

Combinatorics of periodic ellipsoidal billiards

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Abstract

We study combinatorics of billiard partitions which arose recently in the description of periodic trajectories of ellipsoidal billiards in d -dimensional Euclidean and pseudo-Euclidean spaces. Such partitions uniquely codify the sets of caustics, up to their types, which generate periodic trajectories. The period of a periodic trajectory is the largest part while the winding numbers are the remaining summands of the corresponding partition. In order to take into account the types of caustics as well, we introduce weighted partitions. We provide closed forms for the generating functions of these partitions.

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Contents

1	Introduction	1
1.1	(E, m) -representation	2
1.2	Confocal quadrics in Euclidean space \mathbb{R}^d	2
1.3	Weights	3
2	Euclidean Billiard Partitions	4
3	Quadrics and billiards in pseudo-Euclidean spaces and related partitions	8
	References	11

1 Introduction

The aim of this paper is to study combinatorics of billiard partitions which arose recently in the description of periodic trajectories of ellipsoidal billiards in d -dimensional Euclidean and pseudo-Euclidean spaces. Such partitions uniquely codify the sets of caustics, up to their types, which generate periodic trajectories. The period of a periodic trajectory is the largest part while the winding numbers are the remaining summands of the corresponding partition. This

correspondence and the main properties of the partitions in Euclidean case were established in the recent paper [DR]. The key moment in [DR] was relating the periodicity condition for billiard trajectories in d -dimensional space with the Pell functional-polynomial equation for extremal polynomials on d intervals on the real line. In this relationship, the winding numbers of a periodic trajectory correspond to the numbers of internal extremal points of an extremal polynomial on nested intervals, see Theorem 1.1 on (E, m) -representations below, where E is the union of d intervals and m is the period of a billiard trajectory. In order to take into account the types of caustics as well, using Lemma 1.2 below, we introduce weighted partitions. We provide closed forms for the generating functions of such partitions, see Theorem 2.7. Section 3 is devoted to the study of partitions related to ellipsoidal billiards in pseudo-Euclidean spaces and we distinguish space-type, time-type, and light-type partitions. We provide the generating functions for those partitions in Theorem 3.3.

1.1 (E, m) -representation

Given a polynomial $P_{2n}(z)$ of degree $2n$ positive on $E = [a, b] \setminus \cup_{j=1}^g (a_j, b_j)$. It admits an (E, m) -representation if it can be presented in the form

$$P_{2n}(z) = A_m^2(z) + T(z)B_{m-g-1}^2(z), \quad (1.1)$$

where $a < a_1 < b_1 < a_2 < \dots < a_g < b_g < b$, $a(z) = \prod_{j=1}^g (z - a_j)$, $b(z) = \prod_{j=1}^g (z - b_j)$, $T(z) = (z - a)(b - z)a(z)b(z)$, $\deg A_m = m$, $\deg B_{m-g-1} = m - g - 1$, all the zeros of $A_m(z)$, $(z - a)b(z)B_{m-g-1}(z)$ are simple, belong to E and alternate.

Theorem 1.1 (Krein, Levin, Nudelman) *A polynomial $P_{2n}(z)$ of degree $2n$ positive on $E = [a, b] \setminus \cup_{j=1}^g (a_j, b_j)$ admits an (E, m) -representation if and only if the numbers w_k determined by the relation*

$$\frac{1}{2\pi} \int_E \frac{x^j \ln P_{2n}(x)}{\sqrt{T(x)}} dx = \sum_{k=1}^g (-1)^k w_k \int_{a_k}^{b_k} \frac{x^j}{\sqrt{|T(x)|}} dx + (-1)^{g+1} m \int_m^\infty \frac{x^j}{\sqrt{|T(x)|}} dx,$$

are positive integers, for $j = 0, 1, \dots, g - 1$. Here w_k is the number of zeros of $B_{m-g-1}(z)$ in (a, a_k) .

We will consider the situation when $P_{2n}(x) = P_0(x) \equiv 1$. The equation (1.1) then reduces to the Pell equation. The intervals (a, a_k) for $k = 1, \dots, g$ are nested and the polynomial A_m satisfies $|A_m(a) = A_m(b) = A_m(a_i) = A_m(b_j) = 1$ and $B_{m-g-1}|A'_m$. Thus we see that the numbers w_k are the numbers of internal extremal points of the polynomial A_m on nested intervals are these numbers are strictly increasing.

