_{m-1}}=\frac{(2m)_{m-1}}{(m)_{m-1}},$$

and by (1.15)

$$\begin{aligned}\Delta_{2j-1}(2)&=\frac{(2j+2)_{j-1}(2j+\frac{3}{2})_j}{(j)_j(j+\frac{3}{2})_{j-1}}=\frac{(2j+1)_j(2j+\frac{3}{2})_j}{(j+1)_j(j+\frac{1}{2})_j}\\&=\frac{(4j+2)_{2j}}{(2j+1)_{2j}}=\frac{\delta_{2j+1}}{\delta_{2j}},\end{aligned}$$

while by (1.14)

$$\begin{aligned}\Delta_{2j}(2)&=\frac{(2j+4)_j(2j+\frac{5}{2})_{j-1}}{(j)_j(j+\frac{5}{2})_{j-1}}\\&=\frac{(2j+3)_{j+1}(2j+\frac{5}{2})_{j-1}}{(j+1)_j(j+\frac{3}{2})_j}=\frac{(2j+3)_{j+1}(2j+\frac{3}{2})_j}{(j+1)_j(j+\frac{3}{2})_{j+1}}\\&=\frac{(2j+2)_{j+1}(2j+\frac{5}{2})_j}{(j+1)_{j+1}(j+\frac{3}{2})}\times\frac{(2j+1)(3j+3)}{(2j+2)(3j+\frac{3}{2})}\\&=\frac{(4j+2)_{2j+1}}{(2j+2)_{2j+1}}=\frac{\delta_{2j+2}}{\delta_{2j+1}}.\end{aligned}$$

Thus Theorem 10 is established. □

Clearly we can apply Theorem 8 to the case $q=1$ of Theorem 3′; in [5] we discuss this application in more detail.

5. Conclusion

Obviously the final assault on Macdonald's conjecture is the first order of business now. One is tempted to hope that the q-analogs of the results on hypergeometric series utilized in this paper will be adequate to treat both Macdonald's conjecture and the descending plane partitions conjecture. However just as the full q-series proof of MacMahon's conjecture [4] was significantly more complicated than just the case $q=1$ [1], so too is it clear upon inspection that the full conjectures studied here are not obtained purely by direct substitution of q-analogs of Theorems 5 and 6. A discussion of the evidence for these conjectures and related problems occurs in [5].

In [2] the equivalence of the Bender-Knuth conjecture with MacMahon's conjecture was proved using only elementary row operations on determinants. So far similar efforts to prove the equivalence of Macdonald's conjecture with the descending plane partitions conjecture have gone astray.

The assertion in Theorem 5 is the following hypergeometric series identity:

$$\frac{\left(\frac{m}{2}+2j\right)_{a+\omega-1}}{\left(\frac{m}{2}+j+1\right)_{a+\omega-1}}\binom{m+i+j+a-1}{i-j+a-1}$$
$$\times {}_4F_3\left[\begin{matrix}1-j, j, -m-3j-a-\omega+1, -\frac{m}{2}-j; 1\\ i-j+a, -\frac{m}{2}-2j-a-\omega+2, -m-i-j-a+1\end{matrix}\right]$$
$$=(-1)^{i+a}\binom{j+a+\omega-2}{i-j+a-1}$$
$$\times {}_4F_3\left[\begin{matrix}-j+\frac{i+a}{2}, -j+\frac{i+a+1}{2}, j+a+\omega-1, -m-3j-a-\omega+1; 1\\ i-j+a, -\frac{m}{2}-2j+1, -\frac{m}{2}-j+\frac{1}{2}\end{matrix}\right]. \tag{5.1}$$

Since the crucial step of the proof of this identity is obtained from applying (3.6) when the series on the left is not only nearly poised of the second kind but also balanced, it would appear that such series have a significant amount of structure beyond that of merely balanced ${}_4F_3$'s. Recent work of J. Wilson [19] and [20] and R. Askey and J. Wilson [6] has made clear the importance of balanced ${}_4F_3$'s.

Theorem 6 is essentially a contiguous relation among three balanced ${}_4F_3$'s. Consequently it should be derivable from the ${}_4F_3$ contiguous relations described by J. Wilson [19]. Indeed, R. Askey has outlined how the derivation should proceed from the work in Chapter 4 of Wilson's thesis [20]. Equation (3.11) essentially asserts that the tails of the three series in Theorem 6 are indeed summable to the right side of (3.11). Thus we may deduce from (3.11) and (3.9) that the sum of three truncated balanced ${}_4F_3$'s of this type is summable. Askey also suggests that Wilson's work can be used to explain this phenomenon.

Finally we mention that the "massive" computation in (3.12) for L_n which was essential to the proof of Theorem 6 was done by the FORMAC Utility Program due to H.D. Noble of the Pennsylvania State University Computation Center.

References

1. Andrews, G.E.: MacMahon's conjecture on symmetric plane partitions. Proc. Nat. Acad. Sci. USA, **74**, 426–429 (1977)
2. Andrews, G.E.: Plane partitions (II): the equivalence of the Bender-Knuth and MacMahon conjectures. Pac. J. Math., **72**, 283–291 (1977)
3. Andrews, G.E.: The Theory of Partitions, Addision-Wesley, Reading, 1976
4. Andrews, G.E.: Plane partitions (I): the MacMahon conjecture, Advances in Math. (to appear)
5. Andrews, G.E.: On Macdonald's conjecture and descending plane partitions, Proceedings of the Alfred Young Day Conference (to appear)
6. Askey, R., Wilson, J.: A set of orthogonal polynomials that generalize the Racah coefficients or 6-j symbols, S.I.A.M. J. Math. Anal., (to appear)
7. Bailey, W.N.: Generalized Hypergeometric Series. London and New York: Cambridge University Press 1935
8. Carlitz, L.: Rectangular arrays and plane partitions, Acta Arith., **13**, 22–47 (1967)
9. Gansner, E.: Matrix correspondences and the enumeration of plane partitions, Ph.D. Thesis, M.I.T., 1978
10. Macdonald, I.G.: Lecture on plane partitions, Oberwolfach Conference on Combinatorics and Special Functions, May 1977
11. MacMahon, P.A.: Partitions of numbers whose graphs possess symmetry. Trans. Cambridge Phil. Soc., **17**, 149–170 (1898–99)
12. MacMahon, P.A.: Combinatory Analysis, Vol. 2. London and New York: Cambridge University Press 1916
13. Muir, T.: A Treatise on the Theory of Determinants. London: Longmans 1933
14. Stanley, R.P.: Theory and applications of plane partitions I, Studies in Appl. Math., **50**, 167–188 (1971)
15. Stanley, R.P.: Theory and applications of plane partitions II, Studies in Appl. Math., **50**, 259–279 (1971)
16. Whipple, F.J.W.: Well-poised series and other generalized hypergeometric series, Proc. London Math. Soc. (2), **25**, 525–544 (1926)
17. Whipple, F.J.W.: Some transformations of generalized hypergeometric series, Proc. London Math. Soc. (2), **26**, 257–272 (1927)
18. Whittaker, E.T., Watson, G.N.: A Course of Modern Analysis, 4th ed., London and New York: Cambridge University Press 1958
19. Wilson, J.: Three-term contiguous relations and some new orthogonal polynomials, from Pade and Rational Approximation (Saff and Varga, eds.). New York-London: Academic Press, 1977
20. Wilson, J.: Ph.D. Thesis, University of Wisconsin, 1978

Received October 3, 1978