The Ramanujan-Dyson Identities and George Beck's Congruence Conjectures

by

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Abstract

Dyson's famous conjectures (proved by Atkin and Swinnerton-Dyer) gave a combinatorial interpretation of Ramanujan's congruences for the partition function. The proofs of these results center on one of the universal mock theta functions that generates partitions according to Dyson's rank. George Beck has generalized the study of partition function congruences related to rank by considering the total number of parts in the partitions of n. The related generating functions are no longer part of the world of mock theta functions. However, George Beck has conjectured that certain linear combination of the related enumeration functions do satisfy congruences modulo 5 and 7. The conjectures are proved here.

Keywords. Dyson's rank, George Beck's conjectures, partitions, Ramanujan congruences

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1 Introduction

Among the celebrated achievements of Ramanujan, we find [1, p. 159]

$$(1.1) p(5n+4) \equiv 0 \pmod{5},$$

where p(n) is the integer partition function, Dyson [4], seeking to provide a combinatorial setting for (1.1), defined the *rank* of a partition to be the largest part minus the number of parts. For example, the rank of 5 + 5 + 3 + 1 is 5 - 4 = 1.

Let us define (cf. [2]) N(m, k, n) to be the number of partitions of n with rank congruent to m modulo k. Dyson conjectured [4], and Atkin and Swinnerton-Dyer proved [2] that for $0 \le i \le 4$

(1.2)
$$N(i,5,5n+4) = \frac{1}{5}p(5n+4),$$

thus revealing five disjoint, equinumerous subsets of the partitions of 5n + 4. Also treated in this way [4], [2] is the congruence

(1.3)
$$N(i,7,7n+5) = \frac{1}{7}p(7n+5)$$

where $0 \le i \le 6$, and (1.3), of course, implies the Ramanujan congruence

(1.4)
$$p(7n+5) \equiv 0 \pmod{7}.$$

George Beck [3] has made surprising new conjectures along these lines. Now we define NT(m, k, n) to be the total number of parts in the partitions of n with rank congruent to m modulo k. Beck's conjectures are formulated in the following two theorems.

Theorem 1. If i = 1 or 4

(1.5)
$$NT(1,5,5n+i) - NT(4,5,5n+i) + 2NT(2,5,5n+i) - 2NT(3,5,5n+i) \equiv 0 \pmod{5}$$

Theorem 2. If i = 1 or 5,

(1.6)

$$NT(1,7,7n+i) - NT(6,7,7n+i) + NT(2,7,7n+i) - NT(5,7,7n+i) - NT(3,7,7n+i) + NT(4,7,7n+i) = 0 \pmod{7}.$$

Note that if the above assertions concerned N instead of NT, they would trivially be implied by the fact that

(1.7)
$$N(m,k,n) = N(k-m,k,n)$$

which follows by using partition conjugation to transform the partition enumerated by N(m, k, n) into those enumerated by N(k - m, k, n). However, (1.8) is generally false if N is replaced by NT.

As an example of Theorem 1, take n = 0, i = 4. The partitions of 4 are 4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1, with ranks 3, 1, 0, -1, -3. Consequently NT(1, 5, 4) = 2, NT(2, 5, 4) = 4, NT(3, 5, 4) = 1, NT(4, 5, 4) = 3, and

$$NT(1,5,4) - NT(4,5,4) + 2NT(2,5,4) - 2NT(3,5,4)$$

= 2 - 3 + 2 \cdot 4 - 2 \cdot 1 = 10 - 5 = 5 \equiv 0 \quad (mod 5)

For Theorem 2, take n = 0, i = 5, the partitions of 5 are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1, with ranks 4, 2, 1, 0, -1, -2, -4. Sonsequently N(1,7,5) = 2, N(2,7,5) = 2, N(3,7,5) = 5, N(4,7,5) = 1, N(5,7,5) = 4, N(6,7,5) = 3, and

$$NT(1,7,5) - NT(6,7,5) + NT(2,7,5) - NT(5,7,5) - NT(3,7,5) + NT(4,7,5)$$

 $= 2 - 3 + 2 - 4 - 5 + 1 = -7 \equiv 0 \pmod{7}.$

The obstacles to proving Theorems 1 and 2 lie in the nature of the generating function. Recall that the universal mock theta function

(1.8)
$$\sum_{n \ge 0} \frac{q^{n^2}}{(zq;q)_n (q/z;q)_n}$$

where $(A;q) + n = (1 - A)(1 - Aq) \cdots (1 - Aq^{n-1})$ is the generating function in which q marks the number being partitioned and z marks the rank [5, p. 66].