1.2 Confocal quadrics in Euclidean space \mathbb{R}^d

Let an ellipsoid be given by:

$$\mathcal{E} : \frac{x_1^2}{a_1} + \dots + \frac{x_d^2}{a_d} = 1, \quad 0 < a_1 < a_2 < \dots < a_d.$$

The family of quadrics confocal with \mathcal{E} is:

$$\mathcal{Q}_\lambda(x) = \frac{x_1^2}{a_1 - \lambda} + \dots + \frac{x_d^2}{a_d - \lambda} = 1. \quad (1.2)$$

For a point given by its Cartesian coordinates $x = (x_1, \dots, x_d)$, the Jacobi elliptic coordinates $(\lambda_1, \dots, \lambda_d)$ are defined as the solutions of the equation in λ : $\mathcal{Q}_\lambda(x) = 1$. The correspondence between the elliptic and Cartesian coordinates is not injective, since points symmetric with respect to coordinate hyper-planes have equal elliptic coordinates.

The Chasles theorem states that any line ℓ in the space \mathbf{E}^d is tangent to exactly $d - 1$ quadrics from the confocal family. Moreover, any line ℓ' obtained by a billiard reflection off a quadric from the confocal family (1.2) is touching the same $d - 1$ quadrics. These $d - 1$ quadrics are *the caustics* of a given billiard trajectory. The existence of $d - 1$ caustics is a geometric manifestation of integrability of billiards within quadrics. If those quadrics are $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$, then the Jacobi elliptic coordinates $(\lambda_1, \dots, \lambda_d)$ of any point on ℓ satisfy the inequalities $\mathcal{P}(\lambda_j) \geq 0$, $j = 1, \dots, d$, where

$$\mathcal{P}(x) = (a_1 - x) \dots (a_d - x)(\alpha_1 - x) \dots (\alpha_{d-1} - x).$$

Let $b_1 < b_2 < \dots < b_{2d-1}$ be constants such that

$$\{b_1, \dots, b_{2d-1}\} = \{a_1, \dots, a_d, \alpha_1, \dots, \alpha_{d-1}\}.$$

Here, clearly, $b_{2d-1} = a_d$. The possible arrangements of the parameters $\alpha_1, \dots, \alpha_{d-1}$ of the caustics and the parameters a_1, \dots, a_d of the confocal family can be obtained from the following lemma.

Lemma 1.2 ([Aud1994]) *If $\alpha_1 < \alpha_2 < \dots < \alpha_{d-1}$, then $\alpha_j \in \{b_{2j-1}, b_{2j}\}$, for $1 \leq j \leq d - 1$.*

If ℓ is the line containing a segment of a billiard trajectory within \mathcal{E} , then $b_1 > 0$.

Along a billiard trajectory, the Jacobi elliptic coordinates satisfy:

$$b_0 = 0 \leq \lambda_1 \leq b_1, \quad b_2 \leq \lambda_2 \leq b_3, \quad \dots, \quad b_{2d-2} \leq \lambda_d \leq b_{2d-1}.$$

Moreover, along the trajectory, each Jacobi coordinate λ_j fills the whole interval $[b_{2j-2}, b_{2j-1}]$, with local extreme points being only the end-points of the interval. Thus, λ_j takes values b_{2j-2} and b_{2j-1} alternately and changes monotonously between them. Let \mathcal{T} be a periodic billiard trajectory and denote by m_j the number of its points where $\lambda_j = b_{2j-2}$. Based on the previous discussion, m_j is also the number of the points where $\lambda_j = b_{2j-1}$.

Notice that the value $\lambda_1 = 0$ corresponds to an impact with the boundary ellipsoid \mathcal{E} , value $\lambda_j = \alpha_k$ corresponds to a tangency with the caustic \mathcal{Q}_{α_k} , and $\lambda_j = a_k$ corresponds to an intersection with the coordinate hyperplane $x_k = 0$. Since each periodic trajectory intersects any hyperplane even number of times, we get that m_j must be even whenever b_{2j-2} or b_{2j-1} is in the set $\{a_1, \dots, a_d\}$. We can reformulate the last statement and say

Lemma 1.3 (see [DR]) *If m_j is odd then both b_{2j-2} and b_{2j-1} are in the set $\{\alpha_1, \dots, \alpha_{d-1}\}$.*

Following [RR2014], we denote $m_0 = n$, $m_d = 0$, and call $(m_0, m_1, \dots, m_{d-1})$ *the winding numbers* of a given periodic billiard trajectory with period n . It was proven in [DR] that $m_0 > m_1 > \dots > m_{d-1}$. The proof was based on Theorem 1.1 for $P_0(x) \equiv 1$ and the identifications of $g = d - 1$ and $m_0 = n = m$, $m_i = w_{d-i}$. Further, based on Lemma 1.2, Theorem 1.1, and [PS1999], it was proven in [DR], that a partition $m_0 > m_1 > \dots > m_{d-1}$, with m_{d-1} even and not two consecutive m_i, m_{i+1} being both odd, uniquely determines the set of caustics of a given type, which generate periodic trajectories of period m_0 and winding numbers m_i . We will refer to such partitions as the Euclidean billiard partitions.

