Dyson's "Favorite" Identity and Chebyshev Polynomials of the Third and Fourth Kind

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Abstract

The combinatorial and analytic properties of Dyson's "favorite" identity are studied in detail. In particular, a q-series analog of the anti-telescoping method is used to provide a new proof of Dyson's results with mock theta functions popping up in intermediate steps. This leads to the appearance of Chebyshev polynomials of the third and fourth kind in Bailey pairs related to Bailey's Lemma. The natural relationship with L. J. Rogers's pioneering work is also presented.

1 Introduction

Freeman Dyson, in his article, A Walk Through Ramanujan's Garden [11], describes how his study of Rogers-Ramanujan type identities helped to preserve his sanity during the dark days of World War II. Among the results he discovered was his favorite:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n} \prod_{j=1}^{n}(1+q^j+q^{2j})}{(1-q)(1-q^2)\cdots(1-q^{2n+1})} = \prod_{n=1}^{\infty} \frac{(1-q^{9n})}{(1-q^n)} \tag{1.1}$$

Dyson's proof of (1.1) [10, pp. 8-9] and the proof subsequently provided by Slater [17, p. 161, eq.(92)] are based on what has become known as Bailey's Lemma [10, p. 3, eq.(3.1)].

We shall begin in Section 2 by providing a proof of (1.1) and three related identities via q-difference equations. This will necessitate a q-series analog of anti-telescoping [6] with several new intermediate q-series arising.

With an eye to understanding these new intermediate functions, we devote Section 3 to connections between $V_n(x)$ and $W_n(x)$ (the Chebyshev polynomials of the third and fourth kind respectively) and surprising Bailey pairs. For example, in Section 4, we prove

$$\sum_{n\geq 0} \frac{q^{n^2+n}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(1-q)(1-q^2)\cdots(1-q^{2n+1})} = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{n\geq 0} q^{3\binom{n+1}{2}} V_n(x), \tag{1.2}$$

where $V_n(x)$ is the Chebyshev polynomial of the third kind.

Section 4 returns to (1.1) itself to reveal how many other Rogers-Ramanujan type identities are merely specializations of the natural generalization of (1.1).

Section 5 uses the work of Section 3 to establish that many of the new functions are, in fact, mock theta functions or closely related.

Sections 6 and 7 are devoted to ninth order mock theta functions and their generalizations containing Chebyshev polynomials of the third kind. For example,

$$\begin{aligned} &\sum_{n\geq 1} \frac{q^{n^2}\prod_{j=1}^{n-1}(1+2xq^j+q^{2j})}{(1-q)(1-q^2)\cdots(1-q^{2n-1})} \\ &= \prod_{n=1}^{\infty}\frac{1}{1-q^n} \sum_{m\geq 1} q^{2m^2-m}(1-q^{2m}) \sum_{j=0}^{m-1} V_j(x) q^{-j(j+1)/2}. \end{aligned} \tag{1.3}$$

To round out a full treatment of (1.1), we provide a natural interpretation of (1.1) related to sequences in partitions [8] in Section 8.

Section 9 considers a companion to (1.1) arising from the quintuple product identity, and Section 10 considers open questions.

Although he did not continue the terminology of Chebyshev polynomials, L. J. Rogers [15] tacitly used them in his combinations of Fourier series. We shall describe this relationship in Section 10.

2 A New Proof of Dyson's Favorite Identity

The identities to be proved are the following:

$$D_{4,4}(a;q) := \sum_{n\geq 0} \frac{a^n q^{n^2}(aq^{2n};q)_\infty}{(q;q)_n(aq^{3n};q^3)_\infty} = Q_{4,4}(a;q^3) \tag{2.1}$$

$$D_{4,3}(a;q) := \sum_{n\geq 0} \frac{a^n q^{n^2+n}(aq^{2n+2};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} = Q_{4,3}(a;q^3) \tag{2.2}$$

$$D_{4,2}(a;q) := \sum_{n\geq 0} \frac{a^n q^{n^2+2n}(aq^{2n+3};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} = Q_{4,2}(a;q^3) \tag{2.3}$$

$$D_{4,1}(a;q) := \sum_{n\geq 0} \frac{a^n q^{n^2+3n}(aq^{2n+3};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} = Q_{4,1}(a;q^3), \tag{2.4}$$

where

$$Q_{k,i}(a;q) = \sum_{n\geq 0} \frac{(-1)^n a^{kn} q^{\frac{1}{2}(k+1)n(n+1)-in}(1-a^i q^{(2n+1)i})}{(q;q)_n(aq^{n+1};q)_\infty} \tag{2.5}$$

with

$$(A;q)_N = (1-A)(1-Aq)\dots(1-Aq^{N-1}) \tag{2.6}$$

and

$$(A;q)_\infty = \prod_{n=0}^{\infty}(1-Aq^n). \tag{2.7}$$

It has been proved in [1] that the $Q_{k,i}(a;q)$ as doubly analytic functions in a and q are uniquely determined by the following initial conditions and q-difference equations:

$$Q_{k,0}(a;q) = 0, \tag{2.8}$$

$$Q_{k,i}(0;q) = Q_{k,i}(a;0) = 1 \quad \text{for } 1 \leq i \leq k, \tag{2.9}$$

and for $1 \leq i \leq k$

$$Q_{k,i}(a;q) - Q_{k,i-1}(a;q) = (aq)^{i-1} Q_{k,k-i+1}(aq;q). \tag{2.10}$$

Theorem 1. *For* $1 \leq i \leq 4$,

$$D_{4,i}(a;q) = Q_{4,i}(a;q^3). \tag{2.11}$$

Proof. In light of the comments proceeding (2.8), the proof of the theorem merely requires that (2.8)-(2.10) are established (with $q \to q^3$) for $D_{4,i}(a;q)$.

First, we note that (2.8) is by definition and (2.9) follows by inspection. Indeed we see also by inspection that

$$D_{4,1}(a;q) = D_{4,4}(aq^3;q) \tag{2.12}$$

which is (2.10) in the case $k=4$, $i=1$, $q \to q^3$.

Next

$$\begin{aligned} &D_{4,2}(a;q) - D_{4,1}(a;q) \\ &= \sum_{n\geq 0} \frac{a^n q^{n^2+2n}(aq^{2n+3};q)_\infty}{(q;q)_n(aq^{3n+3};q)_\infty} \\ &\quad - \sum_{n\geq 0} \frac{a^n q^{n^2+3n}(aq^{2n+3};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} \end{aligned} \tag{2.13}$$

$$
\begin{aligned}
&=\sum_{n\geq 0}\frac{a^n q^{n^2+2n}(1-q^n)(aq^{2n+3};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty}\\
&=\sum_{n\geq 1}\frac{a^n q^{n^2+2n}(aq^{2n+3};q^3)_\infty}{(q;q)_{n-1}(aq^{3n+3};q^3)_\infty}\\
&=\sum_{n\geq 0}\frac{a^{n+1} q^{n^2+4n+3}(aq^{2n+5};q)_\infty}{(q;q)_n(aq^{3n+6};q^3)_\infty}\\
&=aq^3 D_{4,3}(aq^3;q)
\end{aligned}
$$

which is (2.10) when $k=4$, $i=2$, and $q\to q^3$.