To keep track of the total number of parts, we must add an extra variable x; and the generating function which weights partitions by the by the number of parts while keeping track of the rank is (following Garvan's argument [5, p. 66])

(1.9)
$$\frac{\partial}{\partial x}\Big|_{x=1}\sum_{n\geq 0}\frac{x^nq^{n^2}}{(zq;q)_n(\frac{xq}{z};q)_n}.$$

This is a decidedly more complicated function than that appearing in (1.8).

In section 2, we shall collect the necessary results and notations from the paper of Atkin and Swinnerton-Dyer [2]. Section 3 is devoted to obtaining a tractable representation of the series appearing in (1.9). Section 4 provides a proof of Theorem 1, and Section 5 treats Theorem 2. We conclude with a discussion of further problems.

2 Background

In this section we shall collect the necessary results from the paper of Atkin and Swinnerton-Dyer [2] and a result from classical q-hypergeometric series.

We shall stick closely to the notation in [2, p. 106]. The main change consists of replacing their q by k. Also we shall use the subscript k explicitly to avoid confusion of the cases k = 5 and k = 7.

(2.1)
$$P_k(a) = (q^{ka}; q^{k^2})_{\infty} (q^{k(k-a)}; q^{k^2})_{\infty}$$

(2.2)
$$P_k(0) = (q^{k^2}; q^{k^2})_{\infty}.$$

(2.3)
$$\Sigma_k(a,b) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{kbn+3kn(n+1)/2}}{1-q^{k(kn+a)}}, \qquad a \neq 0 \pmod{k}$$

(2.4)
$$\Sigma_k(0,b) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{kbn+3kn(n+1)/2}}{1-q^{k^2n}}$$

(2.5)
$$S_k(b) = \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} \frac{(-1)^n q^{n(3n+1)/2+bn}}{1-q^{kn}}$$

(2.6)
$$S_k(b) = -S_k(k-1-b)$$

(2.7)
$$S_k(b) - S_k(b+k) = \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} (-1)^n q^{bn+n(3n+1)/2}$$

(2.8) $S_k((k-1)/2) = 0$

Each of (2.6)-(2.8) appears on page 99 of [2, eqs. (6.2)-(6.4)].

(2.9)
$$r_b(d,k) = \sum_{n=0}^{\infty} N(b,k,kn+d)q^{kn},$$

(2.10)
$$r_{bc}(d,k) = r_b(d,k) - r_c(d,k).$$

Beyond definitions, we require several identities from [2].

$$\begin{aligned} (2.11) & 2S_5(1) + S_5(4) \\ &= \left\{ q^5 \frac{\Sigma_5(1,0)}{P_5(0)} + q^2 \frac{P_5(0)}{P_5(2)} - 2q^7 \frac{\Sigma_5(2,0)}{P_5(0)} \\ &- q^3 \frac{P_5(0)P_5(1)}{P_5^2(2)} \right\} (q;q)_\infty \quad ([2, \text{ mid page 102}]) \\ (2.12) & r_{13}(1,7) = 0 \quad ([2, \text{ eq. } (6.27)]) \\ (2.13) & r_{13}(5,7) = 0 \quad ([2, \text{ eq. } (6.39)]) \\ (2.14) &- S_7(1) - 2S_7(4) \\ &= \left\{ -q^{16} \frac{\Sigma_7(3,0)}{P_7(0)} + q^2 \frac{P_7(0)}{P_7(2)} - q^4 \frac{P_7(0)}{P_7(3)} \\ &+ 2q^{13} \frac{\Sigma_7(2,0)}{P_7(0)} + q^6 \frac{P_7(0)P_7(1)}{P_7(2)P_7(3)} \right\} (q;q)_\infty \quad ([2, \text{ eq. } (6.49), \text{ p. 104}]) \end{aligned}$$

