Sequences in Partitions, Double q-Series and the Mock Theta Function $\rho_3(q)$

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Dedicated to an outstanding mathematician and my good friend, Peter Paule.

Keyword: Partitions, q-series, mock theta function

Abstract

The object of this paper is to build on a previous study related to Schur's 1926 partition theorem done by Andrews, Bringmann and Mahlburg. We present generalization of the double series considered by them. Two particular infinite families are identified. Beyond the main theorems, applications are made to Rogers-Ramanujan identities and mock theta functions.

1 Introduction

This paper is devoted to the partition-theoretic aspects of

$$H_{r,s}(k,a,x,q) = \sum_{n,j\geq 0} \frac{(-1)^j x^{aj+n} q^{(aj+n)^2+k\binom{n}{2}+rn+2asj}}{(q;q)_n(q^{2a};q^{2a})_j}, \tag{1.1}$$

where $a \geq 1$ and $k \geq 0$ are integers, and

$$(A;q)_m = (1-A)(1-Aq)\dots(1-Aq^{m-1}).$$

In particular, we study two subfamilies, namely

$$f_{r,s}(a,x,q) = H_{r,s}(1,a,x,q), \tag{1.2}$$

and

$$g_{r,s}(a,x,q) = H_{r,s}(0,a,x,q). \tag{1.3}$$

There are two main theorems. In each of these theorems, the generating function in question will have the exponent of q recording the number being partitioned and the exponent of x will record the number of parts of the partitions being considered.

Theorem 1. *$g_{0,0}(a,x,q)$ is the generating function for partitions wherein the difference between parts is at least 2 and where maximal sequences of consecutive odd parts must be of length congruent to $0,1,2,\ldots,a-1(\mod 2a)$. Odd parts are consecutive if they differ by 2.*

$g_{1,1}(a,x,q)$ is the generating function for the same partition with the added condition that 1 is not allowed as a part.

For example,

$$g_{0,0}(2,1,q) = 1 + q + q^2 + q^3 + q^4 + 2q^5 + 3q^6 + 3q^7 + 3q^8 + 4q^9 + \ldots$$

and the four relevant partition of 9 are 9, $8+1$, $7+2$, $6+3$. Note that $5+3+1$ is excluded because it is a maximal sequence of odd parts of length $3 \not\equiv 0$ or $1(\mod 4)$.

We shall also see that, as a corollary of Theorem 1 (cf. Theorem 8),

$$g_{0,0}(1,1,q) = \prod_{\substack{n=1 \\ n\neq 0,\pm 3(\mod 7)}} (1-q^{2n})^{-1}. \tag{1.4}$$

Our second central result is related to overpartitions, the subject initiated by Corteel and Lovejoy [10]. An overpartition is an ordinary integer partition with the added condition that the first appearance of any given part may be overlined. Thus the eight overpartitions of 3 are 3, $\bar{3}$, $2+1$, $\bar{2}+1$, $2+\bar{1}$, $\bar{2}+\bar{1}$, $1+1+1$, $\bar{1}+1+1$.

Theorem 2. *$f_{0,0}(a,xq^2,q)/(xq^2;q^2)_\infty$ is the generating function for overpartitions subject to the following conditions (i) all parts are ≥ 2, (ii) $\bar{2}$ is never a part, (iii) all odd parts are distinct and overlined, (iv) if a is odd, the following subsequence of parts is not allowed for any $j \geq 0$:*

$$\overline{(2j+3)} + \overline{(2j+6)} + (2j+6) + \overline{(2j+10)} + (2j+10) + \cdots + \overline{(2j+2a)} + (2j+2a), \tag{1.5}$$

(v) if a is even, the following subsequence of parts is not allowed for any $j \geq 0$:

$$\overline{(2j+4)} + (2j+4) + \overline{(2j+8)} + (2j+8) + \overline{(2j+12)} + (2j+12) + \cdot\cdot + \overline{(2j+2a)} + (2j+2a). \tag{1.6}$$

(vi) the difference between overlined parts is ≥ 3.

When $a=1$, Theorem 2 is connected to the Rogers-Ramanujan identities [2, Ch. 7] (see Section 9):

$$f_{0,0}(1,x,q) = \sum_{n\geq 0} \frac{x^n q^{2n^2}}{(q^2;q^2)_n}. \tag{1.7}$$

When $a=3$, Theorem 2 is related to the third order mock theta function $\rho_3(q)$ [19, p. 62] (see Section 10), namely

$$\frac{f_{0,0}(3,q^2,q)}{(q^2;q^2)_\infty} = \frac{(q^3;q^6)_\infty}{(q;q^2)_\infty}\rho_3(q) = \frac{(q^3;q^6)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(q;q^2)_{n+1}}{(q^3;q^2)_{n+1}} \tag{1.8}$$

$$= 1 + q^2 + q^3 + 3q^4 + 2q^5 + 5q^6 + 4q^7 + 9q^8 + \cdots,$$

and the nine relevant partitions of 8 are 8, $\bar{8}$, $6+2$, $\bar{6}+2$, $\bar{4}+4$, $4+4$, $4+2+2$, $\bar{4}+2+2$, $2+2+2+2$. $\rho_3(q)$ is the third order mock theta function[19, p. 62] defined by

$$\rho_3(q) = \sum_{n\geq 0} \frac{(q;q^2)_n q^{2n(n+1)}}{(q^3;q^6)_n}.$$

There is a somewhat scattered history of sequences in partitions dating back to Sylvester [18]. In section 2, we provide a sketch of previous work including the joint paper with Bringmann and Mahlburg [6]. Section 3 provides the necessary q-difference equation satisfied by $H_{r,s}(k,a,x,q)$. The following three sections then develop the theory surrounding $g_{r,s}(a,x,q)$ including Theorem 1 and equation (1.4). Sections 7 through 10 study $f_{r,s}(a,x,q)$ including Theorem 2 and equation (1.5). We conclude with some open questions.