Before we proceed to the $k=4$, $i=3$ case, we shall make some observations about the simplicity of (2.13). Namely, we merely subtracted the two series term by term and the resulting new term was exactly what we wanted. As we move to $k=4$, $i=3$, we see that this simplicity doesn't exist. In order to produce this simplicity, we require the following intermediate functions

$$
M_{320}(a;q):=\sum_{n\geq 0}\frac{a^n q^{n^2+2n}(aq^{2n+2};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty}, \tag{2.14}
$$

$$
M_{321}(a;q):=\sum_{n\geq 0}\frac{a^n q^{n^2}(aq^{2n+1};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty}, \tag{2.15}
$$

$$
M_{322}(a;q):=\sum_{n\geq 0}\frac{a^n q^{n^2+n}(aq^{2n};q)_\infty}{(q;q)_n(aq^{3n};q^3)_\infty}. \tag{2.16}
$$

Hence

$$
\begin{aligned}
&D_{4,3}(a;q)-D_{4,2}(a;q) \qquad (2.17)\\
&=(D_{4,3}(a;q)-M_{320}(a;q))+(M_{320}(a;q)-D_{4,2}(a;q))\\
&=\sum_{n\geq 0}\frac{a^n q^{n^2+n}(1-q^n)(aq^{2n+2};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty}\\
&\quad+\sum_{n\geq 0}\frac{a^n q^{n^2+2n}(aq^{2n+3};q)_\infty((1-aq^{2n+2})-1)}{(q;q)_n(aq^{3n+3};q^3)_\infty}\\
&=\sum_{n\geq 0}\frac{a^{n+1} q^{n^2+3n+2}(aq^{2n+4};q)_\infty}{(q;q)_n(aq^{3n+6};q^3)_\infty}\\
&\quad-\sum_{n\geq 0}\frac{a^{n+1} q^{n^2+4n+2}(aq^{2n+3};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty}\\
&=aq^2(M_{321}(aq^3;q)-M_{322}(aq^3;q))\\
&=aq^2\sum_{n\geq 0}\frac{a^n q^{n^2+3n}(aq^{2n+4};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty}\times((1-aq^{3n+3})-q^n(1-aq^{2n+3}))
\end{aligned}
$$

$$=aq^2\sum_{n\geq 0}\frac{a^nq^{n^2+3n}(1-q^n)(aq^{2n+4};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty}$$
$$=aq^2\sum_{n\geq 0}\frac{a^{n+1}q^{n^2+5n+4}(aq^{2n+6};q)_\infty}{(q;q)_n(aq^{3n+6};q^3)_\infty}$$
$$=(aq^3)^2D_{4,2}(aq^3;q),$$

which establishes (2.10) in the case $k=4$, $i=3$, $q\to q^3$.

Note that in all steps the same simple term by term subtraction occurs. The intermediate functions provided the necessary component to allow this to take place.

To complete the final case of $k=4$, $i=4$, we require four new intermediate functions:

$$M_{432}(a;q)=\sum_{n\geq 0}\frac{a^nq^{n^2+n}(aq^{2n+1};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} \tag{2.18}$$

$$M_{433}(a;q)=\sum_{n\geq 0}\frac{a^nq^{n^2+2n}(aq^{2n+2};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} \tag{2.19}$$

$$M_{434}(a;q)=\sum_{n\geq 0}\frac{a^nq^{n^2+2n}(aq^{2n};q)_\infty}{(q;q)_n(aq^{3n};q^3)_\infty} \tag{2.20}$$

$$M_{435}(a;q)=\sum_{n\geq 0}\frac{a^nq^{n^2+3n}(aq^{2n+2};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} \tag{2.21}$$

This final step is sufficiently intricate that we shall first split it into the several term by term subtractions that are straightforward.

$$D_{4,4}(a;q)-M_{322}(a;q)=aqM_{433}(a;q) \tag{2.22}$$
$$M_{322}(a;q)-M_{432}(a;q)=-a^2q^4M_{434}(aq^3;q) \tag{2.23}$$
$$M_{432}(a;q)-D_{4,3}(a;q)=-aqM_{435}(a;q) \tag{2.24}$$
$$M_{433}(a;q)-M_{435}(a;q)=aq^3M_{432}(aq^3;q) \tag{2.25}$$
$$M_{432}(aq^3;q)-M_{434}(aq^3;q)=aq^5D_{41}(aq^3;q) \tag{2.26}$$

Each of (2.22) is proved simply using term by term subtraction.

$$D_{4,4}(a;q)-M_{322}(a;q)$$
$$=\sum_{n\geq 0}\frac{a^nq^{n^2}(aq^{2n};q)_\infty(1-q^n)}{(q;q)_n(aq^{3n};q^3)_\infty}$$
$$=aq\sum_{n\geq 0}\frac{a^nq^{n^2+2n}(aq^{2n+2};q)_\infty}{(q;q)_n(aq^{3n};q^3)_\infty}$$
$$=aqM_{433}(a;q);$$

$$
\begin{aligned}
&M_{322}(a;q)-M_{432}(a;q)\\
=&\sum_{n\geq 0}\frac{a^nq^{n^2+n}(aq^{2n+1};q)_\infty((1-aq^{2n})-(1-aq^{3n}))}{(q;q)_n(aq^{3n};q^3)_\infty}\\
=&-a^2q^4\sum_{n\geq 0}\frac{a^nq^{n^2+5n}(aq^{2n+3};q)_\infty}{(q;q)_n(aq^{3n+3};q)_\infty}\\
=&-a^2q^4M_{434}(aq^3;q);\\
&M_{432}(a;q)-D_{4,3}(a;q)\\
=&\sum_{n\geq 0}\frac{a^nq^{n^2+n}(aq^{2n+2};q)_\infty((1-aq^{2n+1})-1)}{(q;q)_n(aq^{3n+3};q^3)_\infty}\\
=&-aqM_{435}(a;q);\\
&M_{433}(a;q)-M_{435}(a;q)\\
=&\sum_{n\geq 0}\frac{a^nq^{n^2+2n}(aq^{2n+2};q)_\infty(1-q^n)}{(q;q)_n(aq^{3n+3};q^3)_\infty}\\
=&aq^3\sum_{n\geq 0}\frac{a^nq^{n^2+4n}(aq^{2n+4};q)_\infty}{(q;q)_n(aq^{3n+6};q^3)_\infty}\\
=&aq^3M_{432}(aq^3;q),
\end{aligned}
$$

and finally

$$
\begin{aligned}
&M_{432}(aq^3;q)-M_{434}(aq^3;q)\\
=&\sum_{n\geq 0}\frac{a^nq^{n^2+4n}(aq^{2n+4};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty}((1-aq^{3n+3})-q^n(1-aq^{2n+3}))\\
=&\sum_{n\geq 0}\frac{a^nq^{n^2+4n}(aq^{2n+4};q)_\infty(1-q^n)}{(q;q)_n(aq^{3n+3};q^3)_\infty}\\
=&aq^5\sum_{n\geq 0}\frac{a^nq^{n^2+6n}(aq^{2n+6};q)_\infty}{(q;q)_n(aq^{3n+6};q^3)_\infty}\\
=&aq^5D_{4,1}(aq^3;q).
\end{aligned}
$$

