# ON GENERALIZATIONS OF EULER'S PARTITION THEOREM

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## 1. INTRODUCTION

Sylvester's memoir on partitions contains an interesting generalization of Euler's partition theorem [12, p. 293 (p. 45 in Collected Works)]. In any partition of n into distinct parts, we may count the total number of sequences of consecutive integers appearing. For example, $31 = 10 + 8 + 7 + 3 + 2 + 1$ consists of three such sequences, namely 10; 8, 7; 3, 2, 1. Sylvester's theorem is as follows.

THEOREM 1. *Let* $A_k(n)$ *denote the number of partitions of* n *into odd parts (repetitions allowed) with exactly* k *distinct parts appearing. Let* $B_k(n)$ *denote the number of partitions of* n *into distinct parts such that exactly* k *sequences of consecutive integers appear in each partition. Then*

$$A_k(n) = B_k(n).$$

For example, let $n = 15$, $k = 3$. Then the partitions enumerated by $A_3(15)$ are

$$11 + 3 + 1, \quad 9 + 5 + 1, \quad 9 + 3 + 1 + 1 + 1, \quad 7 + 5 + 3, \quad 7 + 5 + 1 + 1 + 1,$$

$$7 + 3 + 3 + 1 + 1, \quad 7 + 3 + 1 + 1 + 1 + 1 + 1, \quad 5 + 5 + 3 + 1 + 1, \quad 5 + 3 + 3 + 3 + 1,$$

$$5 + 3 + 3 + 1 + 1 + 1 + 1, \quad 5 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1.$$

Hence $A_3(15) = 11$. The partitions enumerated by $B_3(15)$ are

$$11 + 3 + 1, \quad 10 + 4 + 1, \quad 9 + 5 + 1, \quad 9 + 4 + 2, \quad 8 + 6 + 1, \quad 8 + 5 + 2, \quad 8 + 4 + 2 + 1,$$

$$7 + 5 + 3, \quad 7 + 5 + 2 + 1, \quad 7 + 4 + 3 + 1, \quad 6 + 5 + 3 + 1.$$

Hence $B_3(15) = 11$.

This beautiful theorem was proved arithmetically [12, Section (46)]. F. Franklin has deduced the result for $k = 1$ from a study of the generating functions involved [12, Section (25) (C)]; however, there seems to be no known analytic proof for $k > 1$. In Section 2 of this paper, we prove Sylvester's theorem by means of generating functions.

In Section 3, we give a new generalization of Euler's theorem. Let $\Pi_d(n)$ denote the set of partitions of n into distinct parts. If $\pi$ is any partition of n, say $b_1 + \cdots + b_s = n$ $(b_i \geq b_{i+1})$, let $g(\pi)$ denote the number of solutions of the inequality $b_i - b_{i+1} \geq 2$ $(i = 1, \cdots, s$; define $b_{s+1} = 0)$. For example, in the partition $18 = 8 + 6 + 2 + 2$, $g(\pi) = 3$.

THEOREM 2. *Let* $C_k(n)$ *denote the number of partitions of* n *with exactly* k *distinct even parts appearing (all other parts being odd), then*

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$$\sum_{\pi \in \Pi_d(n)} \binom{g(\pi)}{k} = C_k(n).$$

If $k = 0$, the above sum just counts the number of elements of $\Pi_d(n)$, and we have Euler's theorem again. As an example, we take $k = 2$, $n = 13$. The partitions in $\Pi_d(13)$ for which $g(\pi) \geq 2$ are

$$11+2,\quad 10+3,\quad 9+4,\quad 9+3+1,\quad 8+5,\quad 8+4+1,\quad 8+3+2,\quad 7+5+1,$$

$$7+4+2,\quad 6+5+2,\quad 6+4+3,\quad 6+4+2+1.$$

All of these partitions have $g(\pi) = 2$, except $7 + 4 + 2$, which has $g(\pi) = 3$. Thus in this particular case the sum given in the theorem is equal to 14. The partitions enumerated by $C_2(13)$ are

$$10+2+1,\quad 8+4+1,\quad 8+3+2,\quad 8+2+1+1+1,\quad 6+4+3,\quad 6+4+1+1+1,$$

$$6+5+2,\quad 6+3+2+1+1,\quad 6+2+1+1+1+1+1,\quad 7+4+2,\quad 5+4+2+1+1,$$

$$4+3+3+2+1,\quad 4+3+2+1+1+1+1,\quad 4+2+1+1+1+1+1+1+1.$$

Hence $C_2(13) = 14$.

