Partitions and the Minimal Excludant

by
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Abstract

Fraenkel and Peled have defined the minimal excludant or "mex" function on a set S of positive integers to be the least positive integer not in S. For each integer partition π, we define $\operatorname{mex}(\pi)$ to be the least positive integer that is not a part of π.

Define $\sigma \operatorname{mex}(n)$ to be the sum of $\operatorname{mex}(\pi)$ taken over all partitions of n. It will be shown that $\sigma \operatorname{mex}(n)$ is equal to the number of partitions of n into distinct parts with two colors.Finally the number of partitions π of n with $\operatorname{mex}(\pi)$ odd is almost always even.

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1 Introduction

The minimal excludant of an integer partition π to be the smallest positive integer that is *not* a part of π. Thus if π is $5+3+2+2+1$, then $\operatorname{mex}(\pi) = 4$. This paper will be devoted to two arithmetic functions related to $\operatorname{mex}(\pi)$.

$$\sigma \operatorname{mex}(n) = \sum_{\pi \vdash n} \operatorname{mex}(\pi), \tag{1.1}$$

where the sum is over all partitions of n.

We define

$$a(n) = \sum_{\substack{\pi \vdash n \\ \operatorname{mex}(\pi) \text{ odd}}} 1.$$

Thus there are five partitions of 4: 4 with $\operatorname{mex}(\pi) = 1$; $3+1$, with $\operatorname{mex}(\pi) = 2$; $2+2$ with $\operatorname{mex}(\pi) = 1$; $2+1+1$, with $\operatorname{mex}(\pi) = 3$; $1+1+1+1$ with $\operatorname{mex}(\pi) = 2$. Hence $\sigma \operatorname{mex}(4) = 9$ and $a(4) = 3$.

Also $D_2(n)$ is to denote the number of partitions of n into distinct parts using two colors. Thus $D_2(4) = 9$ in light of the fact that the partitions of 4 in question are 4_1, 4_2, $3_1 + 1_1$, $3_1 + 1_2$, $3_2 + 1_1$, $3_2 + 1_2$, $2_2 + 2_1$, $2_2 + 1_2 + 1_1$, $2_1 + 1_2 + 1_1$.

Theorem 1. $\sigma \operatorname{mex}(n) = D_2(n)$.

Theorem 2. *$a(n)$ is almost always odd and is even exactly when $n = j(3j \pm 1)$ for some j.*

The partition statistic $\text{mex}(\pi)$ has been studied explicitly in [3] and implicitly in [2] and [4]. Hopefully, the surprise of these further theorems will inspire further studies of $\text{mex}(\pi)$.

Section 2 is devoted to the proof of Theorem 1; section 3 is devoted to the proof of Theorem 2. The final section considers open questions.

For reference, we note that $a(n)$ is the same as $p_{1,1}(n)$ from [3]. Indeed, the results in [3] for $p_{1,1}(n)$ could have been used to prove Theorem 2, but this would have hidden the relation to Theorem 1.

2 Proof of Theorem 1

First Proof Let us define $M(z,q)$ to be the double series in which the coefficient of $z^m q^n$ is the number of partitions π of n with $\text{mex}(\pi) = m$.

Clearly,

$$\begin{aligned} M(z,q) &= \sum_{m=0}^{\infty} \frac{z^m q^{1+2+\cdots+(m-1)}}{\prod\limits_{\substack{n=1 \\ n\neq m}}^{\infty} (1-q^n)} \\ &= \frac{1}{(q;q)_\infty} \sum_{m=0}^{\infty} z^m q^{\binom{m}{2}} (1-q^m), \end{aligned}$$

where

$$(A;q)_\infty = \prod_{n=0}^{\infty} (1 - Aq^n)$$

Thus

$$\begin{aligned} \sum_{n\geq 0} \sigma\, \text{mex}(n) q^n &= \frac{\partial}{\partial z}\Big|_{z=1} M(z,q) \\ &= \frac{1}{(q;q)_\infty} \sum_{m=0}^{\infty} m q^{\binom{m}{2}} (1-q^m) \\ &= \frac{1}{(q;q)_\infty} \left(\sum_{m=1}^{\infty} m q^{\binom{m}{2}} - \sum_{m=1}^{\infty} (m-1) q^{\binom{m}{2}} \right) \\ &= \frac{\sum\limits_{m=0}^{\infty} q^{\binom{m+1}{2}}}{(q;q)_\infty} \\ &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty (q;q^2)_\infty} \qquad \text{(by [1, p. 23, eq. (2.2.13)])} \end{aligned}$$

$$= (-q;q)_\infty^2 \qquad \text{(by [1, p. 5, eq. (1.2.5)])}$$
$$= \sum_{n\geq 0} D_2(n) q^n.$$

□

Second Proof Let $M_i(n)$ denote the number of partitions π of n for which $\mathrm{mex}(\pi) > i$. We see that

$$M_i(n) = p(n - i(i+1)/2)$$

because each partition π counted by $M_i(n)$ must contain as summands $1, 2, 3, \ldots, i$ (and the sum of these parts is $i(i+1)/2$) with remaining parts being an arbitrary partition of $n - i(i+1)/2$.