Hence

(2.27)
$$
\begin{aligned}
&D_{4,4}(a;q)-D_{4,3}(a;q)\\
=&(D_{4,4}(a;q)-M_{322}(a;q))\\
&+(M_{322}(a;q)-M_{432}(a;q))\\
&+(M_{432}(a;q)-D_{4,3}(a;q))\\
=&aq(M_{433}(a;q)-M_{435}(a;q))\\
&-a^2q^4M_{434}(aq^3;q)\qquad\text{(by (2.22), (2.23) and (2.24))}
\end{aligned}
$$

$$
\begin{aligned}
=&a^2q^4(M_{432}(aq^3;q) - M_{434}(aq^3;q)) \qquad \text{(by (2.25))}\\
=&a^3q^9D_{4,1}(aq^3;q) \qquad \text{(by (2.26))}.
\end{aligned}
$$

□

Now recall that [1, p. 408]

$$
Q_{k,i}(1;q) = \frac{(q^i;q^{2k+1})_\infty(q^{2k+1-i};q^{2k+1})_\infty(q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty}. \tag{2.28}
$$

Hence Dyson's favorite identity follows from Theorem 1.

Corollary 2. *Identity* (1.1) *is valid.*

Proof. Take $a = 1$, $i = 3$ in Theorem 1 and simplify. □

In passing, we note the following instances of Theorem 1; $a = 1$, $i = 4$ yields [17, p. 162, eq. (93)]; $a = 1$, $i = 2$, yields [17, p. 161, eq. (91)], and $a = 1$, $i = 1$, yields [17, p. 161, eq. (90)]. See also [16, p. 109].

These observations raise the question: *Are there identities of interest for the $M_{xyz}(a;q)$ functions when $a = 1$?*

To answer this question requires that we take a short detour to study Chebyshev polynomials of the third and fourth kind.

3 Bailey Pairs and Chebyshev Polynomials

In the past, q-orthogonal polynomials have played an important role in the study of Rogers-Ramanujan type identities and mock theta functions [4], [5], [9].

The surprise here is that classical Chebyshev polynomials (NOT q-Chebyshev) play a central role in studying the $M_{xyz}(a;q)$ introduced in the previous section.

We recall that a sequence of pairs of rational functions $(\alpha_n, \beta_n)_{n\geq 0}$ is called a Bailey pair with respect to a provided [2, p. 278]

$$
\beta_n = \sum_{j=0}^{n} \frac{\alpha_j}{(q;q)_{n-j}(aq;q)_{n+j}}. \tag{3.1}
$$

The identity (3.1) can be inverted [2, p. 278, eq. (4.1)] to the equivalent formulation:

$$
\alpha_n = \frac{(1-aq^{2n})}{(1-a)} \sum_{j=0}^{n} \frac{(a;q)_{n+j}(-1)^{n-j}q^{\binom{n-j}{2}}\beta_j}{(q;q)_{n-j}} \tag{3.2}
$$

In the following we shall also need the q-binomial coefficients;

$$
\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} 0 & \text{if } B < 0 \text{ or } B > A \\ \frac{(q;q)_A}{(q;q)_B(q;q)_{A-B}} & \text{otherwise.} \end{cases} \tag{3.3}
$$

The Chebyshev polynomials of the third kind, $V_n(x)$ are given by [14, p.170]

$$V_n(x) = \begin{cases} 1 & \text{if } n = 0 \\ 2x - 1 & \text{if } n = 1 \\ 2xV_{n-1}(x) - V_{n-2}(x) & \text{if } n > 1 \end{cases} \tag{3.4}$$

The Chebyshev polynomials of the fourth kind, $W_n(x)$ are given by [14, p. 170]

$$W_n(x) = \begin{cases} 1 & \text{if } n = 0 \\ 2x + 1 & \text{if } n = 1 \\ 2xW_{n-1}(x) - W_{n-2}(x) & \text{if } n > 1. \end{cases} \tag{3.5}$$

It is a simple exercise to show that

$$W_n(x) = (-1)^n V_n(-x). \tag{3.6}$$

We choose to use both $W_n(x)$ and $V_n(x)$ for simplicity of notation.

Our object in the section is to fit these Chebyshev polynomials into very natural Bailey pairs.

Theorem 3.

$$\prod_{j=1}^{n}(1 + 2xq^j + q^{2j}) = \sum_{j=0}^{n} q^{\binom{j+1}{2}} V_j(x) \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix}. \tag{3.7}$$

Remark. Identity (3.7) is equivalent to saying that

$$\left(\frac{q^{\binom{n+1}{2}} V_n(x)}{1-q}, \frac{\prod_{j=1}^{n}(1 + 2xq^j + q^{2j})}{(q;q)_{2n+1}} \right) \tag{3.8}$$

forms a Bailey pair at $a = q$.

Proof. Let us denote the left side of (3.7) by $L_n(x)$. Then it is immediate that $L_n(x)$ is uniquely defined by

$$L_n(x) = \begin{cases} 1 & \text{if } n = 0 \\ (1 + 2xq^n + q^{2n})L_{n-1}(x) & \text{if } n > 0. \end{cases} \tag{3.9}$$

Let us denote the right side of (3.7) by $R_n(x)$. Clearly $R_0(x) = 1$. So to conclude that $L_n(x) = R_n(x)$ we need only show that for $n > 0$

$$2xq^n R_{n-1}(x) = R_n(x) - (1 + q^{2n})R_{n-1}(x). \tag{3.10}$$

Now by (3.4),

$$2xV_j(x) = V_{j+1}(x) + V_{j-1}(x).$$

Hence we must show that

$$\begin{aligned}(3.11)\qquad & q^n \sum_{j=0}^{n} q^{\binom{j+1}{2}} (V_{j+1}(x) + V_{j-1}(x)) \begin{bmatrix} 2n-1 \\ n-1-j \end{bmatrix} \\ =& \sum_{j\geq 0} V_j(x) q^{\binom{j+1}{2}} \left(\begin{bmatrix} 2n+1 \\ n-j \end{bmatrix} - (1+q^{2n}) \begin{bmatrix} 2n-1 \\ n-1-j \end{bmatrix} \right)\end{aligned}$$

Now the $V_n(x)$ form a basis for the polynomials in x. Consequently the coefficients of $V_j(x)$ on both sides of (3.11) must coincide. Thus we need only prove

$$\begin{aligned}(3.12)\qquad & q^{n+\binom{j}{2}} \begin{bmatrix} 2n-1 \\ n-j \end{bmatrix} + q^{n+\binom{j+2}{2}} \begin{bmatrix} 2n-1 \\ n-2-j \end{bmatrix} \\ =& q^{\binom{j+1}{2}} \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix} - (1+q^{2n}) \begin{bmatrix} 2n-1 \\ n-j-1 \end{bmatrix},\end{aligned}$$

and multiplying both sides of (3.12) by $(q;q)_{n-j}(q;q)_{n-j+1}/(q;q)_{2n-1}$, we see that we finally must prove

$$\begin{aligned}(3.13)\qquad & q^{n+\binom{j}{2}} (1-q^{n-j})(1-q^{n-j+1}) \\ & + q^{n+\binom{j+2}{2}} (1-q^{n-j})(1-q^{n-j+1}) \\ =& q^{\binom{j+1}{2}} (1-q^{2n+1})(1-q^{2n}) \\ & - (1+q^{2n})(1-q^{n-j})(1-q^{n+j+1}) q^{\binom{j+1}{2}},\end{aligned}$$

and one easily expands the expressions in (3.13) to determine its validity and the truth of (3.7). □