1.3 Weights

We want to go even a step further, and to assign a weight to a Euclidean billiard partition to count the number of possibilities for types of the caustics compatible with the given partition. The key information about the possible types of caustics comes from Lemma 1.2. Namely, if all elements of a partition are even, then there are 2^{d-1} possible different choices of types of caustics. If the period is odd, then the number of choices is two times less, since in this case it has to be $b_1 = \alpha_1$. In other words, for odd-periodic trajectories, the caustic Q_{α_1} has to be an ellipsoid. If any other winding number is odd, then two caustics have to be of a given, fixed type. This means that an odd winding number distinct from the period reduces the number of choices for geometric types of caustics four times. Thus we introduce the following weight function $\phi(n, d, \pi)$ for a given Euclidean billiard partition π of length d with the largest part equal n :

$$\phi(2m, d, \pi) = 2^{d-1-2s}; \quad (1.3)$$

$$\phi(2m+1, d, \pi) = 2^{d-2s}, \quad (1.4)$$

where s is the total number of odd parts in π .

Example 1.4 *Let $d = 2$. Then $\phi(2m+1, 2, \pi) = 1$ and $\phi(2m, 2, \pi) = 2$.*

Example 1.5 *Let $d = 3$.*

$$n = 4 \quad \pi = (4, 3, 2), \quad \phi(4, 3, \pi) = 1.$$

$$n = 5 \quad \pi = (5, 4, 2), \quad \phi(5, 3, \pi) = 2.$$

$$n = 6$$

$$\pi_1 = (6, 5, 4), \quad \phi(6, 3, \pi_1) = 1;$$

$$\pi_2 = (6, 5, 2), \quad \phi(6, 3, \pi_2) = 1;$$

$$\pi_3 = (6, 4, 2), \quad \phi(6, 3, \pi_3) = 4;$$

$$\pi_4 = (6, 3, 2), \quad \phi(6, 3, \pi_4) = 1.$$

$$n = 7$$

$$\pi_1 = (7, 6, 4), \quad \phi(7, 3, \pi_1) = 2;$$

$$\pi_2 = (7, 6, 2), \quad \phi(7, 3, \pi_2) = 2;$$

$$\pi_3 = (7, 4, 2), \quad \phi(7, 3, \pi_3) = 2.$$

In the next Section 2 we are going to study further combinatorial properties of such partitions, including the case of weighted partitions.

2 Euclidean Billiard Partitions

Let \mathcal{D} denote the set of all integer partitions into distinct parts where

(E1) the smallest part is even;

(E2) adjacent parts are never both odd.

Let $p_{\mathcal{D}}(n)$ denote the number of partitions of n that are in \mathcal{D} .

$$1 + \sum_{n \geq 1} p_{\mathcal{D}}(n)q^n = 1 + q^2 + q^4 + q^5 + 2q^6 + q^7 + 2q^8 + 3q^9 + 3q^{10} \\ + 4q^{11} + 4q^{12} + 6q^{13} + 5q^{14} + 9q^{15} + \dots$$

Example 2.1 Thus, $p_{\mathcal{D}}(15) = 9$, and the nine partitions of 15 are $13 + 2$, $11 + 4$, $10 + 3 + 2$, $9 + 6$, $9 + 4 + 2$, $8 + 5 + 2$, $7 + 6 + 2$, $6 + 5 + 4$, $6 + 4 + 3 + 2$.

Additionally, we shall also need to consider weighting the partitions in \mathcal{D} as follows. Suppose $\pi \in \mathcal{D}$ and that π has d parts with largest part n and s odd parts. The weight $\phi(n, d, \pi)$ is given by (1.3) and (1.4).