2 History of Sequences in Partitions

J. J. Sylvester was the first to look at sequences in partitions ([18, Th. 2.12]).

Sylvester's Theorem. *Let $A_k(n)$ denote the number of partitions of n into odds with exactly k different parts. Let $B_k(n)$ denote the number of partitions of n into distinct parts composed exactly k noncontiguous sequences of one or more consecutive integers. Then*

$$A_k(n) = B_k(n).$$

For example $A_3(13) = 5$ enumerating $9+3+1$, $7+5+1$, $7+3+1+1+1$, $5+3+1+1+1+1+1$, and $5+3+3+1+1$; $B_3(13) = 5$ enumerating $9+3+1$, $8+4+1$, $7+5+1$, $7+4+2$, $6+4+2+1$.

P. A. MacMahon [15, Sec. VII, Ch. IV, pp 49-58] was the next to consider sequences in partitions. The most well-known of his theorems is the following

MacMahon's Theorem. *The number of partitions of n without sequences (i.e. no consecutive integers) and no 1's equals the number of partitions of n into parts $\not\equiv \pm 1 (\mod 6)$.*

In 1978, in his unpublished Ph.D. thesis [13, Ch. 5, pp. 51-56] M. D. Hirschhorn proved that the generating function for partitions into distinct parts with all sequences of consecutive integers of length $\leq k$ and with all parts $< n$ is given by

$$\sum_{j\geq 0} x^j q^{\binom{j+1}{2}} \sum_{jl\leq j} (-1)^l q^{k\binom{l}{2}} \begin{bmatrix} n-j \\ l \end{bmatrix}_{q^k} \begin{bmatrix} n-kl-1 \\ j-kl \end{bmatrix}_q, \tag{2.1}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix}_q = \begin{cases} \frac{(q;q)_A}{(q;q)_B (q;q)_{A-B}} & \text{for } 0 \leq B \leq A \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Hirschhorn examined the case when $n \to \infty$ and related the result to the Rogers-Ramanujan identities when $k = 2$.

In 2015, Bringmann et al. [9] rediscovered some of Hirschhorn's theorems (owing to the fact that Hirschhorn's result was never published outside of his Ph.D. thesis).

In 2004, Holroyd, Liggett and Romik [14] looked at $p_k(n)$, the number of partitions of n that do not contain a sequence of consecutive integers of length k. They proved that if

$$G_k(q) = \sum_{n\geq 0} p_k(n)q^n, \tag{2.3}$$

then

$$\log G_k(q) \sim \frac{\pi^2}{6}\left(1 - \frac{2}{k(k+1)}\right)\frac{1}{1-q}, \qquad \text{as } q \to 1^- \tag{2.4}$$

In 2005, a double series representation of $G_k(q)$ was given [3]

$$G_k(q) = \frac{1}{(q;q)_\infty} \sum_{n,j\geq 0} \frac{(-1)^j q^{\binom{k+1}{2}(n+j)^2 + (k-1)\binom{n+1}{2}}}{(q^k;q^k)_j (q^{k+1};q^{k+1})_n}, \tag{2.5}$$

and it was shown that

$$G_2(q) = \frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty} \chi_3(q), \tag{2.6}$$

where $\chi_3(q)$ is one of Ramanujan's third order mock theta functions [18, p. 62] given by

$$\chi_3(q) = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(-q^3;q^3)_n}.$$

An analogous identity will arise in section 10.

There have been subsequent studies related to $p_k(n)$, in [7] and [8].

In 2013, Bringmann et al. [8] studied $\bar{p}_k(n)$, where the concept of sequences in partitions was extended to overpartitions.

Namely, they define *lower k-run overpartitions* to be those overpartitions in which any overlined part must occur within a run of exactly k consecutive overlined parts that terminates below with a gap. More precisely, this means that if some part $\bar{m}$ is overlined, then there is an integer j with $m \in [j+1, j+k]$ such that each of the k overlined parts $\overline{j+1}, \overline{j+2}, \ldots, \overline{j+k}$ appear (perhaps together with non-overlined versions), while no part j (overline or otherwise) appears, and no overlined part $\overline{j+k+1}$ appears.

They proved that if

$$\overline{G_k}(q) = \sum_{n\geq 0} \overline{p_k}(n)q^n, \tag{2.7}$$

then

$$\overline{G_k}(q) = \frac{1}{(q;q)_\infty} \sum_{n,j\geq 0} \frac{(-1)^j q^{\binom{k+1}{2}(n+j)^2+(k+1)\binom{j+1}{2}}}{(q^k;q^k)_n (q^{k+1};q^{k+1})_j}. \tag{2.8}$$

In that paper, the third order mock theta function $\phi_3(q)$ arises [19, p. 62], namely

$$\overline{G_1}(q) = (q;q)_\infty \phi_3(q), \tag{2.9}$$

where

$$\phi_3(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}. \tag{2.10}$$

We now come to the precursor to the current paper, namely [6], a joint work with Bringmann and Mahlburg. One of the main results there (and the one that inspired this article), is, in our current notation

$$\sum_{n,j\geq 0} \frac{(-1)^j x^{n+2j} q^{(n+3j)^2+\binom{n}{2}}}{(q;q)_n (q^6;q^6)_j} = (x;q^3)_\infty \sum_{n=0}^{\infty} \frac{(-q;q^3)_n (-q^2;q^3)_n x^n}{(q^3;q^3)_n}. \tag{2.11}$$

From (2.11), one may deduce Schur's 1926 theorem [16].

Much of [6] is devoted to special cases of

$$R(s,t,l,u,v,w) = \sum_{n,j\geq 0} \frac{(-1)^j q^{s\binom{n+uj}{2}+t(n+uj)+uv\binom{j}{2}+(w+ul)j}}{(q;q)_n (q^{uv};q^{uv})_j}. \tag{2.12}$$

It is easy to see that

$$R(3,1,0,2,3,4) = f_{0,0}(3,1,q),$$

and this fact will lead to interesting identities is section 11.

The only difference between the left side of (2.11) and $f_{00}(3,x,q)$ lies in the exponent on x, and, as we will see in Section 11, this seems to be the subtle difference between Schur's original theorem [16] and what has become known as the Alladi-Schur theorem [5], [6].