Theorem 4.

$$(3.14)\qquad \prod_{j=0}^{n-1} (1+2xq^j+q^{2j}) = \sum_{j=0}^{n} q^{\binom{j}{2}} W_j(x)(1-q^{2j+1}) \frac{(q;q)_{2n}}{(q;q)_n (q;q)_{n+j+1}}$$

Remark. Identity (3.14) is equivalent to saying that

$$(3.15)\qquad \left(\frac{q^{\binom{n}{2}}(1-q^{2n+1})W_n(x)}{1-q}, \frac{\prod_{j=0}^{n-1}(1+2xq^j+q^{2j})}{(q;q)_{2n}} \right)$$

forms a Bailey pair at $a = q$. Also note that this result closely resembles Theorem 3. The change in the second entry of the Bailey pair is the shift in the index j and a shorter product in the denominator.

Proof. As in Theorem 3, we denote the left side of (3.14) by $L_n(x)$. Then it is immediate that $L_n(x)$ is uniquely defined by

$$L_n(x) = \begin{cases} 1 & \text{if } n = 0 \\ (1 + 2xq^{n-1} + q^{2n-2})L_{n-1}(x) & \text{if } n > 0. \end{cases} \tag{3.16}$$

We now denote the right side of (3.14) by $R_n(x)$. Clearly $R_0(x) = 1$. To conclude the proof that $L_n(x) = R_n(x)$ we need only show that for $n > 0$

$$2xq^{n-1}R_{n-1}(x) = R_n(x) - (1 + q^{2n-2})R_{n-1}(x). \tag{3.17}$$

Now by (3.5),

$$2xW_j(x) = W_{j+1}(x) + W_{j-1}(x). \tag{3.18}$$

Hence we must show that

(3.19)

$$\begin{aligned} &q^{n-1}\sum_{j=0}^{n} q^{\binom{j}{2}}(W_{j+1}(x) + W_{j-1}(x))(1 - q^{2j+1})\frac{(q;q)_{2n-2}}{(q;q)_{n-j-1}(q;q)_{n+j}} \\ =&\sum_{j\geq 0} q^{\binom{j}{2}}W_j(x)(1 - q^{2j+1})\left(\frac{(q;q)_{2n}}{(q;q)_{n-j}(q;q)_{n+j+1}} - \frac{(1 + q^{2n-2})(q;q)_{2n-2}}{(q;q)_{n-1-j}(q;q)_{n+j}}\right) \end{aligned}$$

As before, the $W_n(x)$ form a basis for the polynomials in x. Consequently the coefficients of $W_j(x)$ on both sides of (3.19) must coincide. Thus we need only prove that

$$\begin{aligned} &q^{n-1+\binom{j-1}{2}}\left(\frac{(1 - q^{2j-1})(q;q)_{2n-2}}{(q;q)_{n-j}(q;q)_{n+j-1}} + q^{2j-1}\frac{(1 - q^{2j+3})(q;q)_{2n-2}}{(q;q)_{n-1-2}(q;q)_{n+j+1}}\right) \\ =&q^{\binom{j}{2}}(1 - q^{2j+1})\left(\frac{(q;q)_{2n}}{(q;q)_{n-j}(q;q)_{n+j+1}} - \frac{(1 + q^{2n-2})(q;q)_{2n-2}}{(q;q)_{n-1-j}(q;q)_{n+j}}\right) \end{aligned} \tag{3.20}$$

and multiplying both sides of (3.20) by $(q;q)_{n-j}(q;q)_{n+j+1}/(q)_{2n-2}$ we see that we finally must prove

(3.21)

$$\begin{aligned} &q^{n-1+\binom{j-1}{2}}(1 - q^{n+j})(1 - q^{n+j+1})(1 - q^{2j-1}) \\ &+ q^{n-1+\binom{j+1}{2}}(1 - q^{n-j-1})(1 - q^{n-j})(1 - q^{2j+3}) \\ =&q^{\binom{j}{2}}(1 - q^{2j+1})\big((1 - q^{2n})(1 - q^{2n-1}) - (1 + q^{2n-2})(1 - q^{n-j})(1 - q^{n+j+1})\big), \end{aligned}$$

and one easily expands the expressions in (3.21) to determine its validity and the proof of (3.14). □

4 Generalizing Dyson's Favorite Identity

This section will serve as a prototype for the types of identities that can be derived using the Bailey pairs given in section 3.

Let us recall the weak form of Bailey's Lemma in the case $a = q$ [3, Th. 2]. Namely, if (α_n, β_n) form a Bailey pair at $a = q$, then

$$\sum_{n\geq 0} q^{n^2+n}\beta_n = \frac{1}{(q^2;q)_\infty}\sum_{n\geq 0} q^{n^2+n}\alpha_n. \tag{4.1}$$

Inserting the Bailey pair from (3.8) into (4.1), we obtain

$$\begin{aligned}\sum_{n\geq 0}\frac{q^{n^2+n}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q;q)_{2n+1}} \\ =\frac{1}{(q;q)_\infty}\sum_{n\geq 0} q^{3\binom{n+1}{2}}V_n(x).\end{aligned} \tag{4.2}$$

Lemma 5. For $n \geq 0$

$$V_n(-1) = (-1)^n(2n+1) \tag{4.3}$$

$$V_n\left(-\frac{1}{2}\right) = \begin{cases} -2 & \text{if } n\equiv 1 \pmod 3 \\ 1 & \text{otherwise}\end{cases} \tag{4.4}$$

$$V_n(0) = \begin{cases} 1 & \text{if } n\equiv 0,3 \pmod 4 \\ -1 & \text{otherwise}\end{cases} \tag{4.5}$$

$$V_n\left(\frac{1}{2}\right) = \begin{cases} 1 & \text{if } n\equiv 0,5 \pmod 6 \\ 0 & \text{if } n\equiv 1,4 \pmod 6 \\ -1 & \text{if } n\equiv 2,3 \pmod 6\end{cases} \tag{4.6}$$

$$V_n(1) = 1 \tag{4.7}$$

$$V_n\left(\frac{3}{2}\right) = F_{2n+1} \tag{4.8}$$

$$V_n\left(-\frac{3}{2}\right) = (-1)^n L_{2n+1}, \tag{4.9}$$

where F_n and L_n are the Fibonacci and Lucas numbers.