(2.15)
$$\sum_{n \ge 0} N(1,7,n)q^n = \frac{1}{(q;q)_{\infty}} \left(S_7(1) - S_1(0) \right) \quad (by \ [2, p. \ 103])$$

(2.16)
$$\sum_{n\geq 0} N(3,7,n)q^n \frac{1}{(q;q)_{\infty}} S_7(4).$$
 (by [2, p. 103])

Finally we require the following identity which follows from [6, eq. (III.18), p. 242] by letting $n \to \infty$:

$$(2.17) \qquad \sum_{j=0}^{\infty} \frac{(a;q)_j (1-aq^{2j})(b;q)_j (c;q)_j (d;q)_j (e;q)_j (-1)^j \left(\frac{a^2 q^2}{b c d e}\right)^j q^{\binom{j}{2}}}{(q;q)_j (1-a) \left(\frac{aq}{b};q\right)_j \left(\frac{aq}{c};q\right)_j \left(\frac{aq}{d};q\right)_j \left(\frac{aq}{e};q\right)_j}} = \frac{(aq;q)_{\infty} \left(\frac{aq}{d e};q\right)_{\infty}}{\left(\frac{aq}{d};q\right)_{\infty} \left(\frac{aq}{e};q\right)_{\infty}} \sum_{j=0}^{\infty} \frac{\left(\frac{aq}{b c};q\right)_j (d;q)_j (e;q)_j \left(\frac{aq}{d e}\right)^j}{(q;q)_j \left(\frac{aq}{b};q\right)_j \left(\frac{aq}{e};q\right)_j}}$$

3 New Representation of (1.9)

Actually we shall not differentiate the series in (1.9) until late in the proofs of Theorems 1 and 2.

Theorem 3.

(3.1)
$$\sum_{n=1}^{\infty} \frac{x^n q^{n^2}}{(zq;q)_n \left(\frac{xq}{z};q\right)_n} = 1 + \frac{1}{(xq;q)_\infty} \sum_{n=1}^{\infty} \frac{(xq;q)_n}{(q;q)_{n-1}} (-1)^{n-1} q^{n(3n+1)/2} x^n \\ \times \left(\frac{1}{q^n(1-zq^n)} + \frac{x}{z\left(1-\frac{zq^n}{z}\right)}\right)$$

Remark. As noted in the introduction, the series on the left in (3.1) generates the partitions of n where q marks the number being partitioned, z marks the rank, and x marks the number of parts.

Proof. We apply (2.15) with $d = e = \frac{1}{\tau}$, a = x, $b = \frac{x}{z}$ c = z and $\tau \to \infty$. Hence

$$(3.2) \qquad \sum_{n=0}^{\infty} \frac{x^n q^{n^2}}{(zq;q)_n \left(\frac{xq}{z};q\right)_n} \\ = \lim_{\tau \to 0} \sum_{n=0}^{\infty} \frac{(q;q)_n \left(\frac{1}{\tau};q\right)_n^2 (x\tau^2 q)^n}{(q;q)_n (zq;q)_n \left(\frac{xq}{z};q\right)_n} \\ = \frac{1}{(xq;q)_{\infty}} \lim_{t \to 0} \sum_{n=0}^{\infty} \frac{(x;q)_n (1-xq^{2n}) \left(\frac{x}{z};q\right)_n (z;q)_n \left(\frac{1}{\tau};q\right)_n^2}{(q;q)_n (1-x)(zq;q)_n \left(\frac{xq}{z};q\right)_n (xq\tau;q)_n^2} \\ \times \left(\frac{x^2 q^2 \tau^2}{\left(\frac{x}{z}\right) z}\right) (-1)^n q^{\binom{n}{2}} \\ = \frac{1}{(xq;q)_{\infty}} \left(1 + \sum_{n \ge 1} \frac{(xq;q)_{n-1} (1-xq^{2n})}{(q;q)_n} \frac{(1-z) \left(1-\frac{x}{z}\right)}{(1-zq^n) \left(1-\frac{x}{z}q^n\right)} \\ \times x^n (-1)^n q^{n(3n+1)/2}\right) \end{cases}$$