Furthermore

$$\sigma\,\mathrm{mex}(n) = \sum_{i\geq 0} M_i(n)$$

because each partition π with $\mathrm{mex}(\pi) = m$ gets counted once by each $M_0(n), M_1(n), \ldots, M_{m-1}(n)$ and consequently this sum is the sum of all the mex values for the partitions of n.

Hence

$$\begin{aligned}\sum_{n\geq 0} \sigma\,\mathrm{mex}(n) q^n &= \sum_{n\geq 0} q^n \sum_{i\geq 0} M_i(n)\\ &= \sum_{n\geq 0} q^n \sum_{i\geq 0} p(n - i(i+1)/2)\\ &= \sum_{n\geq 0} p(n) q^n \sum_{i\geq 0} q^{i(i+1)/2}\\ &= \frac{\sum_{m=0}^{\infty} q^{m(m+1)/2}}{(q;q)_\infty},\end{aligned}$$

and as in the first proof, we see that this expression equals $(-q;q)_\infty^2$. □

3 Proof of Theorem 2

Lemma 3. $\sigma\,\mathrm{mex}(n)$ is almost always even and is odd exactly when n is of the form $j(3j \pm 1)$.

Proof. By Theorem 1,

$$\sum_{n\geq 0} \sigma\,\mathrm{mex}(n) q^n = \prod_{n=1}^{\infty} (1 + q^n)^2$$

$$\equiv \prod_{n=1}^{\infty}(1-q^{2n})(\mod 2)$$

$$= \sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j+1)} \qquad \text{(by [1, p. 11, eq. (1.3.1)])}$$

$\square$

Clearly

$$\sigma\,\mathrm{mex}(n) \equiv a(n)(\mod 2)$$

because when we add up all the $\mathrm{mex}(\pi)$ the parity is determined precisely by exactly how many of the $\mathrm{mex}(\pi)$ are odd.

Theorem 2 now follows from Lemma 3. $\square$

4 Minimal Odd Excludant

The idea of minimal excludant for the parts of a partition can obviously be restricted to parts in a specific arithmetic progression. As an example, we define $\mathrm{moex}(\pi)$ to be the smallest odd integer that is not a part of π.

$$\sigma\,\mathrm{moex}(n) = \sum_{\pi \vdash n} \mathrm{moex}(\pi).$$

Theorem 4. $\sum_{n\geq 0} \sigma\,\mathrm{moex}(n)q^n = (-q;q)_\infty(-q;q^2)_\infty^2$.

Proof. In analogy with the first proof of Theorem 1,

$$\begin{aligned}
\sum_{n\geq 0}\sigma\,\mathrm{moex}(n)q^n &= \frac{1}{(q;q)_\infty}\sum_{n\geq 0} q^{1+3+\cdots+(2n-1)}(2n+1)(1-q^{2n+1}) \\
&= \frac{1}{(q;q)_\infty}\sum_{n\geq 0}(2n+1)\left(q^{n^2}-q^{(n+1)^2}\right) \\
&= \frac{1}{(q;q)_\infty}\left(\sum_{n\geq 0}(2n+1)q^{n^2}-\sum_{n\geq 1}(2n-1)q^{n^2}\right) \\
&= \frac{1}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}q^{n^2}\right) \\
&= \frac{(q^2;q^2)_\infty(-q;q^2)_\infty^2}{(q;q)_\infty} \\
&= (-q;q)_\infty(-q;q^2)_\infty^2.
\end{aligned}$$

$\square$

5 Conclusion

It would be of great interest to have a bijective proof of Theorem 1. Also combinatorial proofs of Theorem 2 and Lemma 3 would be very interesting.

Finally we note that

$$\begin{aligned}\prod_{n=1}^{\infty}(1+q^n)^2 &= \prod_{n=1}^{\infty}(1+2q^n+q^{2n}) \\ &\equiv \prod_{n=1}^{\infty}(1-2q^n+q^{2n})(\mod 4) \\ &= \prod_{n=1}^{\infty}(1-q^n)^2 \\ &= \left(\sum_{n=-\infty}^{\infty}(-1)^r q^{n(3n+1)/2}\right)^2.\end{aligned}$$

Given the behavior of the number of representations of n as a sum of two fixed quadratic polynomials, it will surely be possible to prove that $\sigma\,\mathrm{mex}(n)$ is almost always divisible by 4, a task left for the reader.

References

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