Let $p_{\mathcal{D}}(m, n)$ denote the number of partitions of n in \mathcal{D} with weight m . Then:

$$1 + \sum_{n \geq 1} p_{\mathcal{D}}(m, n)x^m q^n = 1 + q^2 + q^4 + q^5 + (1+x)q^6 + q^7 + (1+x)q^8 + 3q^9 \\ + (1+2x)q^{10} + (3+x)q^{11} + (1+2x+x^2)q^{12} \\ + (5+x)q^{13} + (2+3x+x^2)q^{14} + (6+3x)q^{15} + \dots$$

Referring back to Example 2.1, we see that three partitions of 15 have weight 1, namely $2+4+9$, $2+6+7$, and $2+3+4+6$ while the remaining six have weight 0. Thus yielding $(6+3x)$ as the coefficient of q^{15} .

Our object is to provide a closed form for these generating functions. To this end we begin by identifying a subset of \mathcal{D} which we will call the irreducible partitions of \mathcal{D} and denote $\bar{\mathcal{D}}$. By irreducible we mean that $\pi \in \bar{\mathcal{D}}$ if no summand of π can be reduced by 2 with the resulting partition remaining in \mathcal{D} . For example $2+4+7$ is not irreducible because $2+4+(7-2) = 2+4+5$ is still in \mathcal{D} . On the other hand $2+4+5$ is irreducible because $2+4+3$ destroys the order of the parts.

Lemma 2.2 The elements of $\bar{\mathcal{D}}$ consists of those elements of \mathcal{D} where

(IE1) the smallest part is 2;

(IE2) adjacent parts are never both odd;

(IE3) the difference between adjacent parts is less or equal 2.

Proof. If $\pi \in \mathcal{D}$ and its smallest part is $\lambda \neq 2$. Since it is even, $\lambda \geq 4$. Thus $\lambda - 2$ is even and $\lambda - 2 \geq 2$. Hence, we have produced a new element of \mathcal{D} by replacing λ by $\lambda - 2$. Thus, it has to be $\lambda = 2$.

Next, suppose $\mathcal{D} \ni \pi = \dots + \lambda_i + \lambda_{i+1} + \dots$ and $\lambda_i - \lambda_{i+1} > 2$. Then the partition obtained by replacing λ_i by $\lambda_i - 2$ is also in \mathcal{D} : $\lambda_i - 2 > \lambda_{i+1}$ and $\lambda_i - 2$ and λ_i are of the same parity. Thus, consecutive parts in partitions in $\bar{\mathcal{D}}$ must differ by not more than 2. \square

Corollary 2.3 Every partition $\pi \in \mathcal{D}$ with d parts can be uniquely represented by $\pi_1 + \pi_2$ where $\pi_1 \in \bar{\mathcal{D}}$ and π_2 is a partition with not more than d parts each even and conversely.

Example 2.4 Consider $9 + 4 + 2 \in \mathcal{D}$: $9 + 4 + 2 = (5 + 4 + 2) + (4 + 0 + 0)$.

Proof. Clearly any partition of the given form $\pi_1 + \pi_2$ is in \mathcal{D} . Now suppose we are given

$$\pi : \lambda_1 + \lambda_2 + \cdots + \lambda_d \in \mathcal{D},$$

with each $\lambda_i > \lambda_{i+1}$. Since λ_d must be even, say $\lambda_d = 2j$ we set the smallest part of π_1 as 2 and the smallest part of π_2 as $2j - 2 = \lambda_d - 2$. If λ_{d-1} is odd, then the next part of π_1 is 3 and that of π_2 is $\lambda_{d-1} - 3$. Observe that $\lambda_{d-1} - 3 \geq \lambda_d - 2$ since $\lambda_{d-1} > \lambda_d$. If λ_{d-1} is even, then the next part of π_1 is 4 and that of π_2 is $\lambda_{d-1} - 4$. Here $\lambda_{d-1} - 4 \geq \lambda_d - 2$ since $\lambda_{d-1} - \lambda_d \geq 2$. We proceed thus up through the parts of π . At each stage there is exactly one choice for the part of π_1 depending on the parity of the original part, and thus the resulting part of π_2 is also unique. \square

One can notice that π and its irreducible partition $\bar{\pi}$ have the same weight:

$$\phi(n, d, \pi) = \phi(n, d, \bar{\pi}),$$

since they have the same number of odd parts.

Let us denote by $s(d, n)$ the generating function for those partitions in $\bar{\mathcal{D}}$ that have exactly d parts and largest part equal n .