As a corollary of our work, we obtain the curious identity

$$\begin{vmatrix} 1 & \beta q & 0 & 0 & 0 & \cdots \\ -1 & 1+q & \beta q^2 & 0 & 0 & \cdots \\ 0 & -1 & 1+q^2 & \beta q^3 & 0 & \cdots \\ 0 & 0 & -1 & 1+q^3 & \beta q^4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \prod_{j=0}^{\infty} \frac{(1+\beta q^{2j+1})}{(1-q^{2j+1})},$$

which resembles certain results of I. Schur [10], [11].

## 2. PROOF OF THEOREM 1

Clearly [12, Section (25) (C)], if $E(a; q)$ denotes the generating function of $A_k(N)$, then

$$E(a;q) = 1 + \sum_{k=1}^{\infty}\sum_{N=1}^{\infty} A_k(N) a^k q^N = \prod_{j=0}^{\infty}\left(1 + \frac{aq^{2j+1}}{1-q^{2j+1}}\right). \tag{2.1}$$

Let $F(a; q)$ be the generating function of $B_k(N)$, and let $F_n(a; q)$ be the generating function of $B_k(n; N)$, where $B_k(n; N)$ denotes the number of partitions of $N$ into distinct parts with exactly $k$ sequences appearing and with no part exceeding $n$.

Thus with $F_0(a; q) = 1$, we get

$$F_1(a;q) = 1 + aq, \qquad F_2(a;q) = 1 + aq + aq^2 + aq^3,$$

and in general,

$$F_n(a; q) = F_{n-1}(a; q) + q^n(F_{n-1}(a; q) - F_{n-2}(a; q)) + aq^n F_{n-2}(a; q). \tag{2.2}$$

Now (2.2) is easily verified. We may divide the partitions enumerated by $B_k(n; N)$ into three disjoint classes: 1) those partitions with largest part less than n; 2) those partitions with n, n - 1 as the two largest parts; 3) those partitions with n as largest part and n - 1 not appearing. The partitions in the first class have $F_{n-1}(a; q)$ as generating function. The partitions in the second class have $q^n(F_{n-1}(a; q) - F_{n-2}(a; q))$ as generating function, and the partitions in the third class have $aq^n F_{n-2}(a; q)$ as generating function.

We may rewrite (2.2) as

$$F_n(a; q) = (1 + q^n) F_{n-1}(a; q) + (a - 1)q^n F_{n-2}(a; q). \tag{2.3}$$

From Tannery's theorem [9, p. 371], it is easily deduced that if $|q| < 1$, then

$$F(a; q) = \lim_{n \to \infty} F_n(a; q).$$

By (2.3) and the remarks preceding (2.2) we have the relation

$$F_n(a; q) = \begin{vmatrix} 1 & (a-1)q & 0 & 0 & \cdots & & \\ -1 & 1+q & (a-1)q^2 & 0 & \cdots & & \\ 0 & -1 & 1+q^2 & (a-1)q^3 & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & (a-1)q^{n-1} & 0 \\ & & & & & 1+q^{n-1} & (a-1)q^n \\ & & & & & -1 & 1+q^n \end{vmatrix}.$$

Define

$$G_n(\beta; x; q) = \begin{vmatrix} 1+x & x\beta q & 0 & 0 & \cdots & & \\ -1 & 1+xq & x\beta q^2 & 0 & \cdots & & \\ 0 & -1 & 1+xq^2 & x\beta q^3 & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & x\beta q^{n-1} & 0 \\ & & & & & 1+xq^{n-1} & x\beta q^n \\ & & & & & -1 & 1+xq^n \end{vmatrix}.$$

Then expansion along the first row yields the formula

$$G_n(\beta; x; q) = (1+x)G_{n-1}(\beta; xq; q) + x\beta q G_{n-2}(\beta; xq^2; q). \tag{2.4}$$

Also, setting $x = 1$, expanding along the first row, and comparing the result with the determinant for $F_n(a; q)$, we find that

$$F_n(a; q) = G_n(a-1; 1; q) - G_{n-1}(a-1; q; q). \tag{2.5}$$

Again by Tannery's theorem, there exists $G(\beta; x; q)$ such that if $|q| < 1$, then

$$\lim_{n \to \infty} G_n(\beta; x; q) = G(\beta; x; q).$$

Hence, by (2.4),

$$G(\beta; x; q) = (1+x)G(\beta; xq; q) + x\beta q G(\beta; xq^2; q), \tag{2.6}$$

and by (2.5),

$$F(a; q) = G(a-1; 1; q) - G(a-1; q; q). \tag{2.7}$$

Now, if

$$G^*(\beta; x; q) = \sum_{\nu=0}^{\infty} \frac{q^{\nu(\nu-1)/2} x^\nu (1+\beta q) \cdots (1+\beta q^\nu)}{(1-q) \cdots (1-q^\nu)},$$