3 q-Difference Equations for $H_{r,s}(k,a,x,q)$

Lemma 3. $H_{r+2,s+1}(k,a,x,q) = H_{r,s}(k,a,xq^2,q)$.

Proof.

$$H_{r+2,s+1}(k,a,x,q) = \sum_{n,j\geq 0} \frac{(-1)^j x^{aj+n} q^{(aj+n)^2+k\binom{n}{2}+(r+2)n+2a(s+1)j}}{(q;q)_n (q^{2a};q^{2a})_j}$$

$$
\begin{aligned}
&= \sum_{n,j\geq 0} \frac{(-1)^j (xq^2)^{aj+n} q^{(aj+n)^2+k\binom{n}{2}+rn+2asj}}{(q;q)_n(q^{2a};q^{2a})_j} \\
&= H_{r,s}(k,a,xq^2,q).
\end{aligned}
$$

□

Lemma 4.

$$H_{r,s}(k,a,x,q) - H_{r+1,s}(k,a,x,q) = xq^{1+r} H_{r+k,s}(k,a,xq^2,q).$$

Proof.

$$
\begin{aligned}
&H_{r,s}(k,a,x,q) - H_{r+1,s}(k,a,x,q) \\
&= \sum_{n,j\geq 0} \frac{(-1)^j x^{aj+n} q^{(aj+n)^2+k\binom{n}{2}+rn+2asj}(1-q^n)}{(q;q)_n(q^{2a};q^{2a})_j} \\
&= \sum_{n,j\geq 0} \frac{(-1)^j x^{aj+n+1} q^{(aj+n+1)^2+k\binom{n+1}{2}+r(n+1)+2asj}}{(q;q)_n(q^{2a};q^{2a})_j} \qquad \text{(by shifting } n \text{ to } n+1\text{)} \\
&= xq^{r+1} H_{r+k,s}(k,a,xq^2,q).
\end{aligned}
$$

□

Lemma 5.

$$H_{r,s}(k,a,x,q) - H_{r,s+1}(k,a,x,q) = -x^a q^{a^2+2as} H_{r,s}(k,a,xq^{2a},q).$$

Proof.

$$
\begin{aligned}
&H_{r,s}(k,a,x,q) - H_{r,s+1}(k,a,x,q) \\
&= \sum_{n,j\geq 0} \frac{(-1)^j x^{aj+n} q^{(aj+n)^2+k\binom{n}{2}+rn+2asj}(1-q^{2aj})}{(q;q)_n(q^{2a};q^{2a})_j} \\
&= \sum_{n,j\geq 0} \frac{(-1)^j x^{aj+n+a} q^{(aj+a+n)^2+k\binom{n}{2}+rn+2asj+2as}}{(q;q)_n(q^{2a};q^{2a})_j} \qquad \text{(by shifting } j \text{ to } j+1\text{)} \\
&= -x^a q^{a^2+2as} H_{r,s}(k,a,xq^{2a},q).
\end{aligned}
$$

□

Lemma 6.

$$
\begin{aligned}
&H_{r,s}(k,a,x,q) - H_{r,s}(k,a,xq^2,q) \\
&= -x^a q^{a^2+2as} H_{r,s}(k,a,xq^{2a},q) \\
&\quad + xq^{r+1} H_{r+1,s+1}(k,a,xq^2,q) \\
&\quad + xq^{r+2} H_{r+k-1,s}(k,a,xq^4,q).
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&H_{r,s}(k,a,x,q) - H_{r,s}(k,a,xq^2,q)\\
=&\sum_{n,j\geq 0} \frac{(-1)^j x^{aj+n} q^{(aj+n)^2+k\binom{n}{2}+rn+2asj}\left((1-q^{2aj})+q^{2aj}(1-q^{2n})\right)}{(q;q)_n(q^{2a};q^{2a})_j}\\
=&-x^a q^{a^2+2as} H_{r,s}(k,a,xq^{2a},q)\\
&+\sum_{n,j\geq 0} \frac{(-1)^j x^{n+1+aj} q^{(aj+n)^2+2(aj+n)+1+k\binom{n}{2}+kn+r(n+1)+2asj+2aj}(1+q^{n+1})}{(q;q)_n(q^{2a};q^{2a})_j}\\
=&-x^a q^{a^2+2as} H_{r,s}(k,a,xq^{2a},q)\\
&+xq^{r+1}H_{r+k,s+1}(k,a,xq^2,q)\\
&+xq^{r+2}H_{r+k+1,s+1}(k,a,xq^2,q)\\
=&-x^a q^{a^2+2as} H_{r,s}(k,a,xq^{2a},q)\\
&+xq^{r+1}H_{r+k,s+1}(k,a,xq^2,q)\\
&+xq^{r+2}H_{r+k-1,s}(k,a,xq^4,q) \qquad \text{(by Lemma 3).}
\end{aligned}
$$

$\square$

4 q-Difference Equation for $g_{0,0}(a,x,q)$

The lemmas of section 3 allow us to obtain defining q-difference equations for $g_{0,0}(a,x,q)$. This will be the foundation for the partition-theoretic interpretation.

Theorem 7.

$$g_{1,1}(a,x,q) = g_{0,0}(a,xq^2,q) + xq^2 g_{1,1}(a,xq^2,q), \tag{4.1}$$

$$
\begin{aligned}
g_{0,0}(a,x,q) = {} & g_{1,1}(a,x,q) + xqg_{0,0}(a,xq^2,q)\\
& - x^a q^{a^2} g_{0,0}(a,xq^{2a},q) + x^{a+1}q^{(a+1)^2} g_{0,0}(a,xq^{2a+2},q).
\end{aligned} \tag{4.2}
$$

Proof. Recall that

$$g_{r,s}(a,x,q) = H_{r,s}(0,a,x,q). \tag{4.3}$$

Hence by Lemma 4 with $r=s=1$, $k=0$

$$
\begin{aligned}
g_{1,1}(a,x,q) &= g_{2,1}(a,x,q) + xq^2 q_{1,1}(a,xq^2,q)\\
&= g_{0,0}(a,xq^2,q) + xq^2 g_{1,1}(a,xq^2,q),
\end{aligned} \tag{4.4}
$$

by Lemma 3, and (4.1) is established.