Example 2.5 Thus

$$\begin{aligned} s(5, 8) &= x^2 q^{23} + x^2 q^{25} + x^2 q^{27} \\ &= x^{5-1-2 \cdot 1} q^{2+3+4+6+8} + x^{5-1-2 \cdot 1} q^{2+4+5+6+8} + x^{5-1-2 \cdot 1} q^{2+4+6+7+8}. \end{aligned}$$

Let us recall the Gaussian polynomials or q -binomial coefficients:

$$\begin{bmatrix} A \\ B \end{bmatrix}_q = \begin{cases} 0, & \text{if } B < 0 \text{ or } B > A \\ \frac{(q; q)_A}{(q; q)_B (q; q)_{A-B}}, & 0 \leq B \leq A \end{cases}$$

and $(x; q)_N = (1 - x)(1 - xq) \cdots (1 - xq^{N-1})$.

Lemma 2.6 The functions $s(d, m)$ can be expressed as follows, depending on parity of m :

a)

$$s(d, 2n) = x^{2n-d-1} q^{2n^2-2dn-n+d^2+2d} \begin{bmatrix} n-1 \\ 2n-d-1 \end{bmatrix}_{q^2}; \quad (2.1)$$

b)

$$s(d, 2n+1) = x^{2n-d} q^{2n^2-2dn+d^2+3n} \begin{bmatrix} n-1 \\ 2n-d \end{bmatrix}_{q^2}. \quad (2.2)$$

Proof. We begin by noting initial values for $s(d, n)$. First

$$s(1, 2) = q^2,$$

otherwise for $n \neq 2$:

$$s(1, n) = 0.$$

We see that these initial values hold for the right hand sides of (2.1) and (2.2) as well. Thus, the lemma is proved for $d = 1$ and all n .

For $d > 1$ we have the following recurrences for $s(d, n)$:

$$s(d, 2n) = q^{2n}(s(d-1, 2n-1) + xs(d-1, 2n-2)), \quad (2.3)$$

$$s(d, 2n+1) = q^{2n+1}s(d-1, 2n). \quad (2.4)$$

These recurrences are immediate. The largest part is supplied by q^{2n} in (2.3) and by q^{2n+1} in (2.4). If the largest part is $2n$ then the next largest part must be either $2n-1$ or $2n-2$. If the largest part is $2n+1$ then the next largest part must be $2n$.

We must also keep track of how the weights change. Suppose $\bar{\pi}$ is a partition treated by $s(d-1, 2n-1)$ and its weight is $(d-1) - 2s$; when we attach $2m$ to $\bar{\pi}$, the weight will become $d-1-2s$, i.e. it does not change. Suppose $\bar{\pi}$ is a partition treated by $s(d-1, 2n-2)$ and its weight is $(d-1) - 1 - 2s$; when we attach $2m$ to $\bar{\pi}$, the weight will become $d-1-2s$, i.e. the weight has increased by 1, thus accounting for the x in the second term in (2.3). Finally, suppose $\bar{\pi}$ is a partition treated by $s(d-1, 2n)$ and its weight is $(d-1) - 1 - 2s$. By attaching $2n+1$ to $\bar{\pi}$, the weight becomes $d-2(s+1)$, i.e. the weight has not changed.

To conclude our proof by induction, we must show that the right hand sides of (2.1) and (2.2) satisfy these same recurrences.

Namely for (2.4) we see that replacing d by $d-1$ in the right side of (2.1) and multiplying the result by q^{2n+1} yields the right side of (2.2). For (2.4) we must evaluate

$$\begin{aligned} & q^{2n} \left(x^{2(n-1)-(d-1)} q^{2(n-1)^2-2(d-1)(n-1)+(d-1)^2+3(n-1)} \begin{bmatrix} n-2 \\ 2n-d-1 \end{bmatrix}_{q^2} \right. \\ & \left. + x \cdot x^{2(n-1)-(d-1)-1} q^{2(n-1)^2-2(d-1)(n-1)-(n-1)+(d-1)^2+2(d-1)} \begin{bmatrix} n-2 \\ 2n-d-2 \end{bmatrix}_{q^2} \right) \\ & = x^{2n-d-1} q^{2n^2-2dn-n+d^2+2d} \left(q^{2(2n-d-1)} \begin{bmatrix} n-2 \\ 2n-d-1 \end{bmatrix}_{q^2} + \begin{bmatrix} n-2 \\ 2n-d-2 \end{bmatrix}_{q^2} \right) \\ & = x^{2n-d-1} q^{2n^2-2dn-n+d^2+2d} \begin{bmatrix} n-1 \\ 2n-d-1 \end{bmatrix}_{q^2}, \end{aligned}$$

by [And1976], p. 35, eq. (3.34). Thus, both recurrences (2.3) and (2.4) are satisfied by the right hand sides of (2.1) and (2.2). Finally, we see that (2.1) and (2.2) are valid by mathematical induction on d . \square

Now we can formulate and prove the main result in this Section.