Now note that

(3.3)
$$\frac{(1-z)(1-\frac{x}{z})(1-xq^{2n})}{(1-zq^n)(1-\frac{x}{z}q^n)}$$
$$= -(1-q^n)(1-xq^n)\left(\frac{1}{q^n(1-zq^n)} + \frac{x}{z(1-\frac{q^nx}{z})}\right)$$
$$+ \frac{(1-xq^{2n})}{q^n}$$

Also we require Sylvester's identity [1, p. 140]

(3.4)
$$1 + \sum_{n=1}^{\infty} \frac{(xq;q)_{n-1}(1-xq^{2n})(-1)^n q^{n(3n-1)/2}}{(q;q)_n}$$

 $=(xq;q)_{\infty}.$

Therefore substituting (3.3) into (3.2), we find

$$(3.5) \qquad \sum_{n=0}^{\infty} \frac{x^n q^{n^2}}{(zq;q)_n (\frac{xq}{z};q)_n} \\ = \frac{1}{(xq;q)_{\infty}} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(xq;q)_n}{(q;q)_{n-1}} (-1)^{n-1} q^{n(3n+1)/2} x^n \\ \times \left(\frac{1}{q^n (1-zq^n)} + \frac{x}{z(1-\frac{xq^n}{z})} \right) \right. \\ \left. + \sum_{n\geq 1} \frac{(-1)^n (xq;q)_{n-1} (1-xq^{2n}) x^n q^{n(3n-1)/2}}{(q;q)_n} \right\} \\ = 1 + \frac{1}{(xq;q)_{\infty}} \sum_{n=1}^{\infty} \frac{(xq;q)_n}{(q;q)_{n-1}} (-1)^{n-1} q^{n(3n+1)/2} x^n \\ \left. \left(\frac{1}{q^n (1-zq^n)} + \frac{x}{z(1-\frac{xq^n}{z})} \right) \right.$$
 (by (3.4)).

It is now an easy matter to expand the terms involving z into a geometric series which leads directly to the generating functions for NT(b, k, n).

Corollary 4. If $0 \le b \le k$,

(3.6)
$$\sum_{n\geq 0} NT(b,k,n)q^{n} = \frac{\partial}{\partial x}\Big|_{x=1} \frac{1}{(xq;q)_{\infty}} \sum_{n\geq 1} \frac{(xq;q)_{n}}{(q;q)_{n-1}} (-1)^{n-1} x^{n} q^{n(3n+1)/2} \\ \left(\frac{q^{n(b-1)}}{1-q^{nk}} + \frac{x^{k-b}q^{(k-1-b)n}}{1-x^{k}q^{kn}}\right)$$

Proof. We write

(3.7)
$$\frac{1}{1 - zq^n} = \sum_{j=0}^{\infty} z^j q^{nj},$$

and

(3.8)
$$\frac{1}{z(1-\frac{xq^n}{z})} = \sum_{j=0}^{\infty} z^{-j-1} x^j q^{nj}.$$

We want only those partitions with ranks $= b \pmod{k}$. So we take exponents in (3.7) of the form kj + b, and in (3.8) of the form kj + (k-1-b).

Thus in the right side of (3.1),

$$\frac{1}{q^n(1-zq^n)}$$
 is replaced by $\frac{q^{n(b-1)}}{1-q^{nk}}$,

and

$$\frac{1}{z(1-\frac{xq^n}{z})} \text{ is replaced by } \frac{x^{k-b}q^{(k-1-b)}}{1-x^kq^{kn}},$$

and this yields (3.6) once we differentiate by x to count each partition with a weight equal to its number of parts.