Proof. Each of (4.3)-(4.9) is a simple exercise in mathematical induction using the initial values and recurrence from (3.4). □

Theorem 6. *Identity* (1.1)*, Dyson's "favorite identity" is valid.*

Proof. By (4.2) with $x = \frac{1}{2}$

$$\begin{aligned}
&\sum_{n\geq 0} \frac{q^{n^2+n} \prod_{j=1}^{n} (1+q^j+q^{2j})}{(q;q)_{2n+1}} \\
=&\frac{1}{(q;q)_\infty} \left(\sum_{n\geq 0} \left(q^{3\binom{6n+1}{2}} + q^{3\binom{6n+6}{2}} - q^{3\binom{6n+3}{2}} - q^{3\binom{6n+4}{2}} \right) \right) \\
=&\frac{1}{(q;q)_\infty} \left(\sum_{n=-\infty}^{\infty} \left(q^{3\binom{6n+1}{2}} - q^{3\binom{6n+3}{2}} \right) \right) \\
=&\frac{1}{(q;q)_\infty} \sum_{n\geq 0} q^{9n(3n+1)/2} (1 - q^{9(2n+1)}) \\
=&\frac{1}{(q;q)_\infty} (q^9;q^9)_\infty \qquad \text{(by [2, p. 22, Cor. 2.9]).}
\end{aligned}$$

□

We isolated (1.1) and gave a detailed proof. The remaining values of $V_n(x)$ given in Lemma 5 yield the following identities.

Theorem 7.

$$\sum_{n\geq 0} \frac{q^{n^2+n}(q;q)_n}{(q^{n+1};q)_{n+1}} = \frac{(q^3;q^3)^3}{(q;q)_\infty}, \tag{4.10}$$

$$\sum_{n\geq 0} \frac{q^{n^2+n}(-q^3;q^3)_n}{(-q;q)_n (q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \left(\psi(q^3) - 3q^3 \psi(q^{27}) \right) \tag{4.11}$$

$$\sum_{n\geq 0} \frac{q^{n^2+n}(-q^2;q^2)_n}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} q^{24n^2+6n} (1 - q^{12n+3}) \tag{4.12}$$

$$\sum_{n\geq 0} \frac{q^{n^2+n}(-q;q)^2}{(q;q)_{2n+1}} = \frac{\psi(q^3)}{(q;q)_\infty} \tag{4.13}$$

$$\sum_{n\geq 0} \frac{q^{n^2+n} \prod_{j=1}^{n} (1+3q^j+q^{2j})}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \sum_{n\geq 0} q^{3\binom{n+1}{2}} F_{2n+1} \tag{4.14}$$

and

$$\sum_{n\geq 0} \frac{q^{n^2+n} \prod_{j=1}^{n} (1-3q^j+q^{2j})}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \sum_{n\geq 0} q^{3\binom{n+1}{2}} (-1)^n L_{2n+1} \tag{4.15}$$

where

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}}. \tag{4.16}$$

Remark. Of these six identities, (4.10) appears in [7], and (4.13) is from [16, p. 154, eq. (22)]. The remaining four appear to be new.

Proof. Each of these identities follows from direct substitution of the values given for $V_n(x)$ in Lemma 5 into (4.2). The only instance where an auxiliary result is use is (4.10) which additionally requires Jacobi's famous result [2, p.176]

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\binom{n+1}{2}} = (q;q)_\infty^3. \tag{4.17}$$

□

5 Implications of Theorem 4

Just as Theorem 3 provided seven corollaries, so too does Theorem 4. We begin by inserting the Bailey pair from (3.15) into (4.1) to obtain

$$\sum_{n\geq 0} \frac{q^{n^2+n}\prod_{j=0}^{n-1}(1+2xq^j+q^{2j})}{(q;q)_{2n}} \tag{5.1}$$

$$= \frac{1}{(q;q)_\infty}\sum_{n\geq 0} q^{n(3n+1)/2}(1-q^{2n+1})W_n(x). \tag{5.2}$$

In light of (3.6), we can use Lemma 5 to provide the special evaluation of $W_n(x)$.

Theorem 8.

(5.3)

$$\sum_{n\geq 0} \frac{q^{n^2+n}\prod_{j=0}^{n-1}(1-3q^j+q^{2j})}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty}\sum_{n\geq 0} q^{n(3n+1)/2}(1-q^{2n+1})(-1)^n F_{2n+1}$$

(5.4)

$$1 = \frac{1}{(q;q)_\infty}\sum_{n\geq 0} q^{n(3n+1)/2}(1-q^{2n+1})(-1)^n$$

(5.5)

$$1+\sum_{n\geq 1}\frac{q^{n^2+n}(-q^3;q^3)_{n-1}}{(-q;q)_{n-1}(q;q)_{2n}}$$

$$=\frac{-1}{(q;q)_\infty}\sum_{n\geq 0}q^{n(3n+1)/2}(1-q^{2n+1})\left(\frac{n-1}{3}\right)$$

$$=\frac{1}{(q;q)_\infty}(1-q-q^7+q^{12}+q^{15}-q^{22}-\cdots)$$

(5.6)

$$\sum_{n\geq 0}\frac{q^{n^2+n}(-1;q^2)_n}{(q;q)_{2n}}$$

$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{n(3n+1)/2}\chi_4(n)$$

(where $\chi_4(n)=+1$ *if* $n\equiv 0,1(\mod 4)$ *and* -1 *otherwise*

(5.7)

$$1+\sum_{n\geq 1}\frac{q^{n^2+n}(q^3;q^3)_{n-1}}{(q;q)_{n-1}(q;q)_{2n}}=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{n(3n+1)/2}\chi_{12}(n),$$

(where $\chi_{12}(n)=+1$ *if* $n\equiv 0,1,7(\mod 12)$, -1 *if* $n\equiv 4,10(\mod 12)$, 0 *otherwise)*

(5.8)

$$\sum_{n\geq 0}\frac{q^{n^2+n}(-1;q)_n^2}{(q;q)_{2n}}=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(2n+1)^{n(3n+1)/2}$$

(5.9)

$$\sum_{n\geq 0}\frac{q^{n^2+n}\prod_{j=0}^{n-1}(1+3q^j+q^{2j})}{(q;q)_{2n}}$$

$$=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{n(3n+1)/2}(1-q^{2n+1})L_{2n+1}$$

Proof. Each of these seven identities follows successively from the instances $x=-\frac{3}{2},-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2}$ of $W_n(x)$ in (5.2). □

6 Generalized Hecke Series

In the previous sections we have restricted attention to results where Chebyshev polynomials have been inserted into the theta-type series, e.g. (4.2) and (5.3). In this section we consider a similar phenomenon related to Hecke-type double series involving indefinite quadratic forms.

Throughout this section we will require instances of the following identity

$$\sum_{n\geq 0}\frac{q^{n^2+\alpha n}}{(q;q)_n(q;q)_{n+\beta}}=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}\sum_{n\geq 0}\frac{(q^{\alpha-\beta};q)_n(-1)^n q^{\beta n+\binom{n+1}{2}}}{(q;q)_n},\tag{6.1}$$

which follows from Heine's second transformation [12, p. 241, eq. (III.2), $a=b=\frac{1}{\tau}$, $z=q^{\alpha+1}\tau^2$, $c=q^{\beta+1}$, and $\tau\to 0$].