Theorem 2.7 *The generating function for the Euclidean billiard partitions has the following formula:*

$$1 + \sum_{n \geq 1, m \geq 0} p_{\mathcal{D}}(m, n) q^n = 1 + \sum_{d=1}^{\infty} \sum_{n=0}^{\infty} \frac{s(d, n)}{(q^2; q^2)_d},$$

where

$$\begin{aligned} s(d, 2n) &= x^{2n-d-1} q^{2n^2-2dn-n+d^2+2d} \begin{bmatrix} n-1 \\ 2n-d-1 \end{bmatrix}_{q^2}; \\ s(d, 2n+1) &= x^{2n-d} q^{2n^2-2dn-n+d^2+3n} \begin{bmatrix} n-1 \\ 2n-d \end{bmatrix}_{q^2}. \end{aligned}$$

Proof. The formulas for $s(d, n)$ have already appeared in Lemma 2.6. To complete the proof we note that by the Corollary 2.3, the generation function for the partitions in \mathcal{D} with d parts and largest part n is given by the product

$$s(d, n) \frac{1}{(1 - q^2)(1 - q^4) \cdots (1 - q^{2d})} = \frac{s(d, n)}{(q^2; q^2)_d},$$

where the product

$$\frac{1}{(q^2; q^2)_d}$$

generates the general partition into at most d even parts. \square

We note that the series given at the beginning of this Section follows directly from Theorem 2.7, just by setting $x = 1$, because

$$\sum_{m \geq 0} p_{\mathcal{D}}(m, n) = p_{\mathcal{D}}(n).$$

In the next Section our interest concerns the instance $x = 1$ in Theorem 2.7.

3 Quadrics and billiards in pseudo-Euclidean spaces and related partitions

We consider pseudo-Euclidean space $\mathbb{R}^{k,l}$, with $d = k + l$. For a given set of positive constants a_1, a_2, \dots, a_d , an ellipsoid is given by:

$$\mathcal{E} : \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \cdots + \frac{x_d^2}{a_d} = 1. \quad (3.1)$$

Let us remark that equation of any ellipsoid in the pseudo-Euclidean space can be brought into the canonical form (3.1) using transformations that preserve the pseudo-Euclidean scalar product.

The family of quadrics confocal with \mathcal{E} is:

$$\mathcal{Q}_\lambda : \frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_k^2}{a_k - \lambda} + \frac{x_{k+1}^2}{a_{k+1} + \lambda} + \cdots + \frac{x_d^2}{a_d + \lambda} = 1, \quad \lambda \in \mathbf{R}. \quad (3.2)$$

Unless stated differently, we are going to consider the non-degenerate case, when set

$$\{a_1, \dots, a_k, -a_{k+1}, \dots, -a_d\}$$

consists of d different values:

$$a_1 > a_2 > \cdots > a_k > 0 > -a_{k+1} > \cdots > -a_d.$$

For $\lambda \in \{a_1, \dots, a_k, -a_{k+1}, \dots, -a_d\}$, the quadric \mathcal{Q}_λ is degenerate and it coincides with the corresponding coordinate hyper-plane.

It is natural to join one more degenerate quadric to the family (3.2): the one corresponding to the value $\lambda = \infty$, that is the hyper-plane at the infinity. The pseudo-Euclidean version of Audin's Lemma 1.2 is the following result from [DR2012]:

Lemma 3.1 *[[DR2012]] In pseudo-Euclidean space $\mathbf{E}^{k,l}$ consider a line intersecting ellipsoid \mathcal{E} (3.1). Then this line is touching $d - 1$ quadrics from (3.2). If we denote their parameters by $\alpha_1, \dots, \alpha_{d-1}$ and take:*

$$\begin{aligned} \{b_1, \dots, b_p, c_1, \dots, c_q\} &= \{\varepsilon_1 a_1, \dots, \varepsilon_d a_d, \alpha_1, \dots, \alpha_{d-1}\}, \\ c_q \leq \dots \leq c_2 \leq c_1 < 0 < b_1 \leq b_2 \leq \dots \leq b_p, \quad p + q &= 2d - 1, \end{aligned}$$

we will additionally have:

- if the line is space-like, then $p = 2k - 1$, $q = 2l$, $a_1 = b_p$, $\alpha_i \in \{b_{2i-1}, b_{2i}\}$ for $1 \leq i \leq k - 1$, and $\alpha_{j+k-1} \in \{c_{2j-1}, c_{2j}\}$ for $1 \leq j \leq l$;
- if the line is time-like, then $p = 2k$, $q = 2l - 1$, $c_q = -a_d$, $\alpha_i \in \{b_{2i-1}, b_{2i}\}$ for $1 \leq i \leq k$, and $\alpha_{j+k} \in \{c_{2j-1}, c_{2j}\}$ for $1 \leq j \leq l - 1$;
- if the line is light-like, then $p = 2k$, $q = 2l - 1$, $b_p = \infty = \alpha_k$, $b_{p-1} = a_1$, $\alpha_i \in \{b_{2i-1}, b_{2i}\}$ for $1 \leq i \leq k - 1$, and $\alpha_{j+k} \in \{c_{2j-1}, c_{2j}\}$ for $1 \leq j \leq l - 1$.

Moreover, for each point on ℓ inside \mathcal{E} , there is exactly d distinct quadrics from (3.2) containing it. More precisely, there is exactly one parameter of these quadrics in each of the intervals:

$$(c_{2l-1}, c_{2l-2}), \dots, (c_3, c_2), (c_1, 0), (0, b_1), (b_2, b_3), \dots, (b_{2k-2}, b_{2k-1}).$$

See [DR2012], [DR2019b], [ADR] for more details about billiards in pseudo-Euclidean spaces.

We will distinguish tree types of partitions related to billiards in pseudo-Euclidean space $\mathbb{R}^{k,l}$, the space-type, the time-type, and the light-type partitions. All of them are of the form $(m_1, m_2, \dots, m_l | n_1, n_2, \dots, n_k)$ such that

$$m_1 > m_2 > \dots > m_l,$$

and

$$n_1 > n_2, \dots > n_k,$$

while there are no assumed relations between n_i and m_j .

Let $\mathcal{S}\mathcal{D}$ denote the set of all such *space-type integer partitions* into parts

$$(m_1, m_2, \dots, m_l | n_1, n_2, \dots, n_k),$$

where

(S1) the smallest part n_k is even;

(S2) adjacent parts m_i, m_{i+1} are never both odd as well as adjacent parts n_j, n_{j+1} .

Let $p_{\mathcal{S}\mathcal{D}}(n)$ denote the number of space-type partitions of $n = n_1 + m_1$ that are in $\mathcal{S}\mathcal{D}$.

Let $\mathcal{T}\mathcal{D}$ denote the set of all such *time-type integer partitions* into parts

$$(m_1, m_2, \dots, m_l | n_1, n_2, \dots, n_k),$$

where

(T1) the smallest part m_l is even;

(T2) adjacent parts m_i, m_{i+1} are never both odd as well as adjacent parts n_j, n_{j+1} .

Let $p_{\mathcal{SD}}(n)$ denote the number of space-type partitions of $n = n_1 + m_1$ that are in \mathcal{SD} .

Let \mathcal{LD} denote the set of all such *light-type integer partitions* into parts

$$(m_1, m_2, \dots, m_l | n_1, n_2, \dots, n_k),$$

where

(L1) the smallest parts m_l and n_k are even;

(L2) adjacent parts m_i, m_{i+1} are never both odd as well as adjacent parts n_j, n_{j+1} .

Let $p_{\mathcal{LD}}(n)$ denote the number of space type partitions of $n = n_1 + m_1$ that are in \mathcal{LD} .

Let us denote by $\tilde{s}(n_1, k)$ the generation function of reduced partitions

$$n_1 > n_2 > \dots > n_k,$$

such that any two adjacent parts n_i, n_{i+1} are never both odd, but without assuming anything about the parity of n_k . In the sequel, we use the instance $x = 1$ of Theorem 2.7 and of the notions from that statement.