We now turn to (4.2). First, by Lemma 5, with $r=1$, $s=0$, $k=0$

$$g_{1,0}(a,x,q) = g_{1,1}(a,x,q) - x^a q^{a^2} g_{1,0}(a,xq^{2a},q). \tag{4.5}$$

By Lemma 4, with $r = s = k = 0$,

$$g_{0,0}(a, x, q) = g_{1,0}(a, x, q) + xqg_{0,0}(a, xq^2, q). \tag{4.6}$$

Utilizing (4.6) to eliminate g_{10} from (4.5), we obtain after simplification

$$\begin{aligned} g_{1,1}(a, x, q) = {} & g_{0,0}(a, x, q) - xqg_{0,0}(a, xq^2, q) \\ & + x^a q^{a^2} \left(g_{0,0}(a, xq^{2a}, q) - xq^{2a+1} g_{0,0}(a, xq^{2a+2}, q) \right), \end{aligned} \tag{4.7}$$

and isolating $g_{0,0}(a, x, q)$ on one side of the equation we find that (4.2) has been proved. □

5 Proof of Theorem 1

Here is what is required, we observe from (1.1), that both $g_{0,0}(a, x, q)$ and $g_{1,1}(a, x, q)$ may (for integer $a \geq 1$) be expanded into double power series in x and q. We want the coefficient of $x^m q^n$ in each series to be the number of partitions of n into m parts as prescribed in Theorem 1. We also note that the initial conditions,

$$g_{r,s}(a, 0, q) = g_{r,s}(a, x, 0) = 1, \tag{5.1}$$

are fully consistent with the assertion that the empty partition of 0 is counted by each class of partitions.

It is also clear that the q-difference equations (4.1) and (4.2) together with (5.1) uniquely define both $g_{0,0}(a, x, q)$ and $g_{1,1}(a, x, q)$. So to conclude the proof of Theorem 1 we only need to show that the generating functions for the partitions described in Theorem 1 fullfill (4.1) and (4.2).

In the following, we note that the replacement of x by xq^j in any of our generating functions, in fact, adds j to each part of each partition being enumerated.

To treat (4.1), we split the partitions asserted to be enumerated by $g_{1,1}(a, x, q)$ into two classes: (i) those partitions that do not contain a 2, and (ii) those that do contain a 2. The partitions in (i) are clearly those enumerated by $g_{0,0}(a, xq^2, q)$. The partitions considered by (ii) must have a 2 (hence xq^2) and the remaining parts must be ≥ 4 (hence $g_{1,1}(a, xq^2, q)$). Note that the last transformation does not alter the parity of parts nor the length of subsequences of consecutive odd integers.

Equation (4.2) is rather more intricate. How does the right hand side of (4.2) account precisely for the partitions being generated by $g_{0,0}(a, x, q)$, the left hand side of (4.2)?

Clearly $g_{1,1}(a, x, q)$ covers the partitions that have no 1 as a part.

The term $xqg_{0,0}(a, xq^2, q)$ correctly generates the partitions that contain 1 as a part with the following exception: (A) we now have 1 in subsequences of consecutive odd parts of length $a(\mod 2a)$, and (B) we do not have 1 in any subsequences of consecutive odd parts of length $0(\mod 2a)$.

To rectify (A) and (B) requires the final two terms on the right side of (4.2). The term

$$-x^a q^{a^2} g_{0,0}(a, xq^{2a}, q) = -x^a q^{1+3+4+\cdots+(2a-1)} g_{0,0}(a, xq^{2a}, q)$$

does subtract off the offending sequences from (A), but it also introduces with a minus sign sequences of odds (starting with 1) of length $a+1, a+2, \ldots, 2a-1(\mod 2a)$.

To correct for this, the term

$$\begin{aligned} &x^{a+1} q^{(a+1)^2} g_{0,0}(a, xq^{2(a+1)}, q) \\ =&x^{a+1} q^{1+3+5+\cdots+(2a+1)} g_{0,0}(a, xq^{2(a+1)}, q) \end{aligned}$$

adds back in sequences of consecutive odd parts (starting with 1) of length $a+1, a+2, \ldots, 2a(\mod 2a)$. I.e. it cancels the newly introduced sequences of length $a+1, a+2, \ldots, 2a-1(\mod 2a)$ and puts back in sequences of length $2a \equiv 0(\mod 2a)$.

Thus (4.2) has been established for $g_{0,0}(a, x, q)$ and $g_{1,1}(a, x, q)$ as generating functions for the partitions described in Theorem 1. Hence Theorem 1 is proved. $\square$

6 Rogers-Ramanujan Aspects of $g_{r,s}(1, x, q)$

We shall now reveal, both via analysis and via partitions, the relation of $g_{r,s}(1, x, q)$ to the Rogers-Ramanujan identities at modulus 14.

Theorem 8.

$$g_{0,0}(1, 1, q) = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 3(\mod 7)}}^{\infty} \frac{1}{1-q^{2n}}, \tag{6.1}$$

$$g_{1,1}(1, 1, q) = \prod_{\substack{n=1 \\ n \not\equiv 0, 0, \pm 2(\mod 7)}}^{\infty} \frac{1}{1-q^{2n}}, \tag{6.2}$$

$$g_{0,0}(1, q^2, q) = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 1(\mod 7)}}^{\infty} \frac{1}{1-q^{2n}}. \tag{6.3}$$

Analytic Proof.

$$\begin{aligned} g_{r,s}(1, x, q) &= \sum_{n,j \geq 0} \frac{(-1)^j x^{n+j} q^{(j+n)^2+rn+2sj}}{(q;q)_n (q^2;q^2)_j} \\ &= \sum_{n \geq 0} \frac{x^n q^{n^2+rn} (xq^{1+2n+2s}; q^2)_\infty}{(q;q)_n} \qquad \text{(by [2, p. 19, eq. (2.2.6)])} \end{aligned}$$

$$=(xq;q^2)_\infty \sum_{n\geq 0} \frac{x^n q^{n^2+rn}}{(q;q)_n (xq;q^2)_{n+s}}.$$