then $G^*(\beta; 0; q) = 1$, and by substitution of $G^*(\beta; x; q)$ into (2.6) and comparison of coefficients of $x^\nu$, we see that (2.6) is satisfied by $G^*(\beta; x; q)$. Therefore, since the relation $G(\beta; 0; q) = 1$ and (2.6) determine $G(\beta; x; q)$ uniquely, we find that

$$G(\beta; x; q) = G^*(\beta; x; q). \tag{2.9}$$

Hence, by Heine's transformation of basic hypergeometric series [7, p. 106], we obtain the formula

$$G(\beta; x; q) = \prod_{h=1}^{\infty} (1+\beta q^h) \prod_{j=0}^{\infty} (1+xq^j) \sum_{m=0}^{\infty} \frac{(-\beta q)^m}{\prod_{s=1}^{m} (1-q^s) \prod_{t=0}^{m-1} (1+xq^t)}. \tag{2.10}$$

Thus

$$G(\beta; 1; q) = \prod_{h=1}^{\infty} (1+\beta q^h) \prod_{j=1}^{\infty} (1+q^j) \sum_{m=0}^{\infty} \frac{(-\beta q)^m (1+q^m)}{\prod_{s=1}^{m} (1-q^s) \prod_{t=1}^{m} (1+q^t)}$$

$$= G(\beta; q; q) + \prod_{h=1}^{\infty} (1+\beta q^h) \prod_{j=1}^{\infty} (1+q^j) \sum_{m=0}^{\infty} \frac{(-\beta q^2)^m}{\prod_{s=1}^{m} (1-q^{2s})}$$

$$= G(\beta; q; q) + \prod_{h=1}^{\infty} (1+\beta q^h) \prod_{j=1}^{\infty} (1+q^j) \prod_{v=1}^{\infty} (1+\beta q^{2v})^{-1} \quad \text{(by [12, Section (4)])}$$

$$= G(\beta; q; q) + \prod_{h=0}^{\infty} (1+\beta q^{2h+1}) \prod_{j=1}^{\infty} (1+q^j).$$

Hence

$$F(a; q) = G(a-1; 1; q) - G(a-1; q; q)$$

$$(2.11) \qquad = \prod_{h=0}^{\infty} (1+(a-1)q^{2h+1}) \prod_{j=1}^{\infty} (1+q^j) = \prod_{h=0}^{\infty} \frac{(1+(a-1)q^{2h+1})}{(1-q^{2h+1})} \quad [9, \text{p. } 13]$$

$$= \prod_{h=0}^{\infty} \left(1 + \frac{aq^{2h+1}}{1-q^{2h+1}}\right) = E(a; q),$$

and therefore $A_k(n) = B_k(n)$.

## 3. PROOF OF THEOREM 2

We shall prove our theorem in a slightly altered form. Define a k-partition of n to be a partition of n of the form

$$n = \sum_{j=1}^{k} a_j + \sum_{\ell=1}^{\nu} b_\ell$$

with

$$a_j - a_{j+1} \geq 2 \quad (j = 1, \cdots, k-1), \qquad a_k \geq 2,$$

$$b_\ell - b_{\ell+1} \geq 1 \quad (\ell = 1, \cdots, \nu - 1),$$

$$a_j \neq b_\ell \text{ for any } j \text{ and } \ell, \qquad a_j - 1 \neq b_\ell \text{ for any } j \text{ and } \ell.$$

We denote such a partition by $a_1, \cdots, a_k \mid b_1, \cdots, b_\nu$.

Thus more briefly, a k-partition $\pi$ of n is a partition of n into distinct parts with $g(\pi) \geq k$; however, we now consider two such partitions distinct if merely the set of $a_i$ (or the set of $b_i$) in one partition differs from that in the other. Thus $6, 2 \mid 4$, $4, 2 \mid 6$, and $6, 4 \mid 2$ are to be considered three distinct 2-partitions of 12. We restate Theorem 2 as follows.

THEOREM 2'. *Let* $C_k(N)$ *denote the number of partitions of* N *with exactly* k *distinct even parts appearing (all other parts being odd). Let* $D_k(N)$ *denote the number of* k*-partitions of* N. *Then*

$$C_k(N) = D_k(N).$$

*Remark.* Clearly

$$D_k(N) = \sum_{\pi \in \Pi_d(N)} \binom{g(\pi)}{k},$$

since we may form exactly $\binom{g(\pi)}{k}$ distinct k-partitions from any given partition of N into distinct parts.