Lemma 3.2 *The generating function \tilde{s} satisfies the following*

$$\tilde{s}(n_1, k) = s(n_1, k) + qs(n_1, k - 1).$$

Proof. We consider two cases depending on parity of n_k . If n_k is even, than the partition belongs to $s(n_1, k)$. If n_k is odd, it has to be $n_k = 1$ and $n_{k-1} = 2$. Thus the partition $n_1 > n_2 > \dots > n_{k-1}$ belongs to $s(n_1, k - 1)$. \square

Theorem 3.3 *The generating functions for the pseudo-Euclidean billiard partitions have the following formulas:*

(S) *for the space-type:*

$$1 + \sum_{n \geq 1} p_{\mathcal{SD}}(n)q^n = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{n_1=0}^{\infty} \sum_{m_1=0}^{\infty} \frac{s(n_1, k)}{(q^2; q^2)_k} \frac{\tilde{s}(m_1, l)}{(q^2; q^2)_l};$$

(T) *for the space-type:*

$$1 + \sum_{n \geq 1} p_{\mathcal{SD}}(n)q^n = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{n_1=0}^{\infty} \sum_{m_1=0}^{\infty} \frac{\tilde{s}(n_1, k)}{(q^2; q^2)_k} \frac{s(m_1, l)}{(q^2; q^2)_l};$$

(L) *for the space-type:*

$$1 + \sum_{n \geq 1} p_{\mathcal{SD}}(n)q^n = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{n_1=0}^{\infty} \sum_{m_1=0}^{\infty} \frac{s(n_1, k)}{(q^2; q^2)_k} \frac{s(m_1, l)}{(q^2; q^2)_l}.$$

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References

- [ADR] A. Adabrah, V. Dragović, and M. Radnović, *Periodic Billiards within conics in the Minkowski plane and Akhiezer Polynomials*, available at [arXiv:1906.0491](#).
- [Akh1932] N. I. Akhiezer, *Über einige Funktionen, welche in zwei gegebenen Interwallen am wenigsten von Null abweichen. I Teil*, Izvestiya Akad. Nauk SSSA, VII ser., Otd. mat. est. nauk **9** (1932), 1163–1202.
- [Akh1933a] ———, *Über einige Funktionen, welche in zwei gegebenen Interwallen am wenigsten von Null abweichen. II Teil*, Izvestiya Akad. Nauk SSSA, VII ser., Otd. mat. est. nauk **3** (1933), 309–344.
- [Akh1933b] ———, *Über einige Funktionen, welche in zwei gegebenen Interwallen am wenigsten von Null abweichen. III Teil*, Izvestiya Akad. Nauk SSSA, VII ser., Otd. mat. est. nauk **4** (1933), 499–536.
- [And1976] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Readings, 1976.
- [Aud1994] M. Audin, *Courbes algébriques et systèmes intégrables: géodesiques des quadriques*, Exposition. Math. **12** (1994), 193–226.
- [BM1962] G. Birkhoff and R. Morris, *Confocal Conics in Space-Time*, The American Mathematical Monthly **69** (1962), no. 1, 1-4.
- [CRR2011] P. S. Casas and R. Ramírez-Ros, *The frequency map for billiards inside ellipsoids*, SIAM J. Appl. Dyn. Syst. **10** (2011), no. 1, 278–324.
- [CRR2012] ———, *Classification of symmetric periodic trajectories in ellipsoidal billiards*, Chaos **22** (2012), no. 2, 026110, 24.
- [Dar1914] G. Darboux, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitesimal*, Vol. 2 and 3, Gauthier-Villars, Paris, 1914.
- [DR2012] V. Dragović and M. Radnović, *Ellipsoidal billiards in pseudo-Euclidean spaces and relativistic quadrics*, Advances in Mathematics **231** (2012), 1173–1201.
- [DR] V. Dragović and M. Radnović, *Periodic ellipsoidal billiard trajectories and extremal polynomials*, Comm. Math. Physics, available at [arXiv:1804.02515\[math.DS\]](#).
- [DR2019a] V. Dragović and M. Radnović, *Caustics of Poncelet polygons and classical extremal polynomials*, Regul. Chaotic Dyn. **24** (2019), no. 1, 1–35.
- [DR2019b] V. Dragović and M. Radnović, *Periodic Billiards within pseudo-Euclidean quadrics and extremal polynomials* (2019).
- [KLN1990] M. G. Kreĭn, B. Ya. Levin, and A. A. Nudel'man, *On special representations of polynomials that are positive on a system of closed intervals, and some applications*, Functional analysis, optimization, and mathematical economics, Oxford Univ. Press, New York, 1990, pp. 56–114. Translated from the Russian by Lev J. Leifman and Tatyana L. Leifman.
- [PS1999] F. Peherstorfer and K. Schiefermayr, *Description of extremal polynomials on several intervals and their computation. I, II*, Acta Math. Hungar. **83** (1999), no. 1-2, 27–58, 59–83.
- [RR2014] R. Ramírez-Ros, *On Cayley conditions for billiards inside ellipsoids*, Nonlinearity **27** (2014), no. 5, 1003–1028.