Theorem 9.

$$\sum_{n\geq 0}\frac{q^{n^2}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q;q)_{2n}}\tag{6.2}$$
$$=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{2n^2+n}(1-q^{6n+6})\sum_{j=0}^{n}V_j(x)q^{-\binom{j+1}{2}}$$

Proof. By (3.7)

$$\sum_{n\geq 0}\frac{q^{n^2}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q;q)_{2n}}$$
$$=\sum_{n\geq 0}\frac{q^{n^2}\sum_{j=0}^{n}q^{\binom{j+1}{2}}V_j(x)\begin{bmatrix}2n+1\\ n-j\end{bmatrix}}{(q;q)_{2n}}\tag{6.3}$$

Thus to prove Theorem 9, we need only identify the coefficients of $V_j(x)$ in (6.3) with those in (6.2). Namely, we must prove

$$\sum_{n\geq j}\frac{q^{n^2+\binom{j+1}{2}}\begin{bmatrix}2n+1\\ n-j\end{bmatrix}}{(q;q)_{2n}}\tag{6.4}$$
$$=\frac{1}{(q;q)_\infty}\sum_{n\geq j}q^{2n^2+n-\binom{j+1}{2}}(1-q^{6n+6})$$

Now

$$\sum_{n\geq j}\frac{q^{n^2+\binom{j+1}{2}}\begin{bmatrix}2n+1\\ n-j\end{bmatrix}}{(q;q)_{2n}}\tag{6.5}$$
$$=q^{j^2+\binom{j+1}{2}}\sum_{n\geq 0}\frac{q^{n^2+2nj}(1-q^{2n+2j+1})}{(q;q)_n(q;q)_{n+2j+1}}$$

$$= \frac{q^{j^2+\binom{j+1}{2}}}{(q;q)_\infty}\left(\sum_{n\geq 0}\frac{(q^{-1};q)_n}{(q;q)_n}(-1)^n q^{(2j+1)n+\binom{n+1}{2}}\right.$$
$$\left. - q^{2j+1}\sum_{n\geq 0}\frac{(q;q)_n}{(q;q)_n}(-1)^n q^{(2j+1)n+\binom{n+1}{2}}\right) \qquad \text{(by (6.1))}$$
$$= \frac{q^{j^2+\binom{j+1}{2}}}{(q;q)_\infty}\left(1+q^{2j+1}-q^{2j+1}\sum_{n\geq 0}(-1)^n q^{\binom{n+1}{2}+(2j+1)n}\right)$$

Comparing the right sides of (6.4) and (6.5), we see (shifting $n \to n+j$ on the right of (6.4)) that to complete the proof, we must show that

$$\sum_{n\geq 0} q^{2n^2+4nj+n}(1-q^{6n+6j+6}) = 1 - q^{2j+1}\sum_{n\geq 1}(-1)^n q^{\binom{n+1}{2}+(2j+1)n}, \tag{6.6}$$

and

$$1 - q^{2j+1}\sum_{n\geq 1}(-1)^n q^{\binom{n+1}{2}+(2j+1)n}$$
$$= 1 + \sum_{n\geq 1} q^{\binom{2n}{n}+2n(2j+1)} - \sum_{n\geq 0} q^{(2j+1)+\binom{2n+3}{2}+(2j+1)(2n+2)}$$
$$= \sum_{n\geq 0} q^{2n^2+4nj+n} - \sum_{n\geq 0} q^{2n^2+4nj+7n+6j+6}$$
$$= \sum_{n\geq 0} q^{2n^2+4nj+n}(1-q^{6n+6j+6}),$$

thus (6.6) is proved and with it (6.2). ☐

Theorem 10.

$$\sum_{n\geq 0}\frac{q^{n^2+2n}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty}\sum_{n\geq 0} q^{2n^2+3n}(1-q^{2n+2})\sum_{j=0}^{n} V_j(x) q^{-\binom{j+1}{2}} \tag{6.7}$$

Proof. This is proved exactly as Theorem 9 is proved; so we suppress some of the more tedious details

$$\sum_{n\geq 0}\frac{q^{n^2+2n}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q;q)_{2n+1}} \tag{6.8}$$

$$= \sum_{n\geq 0} \frac{q^{n^2+2n}}{(q;q)_{2n+1}} \sum_{j=0}^{n} q^{\binom{j+1}{2}} V_j(x) \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix} \qquad \text{(by (3.7))}$$

Thus to prove (6.7) we must check that the coefficients of $V_j(x)$ in (6.7) and (6.8) coincide. Hence we must prove

$$\sum_{n\geq j} \frac{q^{n^2+2n+\binom{j+1}{2}}}{(q;q)_{n-j}(q;q)_{n+j+1}} = \frac{1}{(q;q)_\infty} \sum_{n\geq j} q^{2n^3+3n-\binom{j+1}{2}}(1-q^{2n+2}), \tag{6.9}$$

and identity (6.9) is proved by applying (6.1) to the left side. □

7 Ninth Order Mock Theta Functions

In this section we shall study some of the series arising as intermediate functions in section 2. We now remove the infinite products from the $M_{xyz}(a;q)$. Namely

$$m_{xyz}(a;q) = \frac{(a;q)_\infty}{(a;q^3)_\infty} M_{xyz}(a;q) \tag{7.1}$$

Theorem 11.

$$m_{320}(1;q) := \sum_{n\geq 0} \frac{q^{n^2+2n}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}} \tag{7.2}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{2n^2+3n}(1-q^{2n+2}) \sum_{j=-\lfloor \frac{n+1}{3} \rfloor}^{\lfloor \frac{n}{3} \rfloor} (-1)^j q^{(-3j(3j+1)/2)}$$

Proof. In Theorem 10, set $x = \frac{1}{2}$. Then prove

$$\sum_{j=0}^{n} V_j\left(\frac{1}{2}\right) q^{-j(j+1)/2} = \sum_{j=-\lfloor \frac{n+1}{3} \rfloor}^{\lfloor \frac{n}{3} \rfloor} (-1)^j q^{(-3j(3j+1)/2)} \tag{7.3}$$

by mathematical induction using (4.6) . □

Theorem 12.

$$m_{321}(1;q) := \sum_{n\geq 0} \frac{q^{n^2}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n}} \tag{7.4}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{2n^2+n}(1-q^{6n+6}) \sum_{j=-\lfloor \frac{n+1}{3} \rfloor}^{\lfloor \frac{n}{3} \rfloor} (-1)^j q^{-3j(3j+1)/2} \tag{7.5}$$

Proof. In Theorem 9, set $x = \frac{1}{2}$, and invoke (7.3). □

Lemma 13.

$$\sum_{n\geq 1}\frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_{n-1}(q;q)_{2n-1}}$$
$$=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{2n^2+3n+1}(1-q^{2n+2})\sum_{j=-\lfloor\frac{n+1}{3}\rfloor}^{\lfloor\frac{n}{3}\rfloor}(-1)^j q^{-3j(3j+1)/2}$$

Proof. Shift n to $n-1$ on the left side of (7.2) and multiply by q. ☐

Theorem 14.

$$m_{322}(1;q)=1+\sum_{n\geq 0}\frac{q^{n^2+n}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}}$$
$$=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{2n^2+n}(1-q^{2n+1})\sum_{j=-\lfloor\frac{n}{3}\rfloor}^{\lfloor\frac{n}{3}\rfloor}(-1)^j q^{-3j(3j+1)/2}$$