4 Proof of Theorem 1

To prove Theorem 1, it is sufficient to show that in

(4.1)
$$\sum_{n \ge 0} \left(NT(1,5,n) - NT(4,5,n) + 2NT(2,5,n) - 2NT(3,5,n) \right) q^n$$

the coefficients of q^{5n+1} and q^{5n+4} are congruent to 0 modulo 5. By corollary 4, the expression in (4.1) is equal to

$$(4.2) \qquad \frac{\partial}{\partial x}\Big|_{x=1} \left(\frac{1}{(xq;q)_{\infty}} \sum_{n\geq 1} \frac{(xq;q)_n}{(q;q)_{n-1}} (-1)^{n-1} x^n q^{n(3n+1)/2} \\ \left\{ \left(\frac{1}{1-q^{5n}} + \frac{x^4 q^{3n}}{1-x^5 q^{5n}}\right) \\ - \left(\frac{q^{3n}}{1-q^{5n}} + \frac{x}{1-x^5 q^{5n}}\right) \\ + 2 \left(\frac{q^n}{1-q^{5n}} + \frac{x^2 q^n}{1-x^5 q^{5n}}\right) \\ - 2 \left(\frac{q^{2n}}{1-q^{5n}} + \frac{x^2 q^n}{1-x^5 q^{5n}}\right) \right\} \right) \\ = \frac{\partial}{\partial x}\Big|_{x=1} \frac{1}{(xq;q)_{\infty}} \sum_{n\geq 1} \frac{(xq;q)_n}{(q;q)_{n-1}} (-1)^{n-1} q^{n(3n+1)/2} \\ \times \frac{(1-x)(1-xq^{2n})(1-q^n)(1-xq^n)}{(1-q^{5n})(1-xq^{5n})} \\ \times \left\{ (q^{4n}+3q^{3n}+q^{2n})x^2 + (3q^{3n}+4q^{2n}+3q^n)x \\ + q^{2n}+3q^n+1 \right\}$$

Now we note the simple fact that

(4.3)
$$\frac{\partial}{\partial x}\Big|_{x=1}(1-x)F(x,q,z) = -F(1,q,z).$$

Consequently the expression in (4.1) is equal to

$$(4.4) \qquad -\frac{1}{(q;q)_{\infty}} \sum_{n\geq 1} (1-q^n)(-1)^{n-1} q^{n(3n+1)/2} \frac{(1-q^{2n})(1-q^n)^2}{(1-q^{5n})^2} \\ \times (q^{4n} + 6q^{3n} + 6q^{2n} + 6q^n + 1) \\ \equiv -\frac{1}{(q;q)_{\infty}} \sum_{n\geq 1} \frac{(1-q^n)^2(1-q^{2n})(-1)^{n-1}q^{n(3n+1)/2}}{1-q^{5n}} \pmod{5} \\ = -\frac{1}{(q;q)_{\infty}} (S_5(0) - 2S_5(1)) \\ = -\frac{1}{(q;q)_{\infty}} (-S_5(4) - 2S_5(1)) \qquad (by (2.6)) \\ = \left(q^5 \frac{\Sigma_5(1,0)}{P_5(0)} + q^2 \frac{P_5(0)}{P_5(2)} - 2q^7 \frac{\Sigma_5(2,0)}{P_5(0)} \\ -q^3 \frac{P_5(0)P_5(1)}{P_5^2(2)}\right) \qquad (by (2.11)). \end{cases}$$

We conclude by noting that there are no exponents on $q \equiv 1$ or 4 (mod 5) in this last expression which, in turn, is congruent to the expression in (4.1) modulo 5. Thus Theorem 1 is proved.