Hence

$$\begin{aligned} g_{0,0}(1,1,q) &= (q;q^2)_\infty \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n (q;q^2)_n} \\ &= \frac{(q;q^2)_\infty (q^6;q^{14})_\infty (q^8;q^{14})_\infty (q^{14};q^{14})_\infty}{(q;q)_\infty} \qquad \text{(by [17, p. 158, eq. (61)])} \\ &= \prod_{\substack{n=1 \\ n\not\equiv 0,\pm 3 (\bmod 7)}}^{\infty} \frac{1}{1-q^{2n}}. \end{aligned}$$

Next

$$\begin{aligned} g_{1,1}(1,1,q) &= (q;q^2)_\infty \sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n (q;q^2)_{n+1}} \\ &= \frac{(q;q^2)_\infty (q^4;q^{14})_\infty (q^{10};q^{14})_\infty (q^{14};q^{14})_\infty}{(q;q)_\infty} \qquad \text{(by [17, p. 158, eq, (60)])} \\ &= \prod_{\substack{n=1 \\ n\not\equiv 0,\pm 2 (\bmod 7)}} \frac{1}{1-q^{2n}}. \end{aligned}$$

Finally

$$\begin{aligned} g_{0,0}(1,q^2,q) &= (q^3;q^2)_\infty \sum_{n\geq 0} \frac{q^{n^2+2n}}{(q;q)_n (q^3;q^2)_n} \\ &= \frac{(q;q^2)_\infty (q^2;q^{14})_\infty (q^{12};q^{14})_\infty (q^{14};q^{14})_\infty}{(q;q)_\infty} \qquad \text{(by [17, p. 157, eq. (59)])} \\ &= \prod_{\substack{n=1 \\ n\not\equiv 0,\pm 1 (\bmod 7)}}^{\infty} \frac{1}{1-q^{2n}}. \end{aligned}$$

Proof via Partitions

We recall the following special case of B. Gordon's generalization of the Rogers-Ramanujan identities [2, Ch. 7].

Theorem. *The number of partitions of an even $2N$ into even parts where: (1) none appears more than twice, (2) if a part appears twice then all parts are at least 4 units away, and (3) 2 appears at most j times ($j = 0, 1, 2$) EQUALS the coefficient of q^{2N} in*

$$\prod_{\substack{n=1 \\ n\not\equiv 0,\pm (j+1) (\bmod 7)}}^{\infty} \frac{1}{1-q^{2n}}.$$

Now we need only identify the partitions generated by $g_{0,0}(1,1,q)$, $g_{1,1}(1,1,q)$ and $g_{0,0}(1,q^2,q)$ respectively with the partitions given above in the special case of Gordon's Theorem.

We know that in $g_{0,0}(1,x,q)$ and $g_{1,1}(1,x,q)$ all parts differ by at least 2. Now suppose we have a sequence of consecutive odd integers as parts (note that since $a=1$, the sequence must be of even length):

$$\big(<(2h-3)\big)+(2h-1)+(2h+1)+(2h+3)+(2h+5)+\cdots+(2i-1)+(2i+1)+(>2i+3)$$

and we replace this by

$$\big(<(2h-3)\big)+(2h)+(2h)+(2h+4)+(2h+4)+\cdots+(2i)+(2i)+(>2i+3).$$

Thus whenever odd parts appear they must appear in pairs as indicated, and, as we see, these directly transform into the repeated parts that are allowed in Gordon's theorem.

Finally $g_{0,0}(1,1,q)$ would allow $1+3$ to appear translating into 2 appearing twice. So by the Theorem,

$$g_{0,0}(1,1,q)=\prod_{\substack{n=1\\ n\not\equiv 0,\pm 3(\mod 7)}}^{\infty}\frac{1}{1-q^{2n}};$$

$g_{1,1}(1,1,q)$ allows no 1's so 2 can appear at most once after translation. Hence by the theorem,

$$g_{1,1}(1,1,q)=\prod_{\substack{n=1\\ n\not\equiv 0,\pm 2(\mod 7)}}\frac{1}{1-q^{2n}},$$

and lastly $g_{0,0}(1,q^2,q)$ has smallest part ≥ 3 so no 2's appear at all. Hence by the Theorem

$$g_{0,0}(1,q^2,q)=\prod_{\substack{n=1\\ n\not\equiv 0,\pm 1(\mod 7)}}^{\infty}\frac{1}{1-q^{2n}}.$$

□

7 q-Difference Equations for $f_{00}(a,x,q)$

Paradoxically the q-difference equation is simpler than the one for $g_{00}(a,x,q)$ while the partition theoretic interpretation is a good deal more complicated.

Theorem 9.

$$f_{0,0}(a,x,q)=f_{0,0}(a,xq^2,q)+(xq+xq^2)f_{0,0}(a,xq^4,q)$$
$$+x^2q^5f_{0,0}(a,xq^6,q)-x^aq^{a^2}f_{0,0}(a,xq^{2a},q). \qquad (7.1)$$

Proof. By Lemma 6, with $r = s = 0$, $k = 1$,

$$f_{0,0}(a,x,q) = f_{0,0}(a,xq^2,q) - x^a q^{a^2} f_{0,0}(a,xq^{2a},q) \\ +xqf_{1,1}(a,xq^2,q) + xq^2 f_{0,0}(a,xq^4,q). \tag{7.2}$$

Next by Lemma 4, with $r = s = k = 1$,

$$\begin{aligned} f_{1,1}(a,x,q) &= f_{2,1}(a,x,q) + xq^2 f_{2,1}(a,xq^2,q) \\ &= f_{0,0}(a,xq^2,q) + xq^2 f_{0,0}(a,xq^4,q). \qquad \text{(by Lemma 3)} \end{aligned} \tag{7.3}$$

Using (7.3) to eliminate $f_{1,1}(a,xq^2,q)$ from (7.2), we obtain (7.1). □

It would be lovely if we could use $f_{0,0}(a,x,q)$ directly as a generating function for partitions. However if $a > 1$, the expansion of $f_{0,0}(a,x,q)$ has negative terms. For example

$$f_{0,0}(2,x,q) = 1 + xq + xq^2 + xq^3 + (-x^2 + x)q^4 + \cdots$$

This problem is overcome by introducing the factor $1/(xq^2;q^2)_\infty$ in Theorem 2.