*Proof of theorem.* Define

$$D(k, n; N) = \begin{cases} 0 \text{ if } k < 0, \text{ or } n \leq 0, \text{ or } N \leq 0, \text{ and not } k = n = N = 0, \\ 1 \text{ if } k = n = N = 0, \\ \text{the number of k-partitions of } N \text{ with } n \text{ parts} \\ \qquad \text{(that is, } k + \nu = n) \text{ if } k \geq 0,\ n > 0,\ N > 0. \end{cases}$$

Then

$$\text{(3.1)} \quad D(k, n; N) = D(k, n; N - n) + D(k, n - 1; N - n) + D(k - 1, n - 1; N - 2n).$$

This identity is established as follows. Divide the partitions enumerated by D(k, n; N) into three groups. In group (A), consider those partitions in which $a_k = 2$. In group (B), consider those partitions in which $b_\nu = 1$. In group (C), consider those partitions in which $a_k \neq 2$ and $b_\nu \neq 1$.

If we subtract 2 from every summand of a partition in group (A), we decrease the number being partitioned to N - 2n; the number of parts is decreased by 1, and the number of $a_i$ is reduced by 1. Hence this process establishes a one-to-one correspondence between the partitions enumerated by group (A) and those enumerated by D(k - 1, n - 1; N - 2n).

If we subtract 1 from every summand of a partition in group (B), we decrease the number being partitioned to N - n; the number of parts is decreased by 1, but the number of $a_i$ is not reduced. Hence this process establishes a one-to-one correspondence between the partitions enumerated by group (B) and those enumerated by D(k, n - 1; N - n).

Applying the same process as in the preceding paragraph to the partitions in group (C), we find that the number of partitions in group (C) is just D(k, n; N - n). Hence we have established (3.1). Thus, if

$$\text{(3.2)} \qquad \Gamma(\beta; x; q) = 1 + \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \sum_{N=1}^{\infty} D(k, n; N)\beta^k x^n q^N,$$

then by (3.1)

$$\Gamma(\beta; x; q) = (1 + xq)\,\Gamma(\beta; xq; q) + x\beta q^2\,\Gamma(\beta; xq^2; q).$$

Since $\Gamma(\beta; 0; q) = 1$, we find on comparison with (2.4) that

$$\Gamma(\beta; x; q) = G(\beta; xq; q).$$

Therefore, by (2.10)

$$\begin{aligned}\Gamma(\beta; 1; q) &= \prod_{h=1}^{\infty} (1+\beta q^h) \prod_{j=1}^{\infty} (1+q^j) \sum_{m=0}^{\infty} \frac{(-\beta q)^m}{(1-q^2)\cdots(1-q^{2m})} \\ &= \prod_{h=1}^{\infty} (1+\beta q^h) \prod_{j=1}^{\infty} (1+q^j) \prod_{m=0}^{\infty} (1+\beta q^{2m+1})^{-1} \quad \text{(by [12, Section (4)])} \\ &= \prod_{h=1}^{\infty} (1+\beta q^{2h}) \prod_{j=0}^{\infty} (1-q^{2j+1})^{-1} \quad \text{(by [12, p. 263 (p. 13)])} \\ &= 1 + \sum_{k=0}^{\infty} \sum_{N=1}^{\infty} C_k(N) \beta^k q^N .\end{aligned} \tag{3.3}$$

But by (3.2)

$$\Gamma(\beta; 1; q) = 1 + \sum_{k=0}^{\infty} \sum_{N=1}^{\infty} D_k(N) \beta^k q^N .$$

Hence $C_k(N) = D_k(N)$.

## 4. CONCLUSION

Along with the generalizations of Euler's theorem, we have obtained some interesting side results. First of all, we have shown that each of the expressions in the last identity stated in the introduction is $F(\beta + 1; q)$.

Also, comparing (3.3) with (2.9), we find that

$$\sum_{\nu=0}^{\infty} \frac{q^{\nu(\nu+1)/2} (1+\beta q) \cdots (1+\beta q^\nu)}{(1-q) \cdots (1-q^\nu)} = \prod_{j=1}^{\infty} (1+\beta q^{2j})(1+q^j) . \tag{4.1}$$

Several papers have dealt with (4.1) both in the study of continued fractions and in the study of partition theorems of the Rogers-Ramanujan type [3], [4], [5], [6]; (4.1) was also studied by Bachmann in [2, p. 42], and it is originally due to Lebesgue [5, p. 42].

Finally it would greatly simplify the proof of Sylvester's theorem if one could prove directly that

$$F(a; q) = \sum_{\nu=1}^{\infty} \frac{q^{\nu(\nu-1)/2}(1+(a-1)q) \cdots (1+(a-1)q^\nu)}{(1-q) \cdots (1-q^{\nu-1})} .$$

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