5 Proof of Theorem 2

To prove Theorem 2, it is sufficient to show that in

(5.1)
$$\sum_{n \ge 0} \left(NT(1,7,n) - NT(6,7,n) \right) + NT(2,7,n) - NT(5,7,n) - NT(3,7,n) + NT(4,7,n) q^n$$

the coefficients of q^{7n+1} and q^{7n+5} are congruent to 0 modulo 7. By Corollary 4, the expression in (4.1) is equal to

(5.2)
$$\frac{\partial}{\partial x}\Big|_{x=1} \left(\frac{1}{(xq;q)_{\infty}} \sum_{n\geq 1} \frac{(xq;q)_n}{(q;q)_{n-1}} (-1)^{n-1} x^n q^{n(3n+1)/2} \right) \\ \times \sum_{b=1}^7 \alpha(b) F(b) \right)$$

where

(5.3)
$$\alpha(b) = \begin{cases} 1 & \text{if } b = 1, 2, 4 \\ -1 & \text{if } b = 3, 5, 6 \end{cases}$$

 $\quad \text{and} \quad$

(5.4)
$$F(b) = \frac{q^{n(b-1)}}{1-q^{7n}} + \frac{x^{7-b}q^{(6-b)n}}{1-x^7q^{7n}}.$$

Now

(5.5)
$$\sum_{b=1}^{7} \alpha(b)F(b)$$
$$= -\frac{(1-x)(1-xq^{2n})(1-q^n)(1-xq^n)}{(1-xq^{7n})(1-q^{7n})}$$
$$\times \left\{ x^4(q^{8n}+2q^{7n}+q^{6n}+2q^{5n}+q^{4n}) + x^3(2q^{7n}+3q^{6n}+4q^{5n}+3q^{4n}+2q^{3n}) \\ x^2(q^{6n}+4q^{5n}+4q^{4n}+4q^{3n}+q^{2n}) \\ x(2q^{5n}+3q^{4n}+4q^{3n}+3q^{2n}+2q^n) \\ q^{4n}+2q^{3n}+q^{2n}+2q^n+1 \right\}$$

So if we set x=1 in the { } portion of (5.5), the result is

$$\frac{(1+q^n)^3(1-q^{7n})}{(1-q^n)} + 7(q^{5n}+q^{4n}+q^{3n})$$

Thus if we invoke (4.3), we find that the expression in (5.1) is congruent modulo 7 to

$$(5.6) \quad \frac{1}{(q;q)_{\infty}} \sum_{n \ge 1} \frac{(1-q^n)^2 (1-q^{2n})(1+3q^n+q^{2n})(-1)^{n-1} q^{n(3n+1)/2}}{1-q^{7n}} \\ = \frac{1}{(q;q)_{\infty}} \left(S_7(0,7) + S_7(1,7) - 5S_7(2,7) \right) \\ = \frac{1}{(q;q)_{\infty}} \left(S_7(0) + 2(S_7(1,7) + 2S_7(4,7)) - S_7(1,7) + S_7(4,7) \right) \quad (by (2.6)) \\ = \frac{2}{(q;q)_{\infty}} \left(S_7(1,7) + 2S_7(4,7) \right) \\ - \sum_{n=0}^{\infty} N(1,7,n)q^n + \sum_{n=0}^{\infty} N(3,7,n)q^n \right) \quad (by (2.15) \text{ and } (2.16)) \\ = 2 \left\{ -q^{16} \frac{\Sigma_7(3,0)}{P_7(0)} + q^2 \frac{P_7(0)}{P_7(2)} - q^4 \frac{P_7(0)}{P_7(3)} + 2q^{13} \frac{\Sigma_7(2,0)}{P_7(0)} + q^6 \frac{P_7(0)P_7(1)}{P_7(2)P_7(3)} \right\} \\ \sum_{n=0}^{\infty} \left(N(3,7,n) - N(1,7,n) \right) q^n.$$

Now the expression inside $\{ \}$ has no powers of q congruent to 1 or 5 mod 7. And by (2.12) and (2.13) we see that

$$\sum_{n=0}^{\infty} (N(3,7,n) - N(1,7,n)) q^n$$

also contains no powers of q congruent to 7n + 1 or 7n + 5 modulo 7. Thus the assertion about (5.1) is proved, and Theorem 2 is established.

6 Conclusion

The real surprise in this paper is that anyone would have guessed Theorem 1 and 2 in the first place. For this fact, we are indebted to George Beck's diligence and ingenuity.

George Beck has similar conjectures related to the crank which have been treated by Shane Chern.

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