In addition, the replacement of x by xq^2 yielding $f_{0,0}(a,xq^2,q)/(xq^2;q^2)_\infty$ is done primarily to produce (1.8) in the case $a = 3$.

8 Proof of Theorem 2

Before we undertake this proof, a few comments are in order. First, this is a result about overpartitions, where the generating function is

$$F(a,x,q) := \frac{f_{0,0}(a,xq^2,q)}{(xq^2;q^2)_\infty}. \tag{8.1}$$

At first glance, it appears that the $f_{0,0}(a,xq^2,q)$ produces the overlined parts, and $1/(xq^2;q^2)_\infty$ produces the non-overlined parts; so why mix the two. Of course, the answer lies in conditions (v) and (vi) where interwoven sequences of overlined and non-overlined parts are excluded.

We observe that the xq^2 in $f_{0,0}(a,xq^2,q)$ is necessary for the proof of Theorem 2, but $f_{0,0}(a,x,q)$ would be more natural for the cases $a = 1$ and 3 treated in sections 9 and 10.

We note for subsequent use that the sums in (1.5) and (1.6) are both equal to $a^2 + 2a$ where $j = 0$, and each has exactly a summands.

In order to understand the intricacies of $F(a,x,q)$, we rewrite (7.1) in terms of $F(a,x,q)$ with x replaced by xq^2:

(8.2)

$$F(a,x,q) = \frac{F(a,xq^2,q)}{1-xq^2} + \frac{(xq^{\bar{3}} + xq^{\bar{4}})}{(1-xq^2)(1-xq^4)} F(a,xq^4,q) \\ + \frac{x^2 q^{\bar{3}+\bar{6}} F(a,xq^6,q)}{(1-xq^2)(1-xq^4)(1-xq^6)} - \frac{x^a q^{a^2+2a} F(a,xq^{2a},q)}{(1-xq^2)(1-xq^4)\cdots(1-xq^{2a})}.$$

Now if we let $a \to \infty$, we see from (8.1) and (1.2) that

$$F(\infty, x, q) = \frac{1}{(xq^2; q^2)_\infty} \sum_{n \geq 0} \frac{x^n q^{3n(n+1)/2}}{(q;q)_n}. \tag{8.3}$$

Thus clearly $F(\infty, x, q)$ is the generating function for overpartitions where only even parts can avoid overlines, and the difference between overlined parts is ≥ 3, and all overlined parts are ≥ 3.

Indeed, if we let $a \to \infty$ in (8.2) we see that the final term vanishes and what remains is a q-difference equation that uniquely defines the generating function for the overpartitions listed in the previous paragraph.

In order to complete the proof of Theorem 2 we must determine the effect of the final term in (8.2). This is, indeed, accounted for by condition (iv) and (v) in Theorem 2. There are exactly a summands in each of (1.5) and (1.6). Also as noted previously, when $j = 0$, the numerical sum in both (1.5) and (1.6) is $a^2 + 2a$.

Thus final term in (8.2) excludes either

$$\bar{3} + \bar{6} + 6 + \overline{10} + 10 + \cdots + \overline{2a} + 2a$$

if a is odd, and

$$\bar{4} + 4 + \bar{8} + 8 + \overline{12} + 12 + \cdots + \overline{2a} + 2a$$

if a is even.

In addition, the instances of (1.5) and (1.6) with $j > 0$ are thus also excluded by the action of (8.2) as it generates the partitions of Theorem 2. $\square$

9 Theorem 2 for $a = 1$

We shall provide two proofs of (1.7).

First proof (analytic)

$$f_{0,0}(1, x, q) = \sum_{n,j \geq 0} \frac{(-1)^j x^{n+j} q^{(n+j)^2 + \binom{n}{2}}}{(q;q)_n (q^2;q^2)_j} \tag{9.1}$$

$$= \sum_{N \geq 0} x^N q^{N^2} \sum_{j=0}^{N} \frac{(-1)^j q^{\binom{N-j}{2}}}{(q;q)_{N-j} (q^2;q^2)_j} \tag{9.2}$$

$$= \sum_{N \geq 0} \frac{x^N q^{N^2}}{(q;q)_N} \sum_{j=0}^{N} \frac{(q^{-N};q)_j q^j}{(q;q)_j (-q;q)_j} \tag{9.3}$$

$$= \sum_{N \geq 0} \frac{x^N q^{N^2}}{(q;q)_N} \cdot \frac{q^{N^2}}{(-q;q)_N} \qquad \text{(by [12, p. 236, eq. (II.6)])} \tag{9.4}$$

$$= \sum_{n \geq 0} \frac{x^N q^{2N^2}}{(q^2;q^2)_N} \tag{9.5}$$

which is equivalent to (1.7).

Second Proof (combinatorial)

When $a = 1$, condition (iv) and (1.5) required that no parts are odd. Hence the overlined parts are all even, ≥ 4, and differ by ≥ 3 and thus must differ by ≥ 4. Hence

$$\frac{f_{0,0}(1, xq^2, q)}{(xq^2; q^2)_\infty} = F(1, x, q) = \frac{1}{(xq^2; q^2)_\infty} \sum_{n \geq 0} \frac{q^{2n^2+2n} x^n}{(q^2; q^2)_n} \tag{9.6}$$

where the product generates the non-overlined parts and the series generated the overlined parts.