Proof. Define $d_{4,i}(a;q)=\frac{(a;q)_\infty}{(a;q)_\infty}d_{4,i}(a;q)$. Then by [16, p. 162, eq. (93)]

$$d_{4,4}(1;q)=1+\sum_{n\geq 1}\frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}} \tag{7.6}$$
$$=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}(-1)^n q^{3(9n^2+n)/2}(1-q^{24n+12})$$

So

$$d_{4,4}(1;q^3)-m_{322}(1;q)=\sum_{n\geq 1}\frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_{n-1}(q;q)_{2n-1}}; \tag{7.7}$$

thus by (7.6) and Lemma 13, $m_{322}(1;q)$ is equal to

$$\frac{1}{(q;q)_\infty}\Bigg(\sum_{n\geq 0}(-1)^n q^{(27n^2+3n)/2}(1-q^{24n+12}) \tag{7.8}$$
$$-\sum_{n\geq 0}q^{2n^2+3n+1}(1-q^{2n+2})\sum_{j=-\lfloor\frac{n+1}{3}\rfloor}^{\lfloor\frac{n}{3}\rfloor}(-1)^j q^{-3j(3j+1)/2}\Bigg)$$
$$=\frac{1}{(q;q)_\infty}\Bigg(\sum_{n\geq 0}(-1)^n q^{(27n^2+3n)/2}(1-q^{24n+12})$$
$$-\sum_{n\geq 0}q^{2n^2+3n+1}\sum_{j=-\lfloor\frac{n+1}{3}\rfloor}^{\lfloor\frac{n}{3}\rfloor}(-1)^j q^{-3j(3j+1)/2}$$

$$
\begin{aligned}
&+\sum_{n\geq 1} q^{2n^2+n} \sum_{j=-\lfloor \frac{n}{3} \rfloor}^{\lfloor \frac{n}{3} \rfloor} (-1)^j q^{-3j(3j+1)/2} \Bigg) \\
=&\frac{1}{(q;q)_\infty}\Bigg(\sum_{n\geq 0} (-1)^n q^{(27n^2+3n)/2} (1-q^{24n+12} \\
&-\sum_{n\geq 0} q^{2n^2+3n+1} \sum_{j=-\lfloor \frac{n}{3} \rfloor}^{\lfloor \frac{n}{3} \rfloor} (-1)^j q^{-3k(3j+1)/2} \\
&-\sum_{n\geq 0} q^{2(3n+2^2+3(3n+2)+1} (-1)^{-n-1} q^{-3(-n-1)(3(-n-1)+1)/2} \\
&+\sum_{n\geq 0} q^{2n^2+n} \sum_{j=-\lfloor \frac{n}{3} \rfloor}^{\lfloor \frac{n}{3} \rfloor} (-1)^j q^{-3j(3j+1)/2} \\
&-\sum_{n\geq 0} q^{2(3n)^2+3n} (-1)^n q^{-3n(3n+1)/2} \\
=&\frac{1}{(q;q)_\infty} \sum_{n\geq 0} q^{2n^2+n} (1-q^{2n+1}) \sum_{j=-\lfloor \frac{n}{3} \rfloor}^{\lfloor \frac{n}{3} \rfloor} (-1)^j q^{-3j(3j+1)/2}.
\end{aligned}
$$

□

We remark that Ian Wagner has studied these and many related functions in his Ph.D. thesis (directed by Ken Ono). He observes that some are mock theta functions (Theorem 12 and 13), and some are "near misses" (Theorem 15).

8 Partition Identities

Let us recall B. Gordon's celebrated generalization of the Rogers-Ramanujan identities [13] (cf. [1]).

Gordon's Theorem. *Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i (\mod 2k+1)$. Let $B_{k,i}(n)$ denote the number of partitions of n of the form $\lambda_1 + \lambda_2 + \ldots + \lambda_s$, where $\lambda_j - \lambda_{j+k-1} \geq 2$ and $\lambda_1 \geq \lambda_2 \leq \ldots \lambda_k$ and, in addition, at most $i-1$ of the λ_j are equal to 1. Then for $1 \leq i \leq k$, $n \geq 0$,*

$$A_{k,i}(n) = B_{k,i}(n).$$

The simplest proof [1] of Gordon's theorem reveals that $Q_{k,i}(z;q)$ is the generating function for partitions of the $B_{k,i}$-type where the exponent of z counts the number of parts. Thus

$$Q_{k,i}(1;q) = \sum_{n\geq 0} B_{k,i}(n) q^n,$$

and the result follows by invoking (2.28).

Now one could assume that nothing more needs to be said about $Q_{4,i}(1;q^3)$. After all this is just Gordon's theorem at $k=4$ with all parts multiplied by 3.

However, the work in [8] points to a natural alternative interpretation of partitions generated by $Q_{4,i}(x;q^3)$. In the new interpretation, we begin with R, the set of all integer partitions in which parts differ by at least 2. We also speak of *maximal sequences* of parts in such partitions. We shall say that in the partition

$$\lambda_1 + \lambda_2 + \cdots + \lambda_s \qquad (\lambda_i - \lambda_{i+1} \geq 2)$$

that

$$\lambda_m + \lambda_{m+1} + \ldots + \lambda_{m+j}$$

is maximal if $\lambda_{m-1} - \lambda_m > 2$, $\lambda_{m+j} - \lambda_{m+j+1} > 2$ and $\lambda_{m+i} - \lambda_{m+i+1} = 2$ for $0 \leq i \leq j-1$.

Theorem 15. *Let $C_i(n)$ $(1 \leq i \leq 4)$ denote the number of partitions of n in R with the added condition that*

(8.1) *all parts are* $> 4-i$,

(8.2) *if* $j, j+2, j+4, \ldots, j+(2r-2)$

is a maximal sequence of parts then
when $j \equiv 0 \pmod 3$, *r must be* $\equiv 0, 1 \pmod 3$
when $j \equiv 1 \pmod 3$, *r must be* $\equiv 0 \pmod 3$ *and*
when $j \equiv 2 \pmod 3$, *r must be* $\equiv 0, 2 \pmod 3$,
Then

$$C_i(n) = A_{4,i}\left(\frac{n}{3}\right) = B_{4,i}\left(\frac{n}{3}\right) \tag{8.3}$$

(note that if $\frac{n}{3}$ is not an integer all entries in (8.3) = 0.)

Remark. As an example, in case $n=12$ and $i=4$, $B_{4,4}(4) = 4$ (the partitions considered are 4, $3+1$, $2+2$, $2+1+1$), $C_4(12)$ also equals 4 with the relevant partitions being 12, $3+9$, $5+7$, $2+4+6$.