Clearly (9.6) is equivalent to (1.7). ☐

10 Theorem 2 for $a = 3$

To treat (1.8), we first require two lemmas:

Lemma 10.

$$\rho_3(q) = \frac{(q^2; q^2)_\infty}{\prod_{j=1}^{\infty} (1 + q^{2j-1} + q^{4j-2})} \sum_{n \geq 0} \frac{\prod_{j=1}^{n} (1 + q^{2j-1} + q^{4j-2}) q^{2n}}{(q^2; q^2)_n}. \tag{10.1}$$

Proof. In [11, pg. 61, eq. (26.87)], N. J. Fine proved that if $w = e^{2\pi i/3}$

$$\rho_3(q) = \sum_{n \geq 0} \frac{(w^{-1}q)^n}{(wq; q^2)_{n+1}}. \tag{10.2}$$

Now

$$\begin{aligned}
&\sum_{n \geq 0} \frac{\prod_{j=1}^{n} (1 + q^{2j-1} + q^{4j-2})}{(q^2; q^2)_n} q^{2n} \\
&= \sum_{n \geq 0} \frac{(wq; q^2)_n (w^{-1}q; q^2)_n q^{2n}}{(q^2; q^2)_n} \\
&= \frac{(w^{-1}q; q^2)_\infty (wq^3; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \geq 0} \frac{(w^{-1}q)^n}{(wq^3; q^2)_n} \qquad \text{(by [12, p. 241, (III.1)])} \\
&= \frac{\prod_{j=1}^{\infty} (1 + q^{2j-1} + q^{4j-2})}{(q^2; q^2)_\infty} \rho_3(q),
\end{aligned}$$

by (10.2). $\square$

Lemma 11.

$$f_{0,0}(3,xq^2,q)=(xq^2;q^2)_\infty \sum_{n\geq 0}\frac{\prod_{j=1}^{n}(1+q^{2j-1}+q^{4j-2})x^nq^{2n}}{(q^2;q^2)_n}. \tag{10.3}$$

Proof. It is clear that $F(z,x,q)$ is uniquely determined by the initial conditions

$$f_{0,0}(3,0,q)=f_{0,0}(3,x,0)=1,$$

and the q-difference equation (7.1) which simplifies to

$$\begin{aligned} f_{0,0}(3,xq^2,q)=\ & f_{0,0}(3,xq^4,q)+(xq^3+xq^4)f_{0,0}(3,xq^6,q)\\ &+x^2q^9(1-xq^6)f_{0,0}(3,xq^8,q). \end{aligned} \tag{10.4}$$

Now let

$$f(x)=f(x,q):=\sum_{n\geq 0}\frac{\prod_{j=1}^{n}(1+q^{2j-1}+q^{4j-2})x^nq^{2n}}{(q^2;q^2)_n}.$$

Then clearly $f(0,q)=f(x,0)=1$, and

$$\begin{aligned} f(x)-f(xq^2)&=\sum_{n\geq 0}\frac{\prod_{j=1}^{n}(1+q^{2j-1}+q^{4j-2})x^nq^{2n}(1-q^{2n})}{(q^2;q^2)_n}\\ &=xq^2\sum_{n\geq 0}\frac{\prod_{j=1}^{n}(1+q^{2j-1}+q^{4j-2})x^nq^{2n}(1+q^{2n+1}+q^{4n+2})}{(q^2;q^2)_n}\\ &=xq^2\Big(f(x)+qf(xq^2)+q^2f(xq^4)\Big), \end{aligned}$$

and if

$$f_1(x):=(xq^2;q^2)_\infty f(x), \tag{10.5}$$

then multiplying the above equation by $(xq^4;q^4)_\infty$, we obtain

$$f_1(x)=(1+xq^3)f_1(xq^2)+xq^4(1-xq^4)f_1(xq^4). \tag{10.6}$$

Iterating (10.6), we obtain

$$\begin{aligned} f_1(x)=\ & f_1(xq^2)+xq^3\big((1+xq^5)f_1(xq^4)+xq^6(1-xq^6)f_1(xq^6)\big)\\ &+xq^4(1-xq^4)f_1(xq^4) \end{aligned} \tag{10.7}$$

$$=f_1(xq^2)+(xq^3+xq^4)f_1(xq^4)+x^2q^9(1-xq^6)f_1(xq^6).$$

Comparing (10.7) with (10.4), we see that $f_1(x)$ and $f_{00}(z,xq^2,q)$ satisfy the same q-difference equation and have the same initial value of 1 at $x=0$ dn $q=0$. Hence

$$f_1(x)=f_{0,0}(3,xq^2,q)$$

which is assertion (10.3). $\square$

First proof of (1.8).

Set $x=1$ in (10.3) and compare with (10.1). $\square$

Second proof of (1.8) *(combinatorial).*

We shall provide a combinatorial proof of the assertion

$$\frac{f_{0,0}(z,q^2,q)}{(q^2;q^2)_\infty}=\sum_{n=0}^{\infty}\frac{q^{2n}\prod_{j=1}^{n}(1+q^{2j-1}+q^{5j-2})}{(q^2;q^2)_n} \tag{10.8}$$

which by Lemmas 10 and 11 is equivalent to (1.8).

It is immediate by inspection that the right-hand side of (10.8) is the generating function for partitions in which the largest part is even and odd parts appear at most twice.

Theorem 2 tells us that the left-hand side is the generating function for overpartitions where all parts are >1; 2 is never overlined, odd parts appear at most once and are overlined; overlined parts differ by at least 3, and there is never a sequence of parts of the form $\overline{(2j+3)}+\overline{(2j+6)}+(2j+6)$.

We provide a bijection between these two classes of partitions as follows.

We begin with the overpartitions, and we consider a modified Ferrers graph as follows. Each odd part $2j+1$ is represented by the row

$$\underbrace{2\,2\,2\cdots 2}_{j}\;1$$

Each even, nonoverlined part $2j$ is given a row of j 2's:

$$\underbrace{2\,2\ldots 2}_{j\text{ times}}.$$

Each overlined $\overline{2j}$ is given a row of $j-1$ 2's and two 1's

$$\underbrace{2\,2\cdots 2}_{j-1\text{ times}}\;1\,1.$$

However if both $2j$ and $\overline{2j}$ are parts the two rows are to be:

$$\begin{matrix}2\,2\,2\cdots 2\,2\,1\\ \underbrace{2\,2\,2\cdots 2}_{j-1\text{ times}}\;1\end{matrix}$$

This procedure produces from the given set of overpartitions a unique set of the modified 2-modular Ferrers graphs. The uniqueness is guaranteed by exclusion of part sequences of the form $\overline{(2j+3)} + \overline{(2j+6)} + (2j+6)$ because this sequence would yield

$$\begin{array}{l} 2\,2\,2 \cdots 2\,2\,2\,1 \\ 2\,2\,2 \cdots 2\,2\,1 \\ \underbrace{2\,2\,2 \cdots 2}_{(j+1)\text{ times}}\,1 \end{array}$$

but $\overline{(2j+4)} + (2j+4) + \overline{(2j+7)}$ yields exactly the same component of the modified 2-modular Ferrers graph.