Proof. We need only show that the generating function $K_i(z;q)$ for partitions with m parts among the partitions enumerated by $C_i(n)$ is $Q_{4,i}(z;q^3)$. Clearly the initial conditions (2.8) and (2.9) hold. Thus we need only show that (2.10) with $q \to q^3$ holds for the $K_i(z;q)$. Namely

$$K_4(z;q) - K_3(z;q) = z^3 q^{1+3+5} K_1(zq^3;q) \tag{8.4}$$

$$K_3(z;q) - K_2(z;q) = z^2 q^{2+4} K_2(zq^3;q) \tag{8.5}$$

$$K_2(z;q) - K_1(z;q) = zq^3 K_3(zq^3;q) \tag{8.6}$$

$$K_1(z;q) = K_4(zq^3;q) \tag{8.7}$$

The proof of each of these four identities is similar so we provide full details for (8.4). We note that $K_i(z;q) - K_{i-1}(z;q)$ generates those partitions in which $5-i$ is the smallest part. Thus when $i=4$, we can only consider partitions that have 1 as the smallest part. By (8.2), the shortest allowable sequence starting with 1 is $1+3+5$. The partitions generated by $K_1(zq^3;q)$ are K_1 partitions with 3 added to each part. Thus the smallest part is ≤ 7. Hence either $1+3+5$ attaches to a previously maximal sequence of length now increased by 3 and thus still preserving (8.2) or else $1+3+5$ is itself a legitimate maximal sequence. Hence (8.4) is established. Identities (8.5)-(8.7) follow in the same way.

Therefore since (2.8)-(2.10) uniquely determine $Q_{k,i}(a;q)$, we see that for $1 \le i \le 4$

$$K_i(z;q) = Q_{4,i}(z;q^3), \tag{8.8}$$

and Theorem 16 follows by setting $z=1$ in (8.8) and comparing coefficients of q^n. □

9 One More Identity

We note that Dyson's favorite identity (1.1) may be written in the form of an instance of the quintuple product identity. Namely

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{n^2+n}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}} \\ =\frac{(-q^3;q^9)_\infty(-q^6;q^9)_\infty(q^9;q^9)_\infty(q^3;q^{18})_\infty(q^{15};q^{18})_\infty}{(q;q)_\infty}\end{aligned} \tag{9.1}$$

One may naturally expect that there is an equally elegant quintuple product companion for (9.1), and this is revealed in the following:

Theorem 16.

$$\begin{aligned}&\sum_{n\ge 0} \frac{q^{n^2}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}} \\ =&\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n(9n+1)/2}(1+q^{9n+1}) \\ =&(-q^8;q^9)_\infty(-q;q^9)_\infty(q^9;q^9)_\infty(q^7;q^{18})_\infty(q^{11};q^{18})_\infty/(q;q)_\infty\end{aligned} \tag{9.2}$$

Proof. It is an exercise in mathematical induction to show that

$$\begin{aligned}\sum_{j=0}^{m} \frac{q^{j^2}(q^3;q^3)_j}{(q;q)_j(q;q)_{2j+1}} - \left(1+\sum_{j=1}^{m}\frac{q^{j^2}(q^3;q^3)_{j-1}}{(q;q)_j(q;q)_{2j-1}}\right) \\ -q\sum_{j=0}^{m}\frac{q^{j^2+2j}(q^3;q^3)_j}{(q;q)_j(q;q)_{2j+2}} = \frac{-q^{m^2+4m+3}(q^3;q^3)_m}{(q;q)_m(q;q)_{2m+2}}\end{aligned} \tag{9.3}$$

Letting $m \to \infty$ in (9.3), we see that

$$\begin{aligned}
&\sum_{n\geq 0} \frac{q^{n^2}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}} \\
=&1+\sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}} + q\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+2}} \\
=&\frac{1}{(q;q)_\infty}\Bigg(\sum_{n=-\infty}^{\infty} (-1)^n q^{(27n^2+3n)/2} \\
&+ q\sum_{n=-\infty}^{\infty} (-1)^n q^{(27n^2+21n)/2}\Bigg) \qquad \text{(by [17, pp. 161-162, eqs. (91) and (93)])} \\
=&\frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(9n+1)/2}(1+q^{9n+1}). \\
=&\frac{1}{(q;q)_\infty}(-q^8;q^9)_\infty(-q;q^9)_\infty(q^9;q^9)_\infty(q^7;q^{18})_\infty(q^{11};q^{18})_\infty
\end{aligned}$$

by the quintuple product identity with $q \to q^9$, $z = q$ [12, p. 134, Ex. 5.16]. □

10 Relation to L. J. Rogers's Work

We have treated all the discoveries in this paper using standard polynomial notation. This, in turn, has simplified many of our computations some of which have been extremely intricate. However, it is important to stress that the sorts of results in section 3 are effectively finite versions of theorems of L. J. Rogers [15]. This is not obvious on the surface because Rogers couched his work in terms of Fourier series.

To make this relationship clear, we reprove Theorem 3 in the style of L. J. Rogers.

Second proof of Theorem 3

In (3.7) replace x by $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$. Thus (3.7) becomes

$$\begin{aligned}
&\prod_{j=1}^{n}(1+e^{i\theta}q^j)(1+e^{-i\theta}q^j) \\
&= \sum_{j=0}^{n} q^{\binom{j+1}{2}} V_j(\cos\theta) \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix}
\end{aligned} \tag{10.1}$$

Now noting

$$(-x;q)_{N+1}(-q/x;q)_N = x^{-N}q^{\binom{N+1}{2}}(-xq^{-N};q)_{2N+1}, \tag{10.2}$$

we may expand the right side of (10.2) by the q-binomial theorem [2, p. 36, eq. (3.3.6)] to obtain

$$
\begin{aligned}
(10.3)\qquad &(-xq;q)_N(-q/x;q)_N \\
=&\frac{1}{1+x}\sum_{j=0}^{2N+1} x^j q^{\binom{j}{2}-Nj}\begin{bmatrix}2N+1\\ j\end{bmatrix} \\
=&\sum_{j=0}^{N} q^{\binom{j+1}{2}}\frac{x^{j+\frac{1}{2}}+x^{-j-\frac{1}{2}}}{x^{\frac{1}{2}}+x^{-\frac{1}{2}}}\begin{bmatrix}2N+1\\ N-j\end{bmatrix}
\end{aligned}
$$

Now setting $x=e^{i\theta}$ in (10.3) and noting that

$$
(10.4)\qquad V_j(\cos\theta)=\frac{\cos(\theta(j+\frac{1}{2}))}{\cos\theta},
$$

we deduce (10.1) from (10.4). □

Now if we let $N\to\infty$ in (10.1), we obtain the actual starting point for Rogers in his second proof of the Rogers-Ramanujan identities [15]. A similar treatment can be used for Theorem 4.

11 Conclusion

The entire project began in an attempt to better understand Dyson's mod 27 identities especially the favorite (1.1). The natural appearance of $M_{xyz}(a;q)$ functions naturally led to a quest for proofs of the mock theta specializations. This in turn led to the Chebyshev polynomials.

It is the latter phenomenon that is so surprising. Orthogonal polynomials have arisen several times before in the treatment of q-series (cf. [4], [5], [9]). However, in each instance the orthogonal polynomials were q-analogs of classical orthogonal polynomials.

This is the first instance where classical orthogonal polynomials (namely Chebyshev polynomials of the third and fourth kinds) entered naturally into the world of q. This leaves us with at least three topics worthy of further exploration.

(11.1) Following the lead of Rogers briefly described in Section 10, one should be able to use the other Chebyshev polynomials in further studies of this nature.

(11.2) There are many more explicit results to be obtained for $m_{xyz}(1;q)$. The object here was to illustrate the method without obscuring the project with too many details.

(11.3) In addition to mock theta type results for $m_{xyz}(1;q)$, there should be natural combinatorial interpretations related to the ideas in section 8.

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