Now to complete the bijection we read these Ferrers graphs via columns instead of rows, and the resulting partitions are those generated by right-hand side of (10.8). □

11 The Alladi-Schur Theorem

We remarked at the end of Section 2, that our work here was inspired by the discoveries in [6]. In particular, the identity (eq. (2.11) restated):

$$(11.1) \qquad \sum_{n,j\geq 0} \frac{(-1)^j x^{n+2j} q^{(3j+n)^2+\binom{n}{2}}}{(q;q)_n (q^6;q^6)_j} = (x;q^3)_\infty \sum_{n\geq 0} \frac{x^n(-q;q^3)_n(-q^2;q^3)_n}{(q^3;q^3)n}$$

is naturally related to a proof of Schur's 1926 partition theorem. Namely, as was shown in [1, eq. (2.15)], an application of Abel's lemma reveals

$$\begin{aligned} &\lim_{x\to 1} (x;q^3)_\infty \sum_{n\geq 0} \frac{(-q;q^3)_n(-q^2;q^3)_n x^n}{(q^3;q^3)_n} \\ &= (-q;q^3)_\infty(-q^2;q^3)_\infty = \frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty}. \end{aligned}$$

In the current context, by Lemma 11, with x replaced by xq^{-2}

$$\begin{aligned} f_{0,0}(3,1,q) &= \lim_{x\to 1^-} (x,q^2)_\infty \sum_{n\geq 0} \frac{x^n \prod_{j=1}^{n} (1+q^{2j-1}+q^{4j-2})}{(q^2;q^2)_n} \\ &= \prod_{j=1}^{\infty} (1+q^{2j-1}+q^{4j-2}) \\ &= \frac{(q^3;q^6)_\infty}{(q;q^2)_\infty} \\ &= \frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty}. \end{aligned}$$

Identifying $f_{00}(3,1,q)$ with the left-hand side of (11.1) when $x = 1$, we see that the Alladi formulation of Schur's theorem is naturally related to Lemma 9.

12 Conclusion

The most unsatisfying aspect of this paper is that we have been unable to produce a grand unified treatment of combinatorial aspects of semi-general double series such as the one given in (1.1). If one contrasts the theorems listed in Section 2 with those treated in Theorems 1 and 2, one sees the great diversity of theorems vaguely tied together by the theme of the examination of sequences of parts in partitions.

However, at this stage, one only sees the glimmer of a general theory. Nonetheless, the variety of results found to date suggest that much remains to be found.

References

[1] G. E. Andrews, *On partition functions related to Schurs' second partition theorem*, Proc. Amer. Math. Soc., 19(1968), 441-444.

[2] G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, Cambridge, 1998.

[3] G. E. Andrews, *Partitions with short sequences and mock theta functions*, Proc. Nat. Acad. Sci., 102(2005), 4666-4671.

[4] G. E. Andrews, *A refinement of the Alladi-Schur theorem*, Proc. of Lattice Paths 2015, to appear in Dev. in Math. series.

[5] G. E. Andrews, *The Alladi-Schur polynomials and their factorization*, from Analytic Number Theory, Modular Forms and q-Hypergeometric Serie, G. Andrews and F. Garvan eds., Springer, Cham, Switzerland, 2017, pp. 25-38.

[6] G. E. Andrews, K. Bringmann, and K. Mahlburg, *Double series representations for Schur's partition function and related identities*, J. Comb. Th. (A), 132(2015), 102-119.

[7] K. Bringmann, J. Dousse, J. Lovejoy, and K. Mahlburg, *Overpartitions with restricted odd differences*, Elec. J. Comb., 22(3), (2015), #P3.17.

[8] K. Bringmann, A. E. Holroyd, K. Mahlburg, and M. Vlasenko, *k-Run overpartitions and mock theta functions*, Quart. J. Math., 64(2013), 1009-1021.

[9] K. Bringmann, K. Mahlburg, and K. Nataraj, *Distinct parts partitions without sequences*, Elec. J. Comb., 22(3), (2015), #P3.3.

[10] S. Corteel and J. Lovejoy, *Overpartitions*, Trans. Amer. Math. Soc., 356(2004), 1623-1635.

[11] N. J. Fine, *Basic hypergeometric series and applications*, Amer. Math. Soc., Providence, RI, 1988.

[12] G. Gasper and M. Rahman, *Basic Hypergeometric Series* Cambridge University Press, Cambridge, 1990.

[13] M. D. Hirschhorn, *Developments in the Theory of Partitions*, Ph.D. Thesis, University of New South Wales, 1980.

[14] A. E. Holroyd, T. M. Liggert, and D. Romik, *Integrals, partitions and cellular automata*, Trans. Amer. Math. Soc., 356(2004), 3349-3368.

[15] P.A. MacMahon, *Combinatory Analysis*, Vol. 2, A.M.S. Chelsea Publishing, Providence, RI, 1984.

[16] I. Schur, *Zur additiven Zahlentheorie*, Sitzungsber., Preuss. Akad. Wiss. Phys-Math. Kl. (1926).

[17] L. J. Slater, *Further identities of Rogers-Ramanujan type*, Proc. London Math. Soc. (2), 54(1952), 147-167.

[18] J. J. Sylvester, *A constructive theory of partitions in three acts, an interact and a exodion*, Amer. J. Math., 5(1882), 251-330, 6(1884), 334-336.

[19] G. N. Watson, *The final problem: an account of the mock theta functions*, J. London Math. Soc. 11(1936), 55-80.

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